Testable Implications of the Cournot Model

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Abstract: Consider an industry with N firms producing a homogeneous product. We make t^* observations of this industry, with each observation consisting of the market price and the output and profit of each firm. We identify conditions that such a data set must satisfy so that each observation can be rationalized as the Cournot outcome after a change in the demand function, with each firm having a common cost function across observations.

Very preliminary working paper. Comments welcome.

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1. Introduction

Imagine an industry with N firms producing a homogenous product. We make t^* observations of this industry; each observation t ($1 \leq t \leq t^*$) consists of the market price and the output and profit of each firm. We know that each observation t corresponds to an outcome for a different demand function, while each firm's cost function remains unchanged across observations. How do we test the hypothesis that the firms in this industry are playing a Cournot game at each observation? We identify conditions that such a data set *must* satisfy for it to be compatible with firms playing a Cournot game. We also show that these conditions are *sufficient*: when they hold, it is possible to construct a downward-sloping inverse demand function (defined on R_+) for each observation t and an increasing cost function for each firm i (again defined on R_+), such that each observation t is the outcome of a Cournot game.

This paper is a contribution to the literature on the observable restrictions/testable implications of various canonical economic models. One of the most influential papers in this literature is Afriat (1967), which identified the strong axiom of revealed preference as the necessary and sufficient condition that a finite data set of price and demand observations must satisfy for it to be compatible with the utility maximization hypothesis. This paper has generated a very large empirical literature. It has also been extended in various ways; in particular, see Varian (1982) for an extension to production theory and Brown and Matzkin (1996) for an analysis of observable restrictions in general equilibrium models.

The rest of the paper is divided into three sections. Section 2 sets out some of the major concepts used in the paper. It also considers the case of a monopoly and identifies the restrictions that a data set of price, output and profits must satisfy for it to be compatible with that arising from a single firm maximizing profit at each observation. Section 3, which is the main section of this paper, address the same issue in the context of an oligopoly. Finally, we show in Section 4 that if firm profits are not observed then *any* data set of prices and firm outputs is compatible with the Cournot model.

2. Rationalizability - The Monopoly Case

Consider an experiment in which we make t^* observations of a monopolist. The observations are indexed by t in $T = \{1, 2, ..., t^*\}$; observation t consists of a triple (P_t, Q_t, Π_t), respectively the price charged by the monopolist, the quantity he sells, and the profit he makes. We require $P_t > 0$ and $Q_t > 0$ for all t; we also require the profit Π_t to be (strictly) smaller than total revenue P_tQ_t, so that the total cost C_t incurred by the monopolist in producing Q_t, which equals $P_tQ_t - \Pi_t$, is positive. The value of Π_t may be positive or negative, so we may observe losses.

We say that the set of observations $\{(P_t, Q_t, \Pi_t)\}_{t\in T}$ is *rationalizable* if they are consistent with a profit-maximizing monopolist having a stable cost structure, with each observation corresponding to a different demand condition. Formally, we require that there be a $\mathbb{C}^1$ function $\bar{C} : R_+ \to R$ and $\mathbb{C}^1$ functions $\bar{P}_t : R_+ \to R$, for each t in T, such that

(i) $\bar{C}(q) \geq 0$ and $\bar{C}'(q) > 0$;

(ii) $\bar{P}_t(q) \geq 0$ and $\bar{P}_t'(q) \leq 0$, with the the latter inequality being strict if $\bar{P}_t(q) > 0$;

(iii) $\bar{C}(Q_t) = C_t$ and $\bar{P}_t(Q_t) = P_t$; and

(iv) $\text{argmax}_{q\geq 0}[\bar{P}_t(q)q - \bar{C}(q)] = Q_t$.

Function $\bar{C}$ is the monopolist's *cost function*; condition (i) says that it is positive and strictly increasing.[1] Function $\bar{P}_t$ is the *inverse demand function* at observation t; condition (ii) says that more output can only be sold at a strictly lower price, until the price reaches zero. From this point on, we shall refer to any $\mathbb{C}^1$ cost function satisfying (i) as a *regular* cost function; similarly, a *regular* inverse demand function is a $\mathbb{C}^1$ inverse demand function that obeys (ii). Condition (iii) requires the inverse demand and cost functions to coincide with their observed values at each t. Lastly, condition (iv) requires the observations to be consistent with profit maximization. It is clear that conditions (iii) and (iv) together guarantee that the observed profit is $\Pi_t = \max_{q\geq 0}[\bar{P}_t(q)q - \bar{C}(q)]$. Note that we have allowed for the existence of sunk costs since we do not require $\bar{C}(0) = 0$. This implies that there is no nonnegativity constraint on profits, since the option of producing nothing can still incur a cost.

We say that the observations are *generic* if $Q_t \neq Q_{t'}$ whenever $t \neq t'$. Let $\{(P_t, Q_t, \Pi_t)\}_{t\in T}$ be a generic set of observations. For each t, we define the set $S(t) = \{t' \in T : Q_{t'} < Q_t\}$; in other words, $S(t)$ consists of those observations with output levels lower than Q_t. When $S(t)$ is nonempty, we denote $s(t) = \text{argmax}_{t'\in S(t)} Q_{t'}$; that is, $s(t)$ is the observation corresponding to the highest output level below Q_t. For those observations t with nonempty $S(t)$, we define $\Delta Q_t = Q_t - Q_{s(t)}$ and $\Delta C_t = C_t - C_{s(t)}$. So, ΔC_t is the extra cost incurred by the monopoly when it increases its output from $Q_{s(t)}$ to Q_t. We denote the *average marginal cost* over that output range by $M_t = \Delta C_t / \Delta Q_t$.

The generic set of observations $\{(P_t, Q_t, \Pi_t)\}_{t\in T}$ is said to satisfy the *increasing cost condition* (ICC) if $\Delta C_t > 0$ whenever it is defined. It obeys the *discrete marginal condition* (DMC) if, whenever $S(t)$ is nonempty,

$$P_t Q_{t'} - C_{t'} < P_t Q_t - C_t \text{ for } t' \in S(t). \tag{1}$$

We may re-arrange this inequality to obtain

$$C_t - C_{t'} = \sum_{s\in S(t)\setminus(S(t')\cup\{t'\})} \Delta C_s < P_t(Q_t - Q_{t'}) \text{ for } t' \in S(t). \tag{2}$$

This says that the additional cost incurred by producing at Q_t rather than $Q_{t'}$ is smaller than the added revenue earned if the increased output is sold at price P_t.

Proposition 1: *The generic set of observations $\{(P_t, Q_t, \Pi_t)\}_{t\in T}$ is rationalizable only if it obeys ICC and DMC.*

Proof: If the set of observations is rationalizable, then for any t' in $S(t)$, we have $C_t - C_{t'} = \int_{Q_{t'}}^{Q_t} \bar{C}'(q)dq > 0$, since $C'(q) > 0$.

