Subgame Perfect Implementation of the Nash Rationing Solution

Marco Mariotti*
Queen Mary, University of London

Quan Wen†
Vanderbilt University

February 2008

Abstract

We study a bargaining mechanism that implements the Nash rationing solution of Mariotti and Villar (International Journal of Game Theory, 2005). The bargaining mechanism is an n-person non-cooperative game of perfect information. In each period, one player makes a proposal from a set of feasible alternatives. Even with only two players, there are generally multiple subgame perfect equilibrium outcomes. We show that as the probability of exogenous breakdown does to zero, the limit of any convergent sequence of subgame perfect equilibrium outcomes is a Nash rationing solutions of the underlining rationing problem. However, not every Nash rationing solution can be a limit of subgame perfect equilibrium outcomes.

JEL Classification: C71; C72, C73; C78

Keywords: Rationing problems; bargaining, implementation

*Department of Economics, Queen Mary, University of London, Mile End Road, London E1 4NS, U.K.
†Department of Economics, Vanderbilt University, Nashville, TN 37235-1819, U.S.A.

1 Introduction

A rationing problem describes a situation in which a given amount of a divisible good must be allocated among a group of players, while there is not enough quantity to satisfy their claims. Bankruptcy problems and cost sharing problems are the best known examples of these situations. The standard formulation of a rationing problem is a triple (N, t, x), where N is a set of finite agents, t is a positive real number that represents the amount of resources to be divided among the agents, and $x = (x_i)_{i \in N}$ is an n-vector in R^n_+ that specifies the agents' claims, such $\sum_{i \in N} x_i > t$. The main advantage of this formulation is that all primitives of the problem are observed. In particular, preferences are not explicitly considered. Therefore, once a solution to the problem is accepted, there is no difficulty in implementing it. On the other hand, the lack of an explicit reference to preferences is unsatisfactory, since it stands to reason that a notion of fairness may depend on preferences. It also deviates from the bulk of social choice theory, and a rationing problem is a specific social choice problem. To overcome these difficulties, Mariotti and Villar (2005) have proposed and axiomatized a rationing solution (called the Nash Rationing Solution, or NRS) in a framework of cardinal utility information.[1] How can a planner who would like to ration players according to a Nash rationing solution, but does not know their preferences, accomplish his goal? In this paper we study this issue of *implementation*.

We propose a bargaining mechanism similar to the "alternating offer" bargaining games that requires unanimous agreements,[2] with an exogenous probability of breakdown as in Binmore, Rubinstein and Wolinsky (1986). This bargaining mechanism implements a Nash rationing solution in stationary subgame perfect equilibrium (SPE) in the limit as the probability of breakdown tends to zero. There are (potentially) infinitely many periods, and one player makes a proposal in a set of feasible alternatives. If the proposal is accepted unanimously by the other players, then the game ends. Otherwise, the game may breakdown

[1] See also Mariotti and Villar (2006)

[2] See, for example, Rubinstein (1982), Herrero (1985), Haller (1986), Jun (1987), and Osborne and Rubinstein (1990).

with certain probability. The peculiarity of our mechanism, compared to standard bilateral/multilateral bargaining models, lies in the payoffs allocated to the players in the case of exogenous breakdown after any standing offer is rejected. In our mechanism, the players (the proposing player and the responding players) do not receive their "disagreement payoff" that are resulted from perpetual disagreement. Instead, every responding player receives his *largest* possible utility, namely that associated with receiving their claims in full, while the proposing player receives the residual. In other words, the payoffs associated with the breakdown depend on the identity of the proposer. Of course, if the game does not breakdown, the bargaining will proceed to the following papers where the same process will repeated with a new proposing player.

We show that this bargaining mechanism always has at least one *stationary* subgame perfect equilibrium for all possible values of breakdown probability. Due to the nature of a rationing problem, the issues involved in rationing problems resembles non-convex bargaining problems (Herrero, 1989). Even with only two players, there could be multiple stationary subgame perfect equilibrium outcomes. With more than two players, it is well known that there are generally multiple non-stationary subgame perfect equilibrium outcomes in games that require unanimous agreements. Hence, we concentrate on stationary subgame perfect equilibrium outcomes in this game. We show that as the breakdown probability goes to zero, any convergent sequence of stationary subgame perfect equilibrium outcomes converges to a Nash rationing solution to the underlining problem. Similar to implement Nash bargaining solution in Binmore, Rubinstein and Wolinsky (1986), our result implies that at least one Nash rationing solution can be implemented by stationary subgame perfect equilibrium in the limit as the probability of breakdown goes to zero. We provide an example to demonstrate, however, that not all Nash rationing solution can be the limit of stationary subgame perfect equilibrium outcomes. Our bargaining mechanism for rationing problems provides a simple way of equilibrium selection for Nash rationing solutions.