Suppose that there is a violation of DMC. Then $P_t Q_{t'} - C_{t'} \geq P_t Q_t - C_t$ for t' in $S(t)$. But $\bar{P}_t(Q_{t'}) > \bar{P}_t(Q_t) = P_t$, so $\bar{P}_t(Q_{t'})Q_{t'} - C_{t'} > P_t Q_t - C_t$, which means that the monopolist is better off producing at $Q_{t'}$ rather than at Q_t. *Q.E.D.*

The next result says that ICC and DMC are also sufficient for rationalizability.

THEOREM 1: *Suppose the generic set of observations* $\{(P_t, Q_t, \Pi_t)\}_{t \in T}$ *obeys ICC and DMC, and let* $\{\alpha_t\}_{t \in T}$ *be a set of numbers satisfying* $0 < \alpha_t < P_t$. *Then the observations are rationalizable and the cost function* $\bar{C} : R_+ \to R$ *can be chosen such that* $\bar{C}'(Q_t) = \alpha_t$ *for all* $t \in T$.

Theorem 1 is an immediate consequence of the following two lemmas. Loosely speaking, Lemma 1 provides us with the cost function needed to rationalize the set of observations, while Lemma 2 gives the demand functions corresponding to each observation t.

LEMMA 1: *Suppose the generic set of observations* $\{(P_t, Q_t, \Pi_t)\}_{t \in T}$ *obeys ICC and DMC and let* $\{\alpha_t\}_{t \in T}$ *be a set of numbers satisfying* $0 < \alpha_t < P_t$. *Then, there is a regular cost function* $\bar{C} : R_+ \to R$ *such that, for all* t *in* T,

(i) $\bar{C}(Q_t) = C_t$, *and* $\bar{C}'(Q_t) = \alpha_t$;

(ii) on a neighborhood of Q_t, $\bar{C}$ *is twice differentiable and satisfies that* $\bar{C}''(q) > 0$; *and*

(ii) for all q *in* $[0, Q_t)$,

$$P_t q - \bar{C}(q) < P_t Q_t - \bar{C}(Q_t). \tag{3}$$

Proof: The construction of the cost function can be seen in Figure 1. (An explicit construction is given in the Appendix.) The thick curve corresponds to a piecewise linear function that is increasing and satisfies equation (3); it can be constructed given that the dataset satisfies ICC and DMC. The slope of this function at each Q_t is exactly α_t. Now, since equation (3) is a strict inequality, notice that one can perturb the function around each Q_t to obtain local convexity there, still maintaining the property that equation (3) is satisfied. Again because (3) is strict, one can smooth all kinks to obtain a regular $\bar{C}$. *Q.E.D.*

Note that property (i) in Lemma 1 requires the cost function to obey the specified marginal cost conditions and to agree with the cost data at the observed output levels. Property (iii) in Lemma 1 is a strengthening of DMC: DMC requires (3) to hold at discrete output levels, while (ii) requires it to hold at *all* output levels up to Q_t.

The next result says that, for the cost function guaranteed by Lemma 1, we could find a demand function for each t such that the profit-maximizing output decision is Q_t.

LEMMA 2: *Let $\{\alpha_t\}_{t\in T}$ be a set of numbers satisfying $0 < \alpha_t < P_t$, and let $\bar{C} : R_+ \to R$ be a regular cost function satisfying the three properties of Lemma 1. Then, for any $t \in T$, there is a regular inverse demand function $\bar{P}_t : R_+ \to R$ such that*
(i) $\bar{P}_t(Q_t) = P_t$; and
(ii) $\text{argmax}_{q\geq 0}[\bar{P}_t(q)q - \bar{C}(q)] = Q_t$.

Proof: For an observation t, consider a function of the form $P_t + \gamma(q)(Q_t - q)$, where

$$\gamma(q) = \begin{cases} \Delta, & \text{if } q \leq Q_t - \epsilon; \\ \frac{P_t - \alpha_t}{Q_t}, & \text{if } Q_t - \epsilon \leq q \leq Q_t + \epsilon; \\ \beta, & \text{if } q > Q_t + \epsilon. \end{cases}$$

It is immediate that this function is decreasing, and that its image at Q_t is P_t. Notice that, by property (iii) in Lemma 1, one can find a positive Δ (close enough to 0) such that

$$(P_t - \Delta(Q_t - q))q - \bar{C}(q) < P_t Q_t - C_t$$

for any $q \leq Q_t$. Also, one can always find a large enough β such that

$$(P_t - \beta(Q_t - q))q - \bar{C}(q) < P_t Q_t - C_t$$

for $q \geq Q_t + \epsilon$. Now, for the remaining case, notice from property (i) of Lemma 1 that Q_t satisfies the first-order conditions of the monopolist's maximization problem, which implies, by condition (ii) of that same Lemma 1, that it is the only production level in the interval the interval $[Q_t - \epsilon, Q_t + \epsilon]$ that solves that problem.

Of course, what this function lacks to be a regular demand function is continuity (and hence differentiability) at a finite number of points. In the appendix we provide an explicit construction of $\bar{P}_t$ that satisfies these two properties. In fact, we provide an alternative construction with the property that $\lim_{q\to 0} \bar{P}_t(q) = \infty$, so that the monopolist is not assumed to face a finite reservation price. *Q.E.D.*

It is common practice to assume that firms have monotonic, i.e, either increasing or decreasing, marginal costs. So, it is natural to ask what restrictions on the set of observations are needed to guarantee rationalizability with cost functions of this sort. We first consider the case where marginal costs are increasing; formally, $\bar{C}''(q) > 0$ for $q > 0$. It is trivial to check that this implies that the average marginal cost over a lower range of output must be lower than the average marginal cost over a higher range of output. On the data set $\{(P_t, Q_t, \Pi_t)\}_{t\in T}$, rationalizability by a cost function with increasing marginal costs must imply that $M_{t'} < M_t$, whenever both M_t and $M_{t'}$ are well-defined and $t' \in S(t)$. The next result says that this condition is also sufficient.