The game studied in this paper requires unanimous agreement. Multilateral bargaining models that require unanimous agreements often have multiple equilibrium outcomes. It has been shown that when partial/conditional agreements are allowed, many variations of this kind of model will have a unique subgame perfect equilibrium outcome.[3] However, since a rationing problem resembles a non-convex bargaining problem, even if we allow for partial/conditional agreements, the game would continue to have multiple stationary subgame perfect equilibrium outcomes. Recently, Miyagawa (2002) provides a simple four-stage game that implements a large class of two-person bargaining solutions in subgame perfect equilibrium. However, the solution that can be implemented are those that optimize a monotonic and quasi-concave function of players' utilities. As we will discuss later, a Nash rationing solution may not maximize such a function. Consequently, one may not use a similar mechanism to implement Nash rationing solutions.

The rest of this paper is organized as follows. In Section 2, we present rationing problems and introduce the Nash rationing solutions. In Section 3, we provide a bargaining mechanism for a rationing problem, and establish the existence of a stationary subgame perfect equilibrium in this mechanism. The main result is presented in Section 4: as the probability of exogenous breakdown goes to zero, any limit of convergent stationary subgame perfect equilibrium outcomes is a Nash rationing solution to the underlining rationing problem. In Section 5, we provide an example where some NRS cannot be a limit of stationary subgame perfect equilibrium outcomes.

2 The Nash Rationing Solution (NRS)

A rationing problem for a set N of players, with $|N| = n$, is formulated as a pair (S, c), where $S \subset R^n$ is the set of feasible payoffs (expressed as von Neumann Morgenstern utilities) to the n players and $c \in R^n$ represents their claims. Assume that:

i) S is convex and comprehensive;

ii) $c \notin S$ (the claims are not feasible);

[3] See, for example, Chae and Yang (1988, 1994), Krishna and Serrano (1996), Huang (2002), Suh and Wen (2006).

iii) $c >> s$ for some $s \in S$.[4]

We make one further regularity assumption. For all $i \in N$:

(iv)

$$P(S) \cap \{s \in R | s_{-i} \geq c_{-i}\} \neq \emptyset, \tag{1}$$

where $P(S)$ is the Pareto boundary of S, i.e.,

$$P(S) = \{s \in S \mid s' >> s \Longrightarrow s' \notin S\}.$$

Condition (1) requires that the Pareto boundary of S cuts all axes emanating from c. Let Σ be the set of all rationing problems satisfying the above assumptions.

Let $S^c = \{s \in S| \ s \leq c\}$. We look for allocations that are both efficient and do not exceed any agent's claim. That is, we look for points in the set $P^c(S)$ defined as follows:

$$P^c(S) = P(S) \cap S^c$$

The *Nash Rationing Solution (NRS)* (Mariotti and Villar, 2005) associates with each problem $(S, c) \in \Sigma$ the set of points (called NRS points) $s^* \in P^c(S)$ which have the following properties: there exists $p^* \in$int Δ (the interior of the unit simplex) such that

$$p^*(c - s^*) \leq p^*(c - s) \text{ for all } s \in S, \tag{2}$$

$$p_i^*(c_i - s_i^*) = p_j^*(c_j - s_j^*) \text{ for all } i \neq j \in N. \tag{3}$$

Since S is convex, (2) implies that p^* must be the also the norm vector of a supporting hyperplane to S at s^*. The second property (3) imposes additional "equity" constraints, so that even if there are multiple tangent hyperplanes, p^* is uniquely determined for every s^*. Note, however, that a rationing problem may have multiple Nash rationing solutions.

In the rest of this section, we present two properties of NRS that will be useful in this paper to visualize and interpret the results. First, observe that an NRS s^* is the centre of gravity of the polyhedron determined by the portion of the supporting hyperplane $p^*(c - x) = p^*(c - s^*)$ contained in the negative orthant with origin c.

[4] The convention for vector inequalities is: $\geq, >, >>$, and for subsets we use $\subset$ to denote weak inclusion.