Corollary 1: *Suppose the generic set of observations $\{(P_t, Q_t, \Pi_t)\}_{t\in T}$ obeys ICC and DMC, with $M_{t'} < M_t$ whenever both sides of the inequality are defined and $t' \in S(t)$. Then, the observations are rationalizable and the cost function $\bar{C} : R_+ \to R$ can be chosen to exhibit increasing marginal costs, i.e., $C''(q) > 0$ for all $q > 0$.*

Proof: Again, we use a graph, Figure 2, to illustrate de construction of the cost function, which will be convex, and will further satisfy the conditions of Lemma 1, with $\alpha_t = M_t < P_t$, when M_t is defined, and arbitrary $\alpha_t < \min\{P_t, \min_{t'\neq t}\{M_{t'}\}\}$ when it is not. In Figure 2, we represent the linear interpolation of the points $(Q_{s(t)}, C_{s(t)})$ and (Q_t, C_t); by the assumption that M_t and Q_t are comonotone, we obtain a convex function: the slopes of the linear interpolations are given by M_t.

To complete the proof, we simply need to invoke Lemma 2 again. *Q.E.D.*

3. Rationalizability - The Cournot Model

An industry consists of i^* firms producing a homogeneous good; we denote the set of firms by $I = \{1, 2, ..., i^*\}$. Consider an experiment in which t^* observations are made of this industry. As in the previous section, we index the observations by t in $T = \{1, 2, ..., t^*\}$. For each t, the industry price P_t, the output of each firm $\{Q_{i,t}\}_{i \in I}$ and their profits $\{\Pi_{i,t}\}_{i \in I}$ are observed. We require $P_t > 0$ and $Q_{i,t} > 0$ for all t and i; the profit observations $\Pi_{i,t}$ can take either positive and negative values. Note that the total cost incurred by firm i in producing $Q_{i,t}$, which we denote by $C_{i,t}$, follows immediately from the equation $C_{i,t} = P_t Q_{i,t} - \Pi_{i,t}$.

We say that the set of observations $\{(P_t, \{Q_{i,t}\}_{i \in I}, \{\Pi_{i,t}\}_{i \in I})\}_{t \in T}$ is *Cournot rationalizable* if each observation can be explained as a Cournot equilibrium arising from a different market demand function, keeping the cost function of each firm fixed across observations. Formally, we require that there be a regular cost function $\bar{C}_i : R_+ \to R$ for each firm i and a regular demand function $\bar{P}_t : R_+ \to R$ for each t, such that
(i) $\bar{C}_i(Q_{i,t}) = C_{i,t}$ and $\bar{P}_t(\sum_{j \in I} Q_{j,t}) = P_t$; and
(ii) $Q_{i,t} = \text{argmax}_{q_i \geq 0}[q_i \bar{P}_t(q_i + \sum_{j \neq i} Q_{j,t}) - \bar{C}_i(q_i)]$.

Condition (i) says that the inverse demand and cost functions must coincide with their observed values at each t. Condition (ii) says that, at each observation t, firm i's observed output level $Q_{i,t}$ maximizes its profit given the output of the other firms. It is clear that these conditions imply that the observed profit $\Pi_t = \max_{q_i \geq 0}[q_i \bar{P}_t(q_i + \sum_{j \neq i} Q_{j,t}) - \bar{C}_i(q_i)]$. Note that, as in the previous section, we allow for the existence of sunk costs, since we do not require $\bar{C}_i(0) = 0$.

We say that the observations are *generic* if, for every firm i, we have $Q_{i,t} \neq Q_{i,t'}$ whenever $t \neq t'$. Let $\{(P_t, \{Q_{i,t}\}_{i \in I}, \{\Pi_{i,t}\}_{i \in I})\}_{t \in T}$ be a generic set of observations. For each firm i, we may define $S_{i,t}$, $s_{i,t}$, $\Delta Q_{i,t}$, $\Delta C_{i,t}$, and $M_{i,t}$, in a way analogous to our definitions in the previous section. We say that the set of observations obey the increasing cost condition (ICC) if, for each i, $\{(P_t, Q_{i,t}, \Pi_{i,t})\}_{t \in T}$ obeys the increasing cost condition ICC (in the sense previously defined). Similarly, we say that it obeys the discrete marginal condition (DMC) if, for each i, $\{(P_t, Q_{i,t}, \Pi_{i,t})\}_{t \in T}$ obeys DMC.

It is clear, for exactly the same reasons as the ones given in the monopoly case, that ICC and DMC are *necessary* for a set of observations to be Cournot rationalizable. Specifically, ICC is needed to guarantee that each firm's production cost is increasing in output, and DMC is needed to guarantee that each firm is not strictly better off by producing less than the observed output. The next result says that these conditions are also sufficient for Cournot rationalizability.

THEOREM 2: *Suppose that the generic set of observations* $\{(P_t, \{Q_{i,t}\}_{i\in I}, \{\Pi_{i,t}\}_{i\in I})\}_{t\in T}$ *obeys ICC and DMC. Then the set is Cournot rationalizable.*

Just as Theorem 1 follows from Lemmas 1 and 2, we can prove Theorem 2 with a similar two-step procedure. Note that, at observation t, if firm i is indeed playing its best response for demand function $\bar{P}_t$ and cost function $\bar{C}_i$, then the first order condition

$$\bar{P}_t'(\sum_{j\in I} Q_{j,t})Q_{i,t} + P_t = \bar{C}_i'(Q_{i,t}) \tag{4}$$

must be satisfied. It follows that

$$-\bar{P}_t'(\sum_{j\in I} Q_{j,t}) = \frac{P_t - \bar{C}_1'(Q_{1,t})}{Q_{1,t}} = \frac{P_t - \bar{C}_2'(Q_{2,t})}{Q_{2,t}} = \ldots = \frac{P_t - \bar{C}_{i^*}'(Q_{i^*,t})}{Q_{i^*,t}}. \tag{5}$$

This motivates the condition imposed on the cost functions in the next result, which is loosely analogous to Lemma 1.

LEMMA 3: *Let* $\{(P_t, \{Q_{i,t}\}_{i\in I}, \{\Pi_{i,t}\}_{i\in I})\}_{t\in T}$ *be a generic set of observations obeying ICC and DMC and suppose that the positive numbers* $\{\alpha_{i,t}\}_{(i,t)\in I\times T}$ *satisfy*

$$\frac{P_t - \alpha_{1,t}}{Q_{1,t}} = \frac{P_t - \alpha_{2,t}}{Q_{2,t}} = \ldots = \frac{P_t - \alpha_{M,t}}{Q_{M,t}} > 0 \;\; \textit{for all } t \textit{ in } T. \tag{6}$$

Then, there are regular cost functions $\bar{C}_i : R_+ \to R$ *such that*

(i) $\bar{C}_i(Q_{i,t}) = C_{i,t}$ *and* $\bar{C}_i'(Q_{i,t}) = \alpha_{i,t}$*;*

(ii) on a neighborhood of $Q_{i,t}$*,* $\bar{C}_i$ *is twice differentiable and satisfies that* $\bar{C}_i''(q) > 0$*; and*