Lemma 1 *Let s^i be the intersection point of the supporting hyperplane of S at s^* with the i^{th} axis emanating from c. Then we have*

$$(c - s^*) = \sum_i \frac{1}{n} \left(c - s^i\right). \tag{4}$$

Proof. Note first that s^i can be defined by

$$p^* \left(c - s^i\right) = p_i^* \left(c_i - s_i^i\right) = p^* \left(c - s^*\right) \text{ for all } i \in N. \tag{5}$$

so that

$$p_i^* = \frac{p_j^*(c_j - s_j^*)}{(c_i - s_i^*)} \quad \text{for all } i \neq j$$

(by (3)) into (5), we obtain that for all $i \neq j$,

$$\begin{aligned} p^* \left(c - s^*\right) &= \frac{p_j^*(c_j - s_j^*)}{(c_i - s_i^*)} \left(c_i - s_i^i\right), \\ &\Rightarrow \left(c_i - s_i^*\right) = \frac{p_j^*(c_j - s_j^*)}{p^* \left(c - s^*\right)} \left(c_i - s_i^i\right), \\ &\Rightarrow \sum_{j \in N} (c_i - s_i^*) = \sum_{j \in N} \frac{p_j^*(c_j - s_j^*)}{p^* \left(c - s^*\right)} \left(c_i - s_i^i\right), \\ &\Rightarrow \left(c_i - s_i^*\right) = \frac{1}{n} \left(c_i - s_i^i\right), \\ &\Leftrightarrow \quad \left(c - s^*\right) = \sum_i \frac{1}{n} \left(c - s^i\right) \end{aligned}$$

as claimed. ■

Given any rationing problem (S, c), we define that for all $i \in N$,

$$f_i(s_{-i}) = \max s_i \quad \text{s.t.} \quad (s_i, s_{-i}) \in S.$$

The closeness and convexity of S implies that $f_i(\cdot)$ is weakly monotonic, concave, and continuous. With loss of generality, we normalize the problem so that $c = 0$ and $r_i = f_i(c_{-i}) = -1$. Under such normalization, the most relevant part of feasible set will be $S \cap [0, 1]^n$, as illustrated in Figure 1.

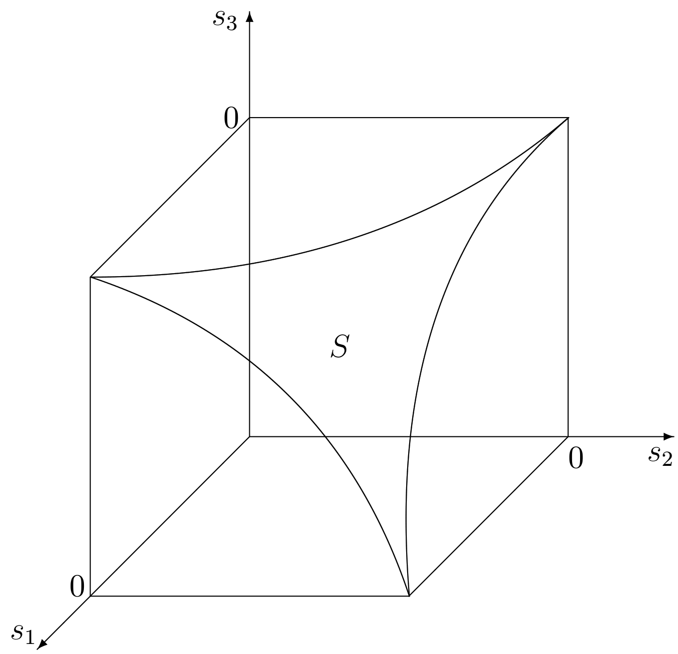

Figure 1: A normalized rationing problem.

For a normalized rationing problem (S, c), $s^* \in S$ is a Nash rationing solution (NRS) if and only if there exists $p^* \in \Delta$ such that

$$p^* s^* \geq p^* s \qquad \text{for all } s \in S, \tag{6}$$

$$p_i^* s_i^* = p_j^* s_j^* \qquad \text{for all } i \neq j \in N. \tag{7}$$

Note that all Nash rationing solutions must be in $S \cap [0,1]^n$.

3 A Bargaining Mechanism

Consider the following bargaining mechanism with infinite horizon where players take turns in making proposals. In period $t = kN + i$ for $k \in K$ (non-negative integers) and $i \in N$, player i proposes a feasible alternative $s^i \in S$. Then the other players simultaneously decide whether to accept or to reject player i's proposal $s^i \in S$. If player i's proposal is accepted unanimously, then it will be implemented immediately. Otherwise, with probability $\rho \in (0, 1)$, the mechanism proceeds to the next period $t + 1$, in which player $i + 1$ proposes (define $n+1=1$); and with probability $1-\rho\in(0,1)$, the mechanism is terminated exogenously. In the latter case, every responding player $j\neq i$ receives his full claim $c_j=0$, and the proposing player i receives the residual $r_i=f_i(c_{-i})=-1$.