(iii) for all q_i *in* $[0, Q_{i,t})$*,*

$$P_t q_i - \bar{C}_i(q_i) < P_t Q_{i,t} - \bar{C}_i(Q_{i,t}). \tag{7}$$

Proof: The construction of the cost functions of this lemma is identical to the one of the monopolist (Lemma 1), for each of the firms in the industry. *Q.E.D.*

It is important to notice that for *any* P_t and $\{Q_{i,t}\}_{i \in I}$ there *always exist* positive numbers $\{\alpha_{i,t}\}_{i \in I}$ such that equation (6) holds. Suppose that firm k produces more than any other firm at observation t, i.e., $Q_{k,t} \geq Q_{i,t}$ for all i in I. Let $\alpha_{k,t}$ be any positive number smaller than P_t, and define $\beta = (P_t - \alpha_{k,t})/Q_{k,t}$. Then,

$$\alpha_{i,t} = P_t - \beta Q_{i,t} \geq P_t - \beta Q_{k,t} = \alpha_{k,t} > 0.$$

The next result is analogous to Lemma 2. It is clear that this result together with Lemma 3 proves Theorem 2.

LEMMA 4: *Let $\{\alpha_{i,t}\}_{(i,t) \in I \times T}$ be a set of positive numbers satisfying equation (6) and suppose that the cost functions $\bar{C}_i : R_+ \to R$ satisfy the properties in Lemma 3. Then, there are regular demand functions $\bar{P}_t : R_+ \to R$ such that, for every firm i,*

$$Q_{i,t} = \operatorname{argmax}_{q_i \geq 0} [q_i \bar{P}_t(q_i + \sum_{j \neq i} Q_{j,t}) - \bar{C}_i(q_i)].$$

Now, let J be a subset of firms in the industry with the property that $M_{i,t'} < M_{i,t}$, whenever both $M_{i,t}$ and $M_{i,t'}$ are well-defined and $t' \in S_i(t)$. In other words, data from these firms show that their average marginal costs are increasing with their output levels. In this case, we may wish to require that the cost function used to rationalize each of these firms' behaviors should exhibit increasing marginal costs. What conditions are needed to guarantee rationalizability with this stronger requirement?

We have already noted that if a rationalization exists, then the firms' marginal costs must obey (5), but for a firm i in J with increasing marginal cost, we can say more about $\bar{C}'_i(Q_{i,t})$. For a given observation t, there may or may not exist another observation t' such that $Q_{i,t'} > Q_{i,t}$. If such an observation exists, then we define $h(t)$ to be the observation producing the lowest output level higher than $Q_{i,t}$; in other words, $s(h(t)) = t$. It is not hard to check that if $\bar{C}_i$ has increasing marginal cost, then $M_{i,h(t)} > M_{i,t}$ and $\bar{C}'(Q_{i,t})$ must be in the open interval $(M_{i,t}, M_{i,h(t)})$.[2]

So, a *necessary* condition for rationalizability is that there exists $\{\alpha_{i,t}\}_{(i,t)\in I\times T}$ such that (6) holds, and for every firm i in J, we also require that $M_{i,t} < \alpha_{i,t} < M_{i,h(t)}$.[3] These requirements can be restated succinctly as the following *common ratio condition* (CRC): the set $\cap_{i\in I} A_{i,t}$ is nonempty for all t in T, where

$$\begin{aligned} A_{i,t} &= \{(P_t - x_i)/Q_{i,t} : M_{i,t} < x_i < \min\{P_t, M_{i,h(t)}\}\} \text{ for } i \in J, \text{ and} \\ A_{i,t} &= \{(P_t - x_i)/Q_{i,t} : 0 < x_i < P_t\} \text{ for } i \in I \setminus J. \end{aligned}$$

The next result says that rationalizability with cost functions having increasing marginal costs is guaranteed by the addition of CRC to the usual conditions.

Corollary 2: *Let $\{(P_t, \{Q_{i,t}\}_{i\in I}, \{\Pi_{i,t}\}_{i\in I})\}_{t\in T}$ be a generic set of observations such that for any firm i in $J \subseteq I$, we have $M_{i,t'} < M_{i,t}$, whenever both $M_{i,t}$ and $M_{i,t'}$ are well-defined and $t' \in S_i(t)$. If this set of observations obeys ICC, DMC, and CRC (the last with respect to J), then it is Cournot rationalizable and the cost functions for firms in J can be chosen to have increasing marginal cost.*

Proof: The construction of the cost functions is as in corollary 1. The construction of the demand functions is as in Theorem 2. *Q.E.D.*

4. Rationalizability Without Observing Costs

In this section, we consider the problem of Cournot rationalizability in the case where the only information available to the observer are prices and firm output levels; in particular, the profits earned - and thus the costs incurred - by each firm are not known. Formally, the dataset reduces to $\{P_t, (Q_{i,t})_{i\in I}\}_{t\in T}$, namely a price level for each t in T, and a production level for each i in I and each t in T. As before, we will call the dataset *generic* if $Q_{i,t'} \neq Q_{i,t}$ whenever $t \neq t'$, and we will say that it is *Cournot rationalizable* if we can find a regular demand function, $\bar{P}_t$, for each observation t, and a regular cost function, $\bar{C}_i$, for each firm i, such that

(i) $\bar{P}_t(\sum_{i\in I} Q_{i,t}) = P_t$; and

(ii) $Q_{i,t} = \text{argmax}_{q_i \geq 0}[q_i \bar{P}_t(q_i + \sum_{j\neq i} Q_{j,t}) - \bar{C}_i(Q_i)]$.

In other words, the tth observation $(P_t, (Q_{i,t})_{i\in I})$ is the Cournot outcome when firm i has cost function $\bar{C}_i$ (for all i) and the market inverse demand function is $\bar{P}_t$. The following result says that Cournot competition imposes no restriction on the observations $\{P_t, (Q_{i,t})_{i\in I})\}_{t\in T}$. We conclude that cost information is crucial to the refutability of the Cournot model in such a context.

Corollary 3: *Any generic set of observations $\{P_t, (Q_{i,t})_{i\in I})_{t\in T}\}$, is Cournot rationalizable. In the rationalization, we may require the cost functions of all firms to exhibit increasing marginal costs.*

Proof: By Theorem 2, all we need to show is that we can find a hypothetical array of individual costs, $\{C_{i,t}\}_{(i,t)\in I\times T}$, which, if added to the observed data, would give a set of observations that obeys MCC and DMC (and CRC, if desired). To see that this is indeed the case, let $\mu_i = \min_{t:S(t)\neq\varnothing}\{P_t(Q_{i,t} - Q_{i,s_i(t)})\}$, a strictly positive number, and pick any number $0 < C_{i,t} < \mu_i$ for each t in T. Then, we immediately have that, whenever $Q_{i,t'} < Q_{i,t}$,

$$C_{i,t} + P_t(Q_{i,t'} - Q_{i,t}) \leq C_{i,t} + P_t(Q_{i,s_i(t)} - Q_{i,t}) \leq C_{i,t} - \mu_i < 0 < C_{i,t'},$$

which suffices to imply condition DMC. Of course, in order to guarantee rationalizability, it now suffices to pick $\{C_{i,t}\}_{(i,t)\in I\times T}$ such that $\Delta C_{i,t} > 0$, which is straightforward.

If so desired, one can also pick the latter array so that $M_{i,t}$ is co-monotone with $Q_{i,t}$. *Q.E.D.*

Appendix: More Explicit Constructions of Rationalizing Functions

The Monopoly Case

A cost function: For notational simplicity, let us assume, without any loss of generality, that

$$Q_1 < Q_2 < \ldots < Q_{t^*}.$$

In order to construct the different pieces of the cost function, we need to define the intervals of its domain. Let us define $\bar{\epsilon} = \frac{1}{4}\min_{t\geq 2}\{Q_t - Q_{t-1}\}$, which is a strictly positive number (since the observation is generic).

Since the observation obeys ICC, we can fix, for each $t \geq 2$, a strictly positive number $\hat{\epsilon}_t$ such that

$$C_{t-1} + \hat{\epsilon}_t \alpha_{t-1} < C_t - \hat{\epsilon}_t \alpha_t,$$

and since the observation obeys DMC, we can fix, for each $t \leq t^* - 1$, a strictly positive $\tilde{\epsilon}_t$ such that

$$C_t + \alpha_t(q - Q_t) > C_{t'} + P_{t'}(q - Q_{t'}),$$

for all $t' > t$ and all $q \in [Q_t - \tilde{\epsilon}_t, Q_t + \tilde{\epsilon}_t]$.

Define

$$\epsilon = \min\left\{\bar{\epsilon}, \min_{t\geq 2}\{\hat{\epsilon}_t\}, \min_{t\leq t^*-1}\{\tilde{\epsilon}_t\}, \frac{C_1}{2\alpha_1}, \frac{Q_1}{2}\right\}$$

and

$$\gamma = \min\left\{\min_t\{P_t\}, \frac{C_1 - \alpha_1\epsilon}{Q_1 + \epsilon}\right\}.$$

By construction, $\epsilon > 0$ and $\gamma > 0$.

Now, we can define the cost function $\bar{C}$, piecewise, as follows:
(i) if $q < Q_1 - \epsilon$, then $\bar{C}(q) = C_1 - \alpha_1\epsilon + \gamma(q - Q_1 + \epsilon)$;
(ii) if $Q_t - \epsilon \leq q \leq Q_t + \epsilon$ for some t, then $\bar{C}(q) = C_t + \alpha_t(q - Q_t)$;
(iii) if $q \geq Q_{t^*}$, then $\bar{C}(q) = C_{t^*} + \alpha_{t^*}(q - Q_{t^*})$.
(Note that since $\epsilon \leq \bar{\epsilon}$, function $\bar{C}$ is so far well defined.) Now, we can complete the construction by using linear interpolation of subsequent subdomains: if $Q_{t-1} + \epsilon < q < Q_t - \epsilon$, for some $t \geq 2$, then we just let

$$\bar{C}(q) = \frac{Q_t - \epsilon - q}{Q_t - Q_{t-1} - 2\epsilon}(C_{t-1} + \alpha_{t-1}\epsilon) + \frac{q - Q_{t-1} - \epsilon}{Q_t - Q_{t-1} - 2\epsilon}(C_t - \alpha_t\epsilon).$$

Note also that $\bar{C}$ is strictly increasing, since $\epsilon \leq \hat{\epsilon}_t$, for all $t \geq 2$, and $\alpha_t > 0$ for all t. Also, since $\gamma \leq \frac{C_1 - \alpha_1\epsilon}{Q_1 + \epsilon}$, it is true that $\bar{C}(0) > 0$, and hence $\bar{C}(q) > 0$ for any

$q > 0$. It also follows by construction that $\bar{C}$ is continuous, and that $\bar{C}(Q_t) = C_t$ and $\bar{C}'(Q_t) = \alpha_t$for all t in T.

Now, we show that $\bar{C}(q) > C_t + P_t(q - Q_t)$ for any t and any $q < Q_t$. Fix t and $q < Q_t$, and consider the following cases:

1. If $t = 1$ and $q < Q_1 - \epsilon$. Then, by construction,

$$\begin{aligned} \bar{C}(q) &= C_1 - \alpha_1\epsilon + \gamma(q - Q_1 + \epsilon) \\ &> C_1 - P_1\epsilon + \gamma(q - Q_1 + \epsilon) \\ &\geq C_1 - P_1\epsilon + P_1(q - Q_1 + \epsilon) \\ &= C_1 + P_1(q - Q_1), \end{aligned}$$

   where the first inequality follows since $\alpha_1 < P_1$ and the second inequality since $\gamma \leq P_1$.

2. If $t \geq 2$ and $q < Q_1 - \epsilon$. Then,

$$\begin{aligned} \bar{C}(q) &= C_1 - \alpha_1\epsilon + \gamma(q - Q_1 + \epsilon) \\ &> C_t + P_t(Q_1 - \epsilon - Q_t) + \gamma(q - Q_1 + \epsilon) \\ &\geq C_t + P_t(Q_1 - \epsilon - Q_t) + P_t(q - Q_1 + \epsilon) \\ &= C_t + P_t(q - Q_t), \end{aligned}$$

   where the first inequality follows since $\epsilon \leq \tilde{\epsilon}_1$ and the second inequality since $\gamma \leq P_t$.

3. If for some $t' \neq t$, it is true that $Q_{t'} - \epsilon \leq q \leq Q_{t'} + \epsilon$. Then, it must be that $t' < t$ and, by construction,

$$\bar{C}(q) = C_{t'} + \alpha_{t'}(q - Q_{t'}) > C_t + P_t(q - Q_t),$$

   which follows since $\epsilon < \tilde{\epsilon}_t$.

4. If $Q_t - \epsilon \leq q$. Then,

$$\bar{C}(q) = C_t + \alpha_t(q - Q_t) > C_t + P_t(q - Q_t),$$

   because $\alpha_t < P_t$.