The non-cooperative game described above closely resembles the multilateral bargaining game of Haller (1986), that generalizes the bilateral bargaining model of Rubinstein (1982). The key feature in this mechanism is, however, the allocation in the case of exogenous termination depends on the identity of the proposing player in the current period. From the setup of the problem, this game is also related to the bargaining game with non-convex feasible set of Herrero (1989). This bargaining mechanism is a non-cooperative game of perfect information. Histories and strategies are defined as usual. After any finite history (no agreement has reached and the game has not been terminated), a strategy profile specifies a proposal by the proposing player, and responding players' votes to every possible proposal. Players' preferences are represented by their von Neumann Morgenstern utility functions with no time discounting.

As in multilateral bargaining games with non-convex feasible sets, there are multiple subgame perfect equilibrium outcomes. In this paper, we are interested in stationary subgame perfect equilibrium (SSPE) outcomes. A stationary strategy profile can be simply described by a list of n feasible allocations $\{s^1,\ldots,s^n\}\subset S$, such that player i always proposes $s^i\in S$ whenever he proposes, and accept player j's proposal if and only if his payoff in the proposal is not less than s^j_i for all $j\neq i$. A subgame perfect equilibrium is a SSPE if its strategy profile is stationary. Our next proposition provides a set of necessary and sufficient conditions for such a stationary strategy profile to be a SSPE.

Proposition 2 *The stationary strategy profile $\langle s^1,\ldots,s^n\rangle$ constitutes a SSPE if and only if, for all $i,j\in N$,*

$$s^i_i=f_i(s^i_{-i})\quad \textit{and}\quad s^i_j=\rho^{\langle j-i\rangle}\cdot s^j_j, \tag{8}$$

where

$$\langle j-i\rangle = \begin{cases} j-i, & \textit{if } j-i \geq 0, \\ N+j-i, & \textit{otherwise.} \end{cases}$$

Proof. According to the mechanism, $\langle j-i\rangle$ is the number of periods that player j will propose after player i's proposal is rejected in the current period. By (8), s_j^i is player j's expected continuation payoff after player i's proposal is rejected. Note that after player i's proposal is rejected, player j's continuation payoff is

$$(1-\rho)0 + \rho s_j^{i+1} = \rho\left((1-\rho)0 + \rho s_j^{i+2}\right) = \cdots = \rho^{\langle j-i\rangle} s_j^j.$$

Therefore, it is sequentially rational for player j to accept player i's proposal if and only if player j's payoff is not less than s_j^i.

Now we show that whenever player i proposes, player i will propose s^i by (8), rather than demand more than s_i^i. Suppose that player i deviates from such a stationary strategy profile by demanding more than s_i^i. It is then necessary to offer less to some other players so that such a proposal will be rejected. According to the mechanism, after player i's proposal is rejected, his expected continuation payoff will be

$$(1-\rho)r_i + p s_i^{i+1} \tag{9}$$

On the other hand, if player i proposes s^i as specified by the strategy profile, he will receive

$$s_i^i = f_i(s_{-i}^i) = f_i\left((1-\rho)\mathbf{0}_{-i} + \rho s_{-i}^{i+1}\right) \geq (1-\rho) f_i\left(\mathbf{0}_{-i}\right) + \rho f_i\left(s_{-i}^{i+1}\right) \tag{10}$$

due to the concavity of $f_i(\cdot)$. The fact that $s^{i+1} = (s_i^{i+1}, s_{-i}^{i+1}) \in P^{\mathbf{0}}(S)$ implies that $s_i^{i+1} \leq f_i\left(s_{-i}^{i+1}\right)$. (9) and (10) then imply that player i will propose s^i whenever he proposes, which concludes this proof. ■

Our next proposition ensures the existence of a solution to the equation system of (8), and hence the existence of a SSPE in this mechanism. In addition, the result also shows that in any SSPE, every player receives a negative payoff. In other words, nobody receives his full claim in any SSPE.

Proposition 3 *Equation system (8) admits, at least, one solution. Moreover, for any solution* $\{s^1, \ldots, s^n\}$ *of (8), we have* $s_i^i < 0$ *for all* $i \in N$.