5. In any other case, it must be that for some $t' \leq t$, $Q_{t'-1} + \epsilon < q < Q_{t'} - \epsilon$. Then, by construction,

$$\bar{C}(q) = \frac{Q_t - \epsilon - q}{Q_t - Q_{t-1} - 2\epsilon}\bar{C}(Q_{t'-1} + \epsilon) + \frac{q - Q_{t-1} - \epsilon}{Q_t - Q_{t-1} - 2\epsilon}\bar{C}(Q_{t'} - \epsilon).$$

By cases 3 and 4 above, $C(Q_{t'-1}+\epsilon) > C_t + P_t(Q_{t'-1}+\epsilon-Q_t)$ and $C(Q_{t'}-\epsilon) > C_t + P_t(Q_{t'}-\epsilon-Q_t)$. It follows that

$$\begin{aligned}\bar{C}(q) &> \frac{Q_t-\epsilon-q}{Q_t-Q_{t-1}-2\epsilon}(C_t+P_t(Q_{t'-1}+\epsilon-Q_t)) \\ &\quad +\frac{q-Q_{t-1}-\epsilon}{Q_t-Q_{t-1}-2\epsilon}(C_t+P_t(Q_{t'}-\epsilon-Q_t)) \\ &= C_t+P_t(q-Q_t),\end{aligned}$$

where the equality follows by direct computation.

Now, we can obtain convexity of $\bar{C}$ in the intervals around each Q_t: we can simply redefine the function at that step as

$$\bar{C}(q) = C_1 - \alpha_1\epsilon + \gamma(q - Q_1 + \epsilon) + \delta_t\sqrt{(}(Q_t-q)^2+1) - 1;$$

with δ_t positive, but small enough, we preserve all the properties above, and obtain a strictly positive second derivative near Q_t. The function $\bar{C}$ is differentiable everywhere except for finitely many points, but these can be smoothed out.

The demand functions: We now construct each demand function independently, given a common cost function satisfying the properties of Lemma 1.

Fix an observation t in T. As before, we start the construction by subdomains: given $\epsilon > 0$ be such that $\bar{C}''(q) > 0$ for all $q \in [Q_t-\epsilon, Q_t+\epsilon]$, we are going to construct the demand over the intervals $[0, Q_t-\epsilon]$, $[Q_t-\epsilon/2, Q_t+\epsilon]$ and $[Q_t+\epsilon, \infty)$.

For the first interval, let us define the function $f: R\times R_+ \to R$ by

$$f(\Delta, q) = (P_t + \Delta(Q_t-q))q - \bar{C}(q) - (P_tQ_t - C_t),$$

which is continuous. Function $F: R \to R; \Delta \mapsto \max_{0\leq q\leq Q_t/2} f(\Delta,q)$ is well defined, and continuous. By the properties of $\bar{C}$, we know that $F(0) < 0$, so, by continuity, we can find some $\Delta > 0$ such that $F(\Delta) < 0$. With one such Δ, function $d_1(q) = P_t + \Delta(Q_t-q)$ is strictly decreasing, and has the property that

$$d_1(q)q - \bar{C}(q) < P_tQ_t - \bar{C}(Q_t) \tag{8}$$

for all $q \leq Q_t - \epsilon/2$.

Now, define $\gamma = \frac{P_t-\alpha_t}{Q_t}$, a strictly positive number. Define $d_2(q) = P_t + \gamma(Q_t-q)$, and notice that

$$d_2'(q)q + d_2(q) - \bar{C}'(q) = P_t + \gamma(Q_t-2q) - \alpha_t,$$

and that, if $q \in [Q_t - \epsilon, Q_t + \epsilon]$,

$$d_2''(q)q + 2d_2'(q) - \bar{C}''(q) = -2\gamma - \bar{C}''(q) < 0.$$

This suffices to show that

$$d_2(q)q - \bar{C}(q) < P_t Q_t - \bar{C}(Q_t) \tag{9}$$

for any $q \in (Q_t - \epsilon, Q_t + \epsilon)$.

Thirdly, let

$$\mu = \min_{q \in [Q_t+\epsilon, Q_t+2\epsilon]} \bar{C}'(q)$$

again a strictly positive number. Define the strictly negative number

$$\beta = \min\left\{-1, \frac{\mu - P_t + \epsilon\gamma}{Q_t + \epsilon}, \frac{-P_t + \epsilon\gamma}{\epsilon}\right\},$$

and let

$$d_3(q) = P_t - \epsilon\gamma + \beta(q - Q_t - \epsilon)$$

be defined over $[Q_t + \epsilon, \infty)$. By direct computation, for any $q \in [Q_t + \epsilon, Q_t + 2\epsilon]$, one has that

$$\begin{aligned} d_3'(q)q + d_3(q) &= P_t - \epsilon\gamma + 2\beta q - \beta(Q_t + \epsilon) \\ &\leq P_t - \epsilon\gamma + \beta(Q_t + \epsilon) \\ &\leq P_t - \epsilon\gamma + \frac{\mu - P_t + \epsilon\gamma}{Q_t + \epsilon}(Q_t + \epsilon) \\ &= \mu \\ &\leq \bar{C}'(q). \end{aligned}$$

It follows that

$$\begin{aligned} d_3(q)q &= d_3(Q_t + \epsilon)(Q_t + \epsilon) + \int_{Q_t+\epsilon}^{q} (d_3'(v)v + d_3(v))dv \\ &\leq d_3(Q_t + \epsilon)(Q_t + \epsilon) + \int_{Q_t+\epsilon}^{q} \bar{C}'(v)dv \\ &= d_3(Q_t + \epsilon)(Q_t + \epsilon) + \bar{C}(q) - \bar{C}(Q_t + \epsilon) \\ &= d_2(Q_t + \epsilon)(Q_t + \epsilon) + \bar{C}(q) - \bar{C}(Q_t + \epsilon), \end{aligned}$$

which implies, by (9), that

$$d_3(q)q - \bar{C}(q) < d_2(Q_t)Q_t - \bar{C}(Q_t). \tag{10}$$

The three functions constructed before are pieces of the inverse demand function. We guarantee continuity by constructing $\bar{P}_t$ as follows:
(i) if $q < Q_t - \epsilon$, then $\bar{P}_t(q) = d_1(q)$;
(ii) if $Q_t - \frac{\epsilon}{2} \leq q < Q_t + \epsilon$, then $\bar{P}_t(q) = d_2(q)$;
(iii) if $q \geq Q_t + \epsilon$, then $\bar{P}_t(q) = \max\{d_3(q), 0\}$; and
(iv) if $Q_t - \epsilon \leq q < Q_t - \frac{\epsilon}{2}$,

$$\bar{P}_t(q) = \frac{2}{\epsilon}((q - Q_t + \epsilon)d_2(q) + (Q_t - \frac{\epsilon}{2} - q)d_1(q)).$$

Function $\bar{P}_t$ is continuous, nonnegative and strictly decreasing when positive. By equations (8), (9), and (10), it follows that $\max_{q\geq 0} \bar{P}_t(q)q - \bar{C}(q) = Q_t$. To complete the proof, notice that this function is differentiable everywhere except at $Q_t - \epsilon$, $Q_t - \frac{\epsilon}{2}$ and $Q_t + \epsilon$, but, since (8), (9) and (10) are strict inequalities, we can again obtain differentiability at these points, using a convolution, without affecting the previous result.

An alternative demand function: The construction above suffices for the purposes of Lemma 2 and Theorem 1. The demand function, however, has the property that demand becomes null at a finite price, which would contradict, for instance, an assumption that the commodity being considered is essential. We now construct an alternative demand where there is no finite reservation price, namely that $\lim_{q\to 0} \bar{P}_t(q) = \infty$, without requiring any further assumptions.

All we need to do is redefine the d_1 constructed above. For this, define the following function:

$$\phi(q) = \begin{cases} \frac{C_t - \bar{C}(q)}{Q_t - q}, & \text{if } q \neq Q_t; \\ \alpha_t, & \text{if } q = Q_t. \end{cases}$$

This function is continuous and satisfies that $\phi(q) < P_t$ for any $q \leq Q_t$, so we can define $\Delta = \frac{1}{2}(P_t - \max_{q\leq Q_t} \phi(q))$, a strictly positive number. Define the function $d_1(q) = P_t + \Delta(\frac{Q_t}{q} - 1)$, for $q \leq Q_t$, and notice that this function is strictly decreasing. By construction, given $q < Q_t$,

$$\frac{d_1(q) - P_t}{Q_t - q} q = \Delta < P_t - \max_{q'\leq Q_t} \phi(q') \leq P_t - \phi(q),$$

which implies that $d_1(q)q - \bar{C}(q) < P_t Q_t - \bar{C}(Q_t)$, so that equation (8) continues to hold.

Suppose that we have regular cost functions $\{\bar{C}_i\}_{i\in I}$ satisfying the conditions of Lemma 3. We are interested in constructing the demand function corresponding to observation t in T.

We start the construction by fixing an $\epsilon > 0$ such that $\bar{C}_i''(q) > 0$ for all $q_i \in [Q_{i,t} - \epsilon, Q_{i,t} + \epsilon]$. We are going to construct the demand over the intervals $[0, \sum_{i\in I} Q_{i,t} - \epsilon]$, $[\sum_{i\in I} Q_{i,t} - \epsilon/2, \sum_{i\in I} Q_{i,t} + \epsilon]$ and $[\sum_{i\in I} Q_{i,t} + \epsilon, \infty)$.

For the first interval, let us define the functions

$$f_i(\Delta_i, q_i) = (P_t + \Delta_i(Q_{i,t} - q_i))q_i - \bar{C}_i(q_i) - (P_t Q_{i,t} - \bar{C}_i(Q_{i,t}),$$

and

$$F_i(\Delta_i) = \max_{0\le q_i \le Q_{i,t}/2} f_i(\Delta_i, q_i).$$

By the properties of $\bar{C}_i$, we know that $F_i(0) < 0$, so, by continuity and finiteness of i^*, we can find some $\Delta > 0$ such that $F_i(\Delta) < 0$ for all i. With one such Δ, construct the first function

$$d_1(q) = P_t + \Delta(\sum_{i\in I} Q_{i,t} - q),$$

which is strictly decreasing. Notice that for each i and each $q_i \le Q_{i,t} - \epsilon/2$ it is true that

$$d_1(q_i + \sum_{j\ne i} Q_{j,t})q_i - \bar{C}_i(q_i) = (P_t + \Delta(Q_{i,t} - q_i))q_i - \bar{C}_i(q_i) < P_t Q_{i,t} - C_{i,t}. \quad (11)$$

Now, for the second piece, define

$$\gamma = \frac{P_t - \alpha_{1,t}}{Q_{1,t}} = \frac{P_t - \alpha_{2,t}}{Q_{2,t}} = \ldots = \frac{P_t - \alpha_{i^*,t}}{Q_{i^*,t}},$$

a well-defined and strictly positive number. Define

$$d_2(q) = P_t + \gamma(\sum_{i\in I} Q_{i,t} - q),$$

and notice that, for every i,

$$d_2'(q_i + \sum_{j\ne i} Q_{j,t})q_i + d_2(q_i + \sum_{j\ne i} Q_{j,t}) - \bar{C}_i'(q_i) = P_t + \gamma(Q_{i,t} - 2q) - \alpha_{i,t},$$

and that, if $q_i \in [Q_{i,t} - \epsilon, Q_{i,t} + \epsilon]$,

$$d_2''(q_i + \sum_{j\ne i} Q_{j,t})q_i + 2d_2'(q_i + \sum_{j\ne i} Q_{j,t}) - \bar{C}_i''(q_i) = -2\gamma - \bar{C}_i''(q_i) < 0,$$

which suffices to show that

$$d_2(q_i + \sum_{j \neq i} Q_{j,t})q_i - \bar{C}_i(q_i) < P_t Q_{i,t} - \bar{C}_i(Q_{i,t}) \tag{12}$$

for any $q_i \in (Q_{i,t} - \epsilon, Q_{i,t} + \epsilon)$.

Thirdly, let

$$\mu = \min_{i \in I} \left\{ \min_{q_i \in [Q_{i,t}+\epsilon, Q_{i,t}+2\epsilon]} \bar{C}_i'(q_i) \right\}$$

again a strictly positive number. Define the strictly negative number

$$\beta = \min \left\{ -1, \min_{i \in I} \{ \frac{\mu - P_t + \epsilon\gamma}{Q_{i,t} + \epsilon} \}, \frac{-P_t + \epsilon\gamma}{\epsilon} \right\},$$

and let

$$d_3(q) = P_t - \epsilon\gamma + \beta(q - \sum_{i \in I} Q_{i,t} - \epsilon)$$

be defined over $[\sum_{i \in I} Q_t + \epsilon, \infty)$. By direct computation, for any i and any $q_i \in [Q_{i,t} + \epsilon, Q_{i,t} + 2\epsilon]$, we have that

$$\begin{aligned} d_3'(q_i + \sum_{j \neq i} Q_{j,t})q_i + d_3(q_i + \sum_{j \neq i} Q_{j,t}) &= P_t - \epsilon\gamma + 2\beta q_i - \beta(Q_{i,t} + \epsilon) \\ &\leq P_t - \epsilon\gamma + \beta(Q_{i,t} + \epsilon) \\ &\leq P_t - \epsilon\gamma + \frac{\mu - P_t + \epsilon\gamma}{Q_{i,t} + \epsilon}(Q_{i,t} + \epsilon) \\ &= \mu \\ &\leq \bar{C}_i'(q_i); \end{aligned}$$

it follows that

$$\begin{aligned} d_3(q_i + \sum_{j \neq i} Q_{j,t})q_i &= d_3(\sum_{j \in I} Q_{j,t} + \epsilon)(Q_{i,t} + \epsilon) \\ &\quad + \int_{Q_{i,t}+\epsilon}^{q_i} (d_3'(v + \sum_{j \neq i} Q_{j,t})v + d_3(v + \sum_{j \neq i} Q_{j,t}))dv \\ &\leq d_3(\sum_{j \in I} Q_{j,t} + \epsilon)(Q_{i,t} + \epsilon) + \int_{Q_{i,t}+\epsilon}^{q_i} \bar{C}_i'(v)dv \\ &= d_3(\sum_{j \in I} Q_{j,t} + \epsilon)(Q_{i,t} + \epsilon) + \bar{C}_i(q_i) - \bar{C}_i(Q_{i,t} + \epsilon) \\ &= d_2(\sum_{j \in I} Q_{j,t} + \epsilon)(Q_{i,t} + \epsilon) + \bar{C}_i(q_i) - \bar{C}_i(Q_{i,t} + \epsilon), \end{aligned}$$

which implies, by (12), that

$$d_3(q_i + \sum_{j \neq i} Q_{j,t})q_i < d_2(\sum_{j \in I} Q_{j,t})Q_{i,t} - \bar{C}_i(Q_{i,t}). \tag{13}$$

Finally, we construct the demand function as follows:
(i) if $q < \sum_{i \in I} Q_{i,t} - \epsilon$, then $\bar{P}_t(q) = d_1(q)$;
(ii) if $\sum_{i \in I} Q_{i,t} - \frac{\epsilon}{2} \leq q < \sum_{i \in I} Q_{i,t} + \epsilon$, then $\bar{P}_t(q) = d_2(q)$;
(iii) if $q \geq \sum_{i \in I} Q_{i,t} + \epsilon$, then $\bar{P}_t(q) = \max\{d_3(q), 0\}$; and, finally,
(iv) if $\sum_{i \in I} Q_{i,t} - \epsilon \leq q < \sum_{i \in I} Q_{i,t} - \frac{\epsilon}{2}$, then

$$\bar{P}_t(q) = \frac{2}{\epsilon}((q - \sum_{i \in I} Q_{i,t} + \epsilon)d_2(q) + (\sum_{i \in I} Q_{i,t} - \frac{\epsilon}{2} - q)d_1(q)).$$

This function is continuous, nonnegative and strictly decreasing when positive. By equations (11), (12) and (13), we have that

$$\max_{q_i \geq 0} \bar{P}_t(q_i + \sum_{i \in I} Q_{i,t})q_i - \bar{C}_i(q_i) = Q_{i,t}.$$

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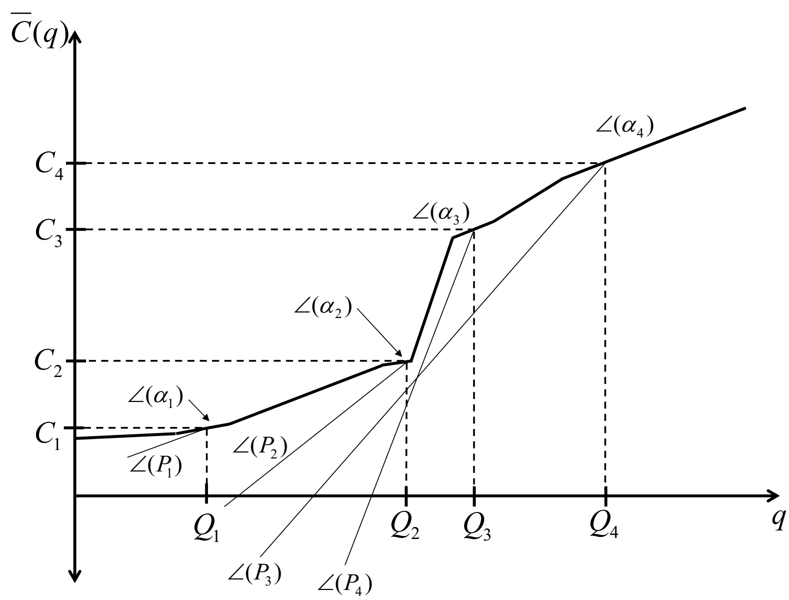

Figure 1: GRAPHICAL CONSTRUCTION OF THE COST FUNCTION. The notation $\angle(\delta)$ is used to denote the fact that the contiguous line has slope δ. The straight lines represent the functions $C_t + P_t(q - Q_t)$. Condition DMC guarantees that if $Q_{t'} < Q_t$, then the point $(Q_{t'}, C_{t'})$ lies above the line defined by observation t, which allows the construction of the thick curve.

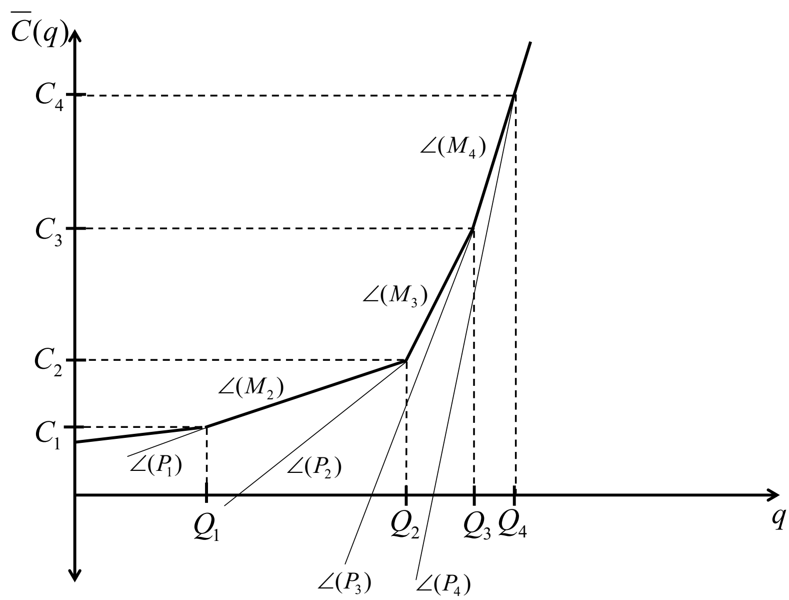

Figure 2: Graphical construction of a convex cost function. The resulting function will be convex, given that $M_2 < M_2 < M_3$, as long as $\alpha_1 < M_2$. The function is increasing, and lies above the straight lines, given that DMC guarantees that $0 < M_t < P_t$, whenever M_t is defined.