Proof. Since $f_i(\cdot)$, for all $i \in N$, is continuous and maps any point in $[-1,0]^{n-1}$ into $[-1,0]$,

$$F(x) \equiv \begin{pmatrix} f_1\left(\rho x_2, \rho^2 x_3, ..., \rho^{n-1} x_n\right) \\ f_2\left(\rho^{n-1} x_1, \rho x_3, ..., \rho^{n-2} x_n\right) \\ \vdots \\ f_n\left(\rho x_1, \rho^2 x_2, ..., \rho^{n-1} x_{n-1}\right) \end{pmatrix}$$

is continuous mapping that maps from $[-1,0]^n$ into itself. Brower's fixed point theorem then implies that there exists a fixed point $x^* = F(x^*) \in [-1,0]^n$. Let $s_i^i = x_i^*$ and $s_i^j = \rho^{\langle i-j \rangle} \cdot s_i^i$ for all $i \neq j \in N$. It is straightforward that $\{s^1, \ldots, s^n\}$ is a solution to equation system (8).

We now prove that $s_i^i < 0$ for all $i \in N$ by contradiction. Note that $(0, \ldots, 0) \notin S$ implies that $(0, \ldots, 0)$ cannot be a solution to (8). Without loss of generality, suppose that $x_1^* = 0$ and $x_2^* < 0$. Then $s_1^i = 0$ for all $i \in N$ by construction. Note that

$$s^1 = (0, \rho x_2^*, \ldots, \rho^{n-1} x_n^*) \geq (0, x_2^*, \ldots, \rho^{n-2} x_n^*) = s^2.$$

The last inequality implies that if player 2 proposes $s^1 \in S$ instead of $s^2 \in S$, $s^1 \in S$ will be accepted by all other players and player 2 he will receive ρx_2^*. Since $x_2^* < 0$ and $\rho \in (0,1)$, we have $\rho x_2^* > x_2^*$, which is contradictory for $\{s^1, \ldots, s^n\}$ being the equilibrium proposals. ■

For convenience, a SSPE can be represented by vector $(s_1^1, \ldots, s_n^n)$ and, by Proposition 2, the corresponding equilibrium proposals are

$$\left(s^1 \cdots s^n\right) = \begin{pmatrix} s_1^1 & \rho^{n-1} s_1^1 & \cdots & \rho s_1^1 \\ \rho s_2^2 & s_2^2 & & \rho^2 s_2^2 \\ \vdots & & \ddots & \vdots \\ \rho^{n-1} s_n^n & \rho^{n-2} s_n^n & \cdots & s_n^n \end{pmatrix}. \tag{11}$$

We write a set of equilibrium proposals as a $n \times n$ matrix for later reference.

4 Implementation

Even when $n = 2$, there could be multiple SSPE outcomes. For example, consider a normalized rationing problem $(S, 0)$, where

$$S = \left\{(s_1, s_2) \in R^2_- : s_2 \le -1 - 3s_1, s_1 \le -1 - 3s_2 \right\}.$$

For sufficiently large $\rho \in (0, 1)$, there are three stationary equilibria:

x_1	$-\frac{1}{3+3\rho}$	$-\frac{1}{1+\rho}$	$-\frac{1}{3\rho+1}$
x_2	$-\frac{1}{1+\rho}$	$-\frac{1}{3+3\rho}$	$-\frac{1}{3\rho+1}$

Note that as $\rho \to 1$, these three SSPE outcomes converges to the three Nash rationing solutions of this rationing problem, namely, $\left(-\frac{1}{6}, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{6}\right)$ and $\left(-\frac{1}{4}, -\frac{1}{4}\right)$. This example demonstrates two important facts: First, for a rationing problem, there could be multiple SSPE outcomes. In fact, there can be even continuum stationary subgame perfect equilibrium outcomes if the Pareto frontier of S coincides part of a rectangular hyperbola. Second, as ρ goes to one, the limit of any convergent sequence of stationary subgame equilibrium outcomes is a Nash rationing solution of the underlining rationing problem.

To order to present these results formally, let $\varphi(\rho) \subset S$ denote the set of SSPEs outcomes (player 1's proposals) in the bargaining mechanism with continuation probability $\rho \in (0, 1)$, and $\phi(1)$ to be the set of NRSs to the normalized rationing problem (S, c). One may consider $\varphi(\cdot) : (0, 1] \mapsto S$ as a correspondence. Our main result asserts that correspondence $\varphi(\cdot)$ is left upper hemi continuous at $\rho = 1$. In other words, any limit of SSPE outcomes is a NRS.

Proposition 4 *If $s^1(\rho) \in \varphi(\rho)$ converges as $\rho \to 1$, i.e., $\lim_{\rho \to 1} s^1(\rho) = s^*$, then $s^* \in \phi(1)$.*

Proof. For any $\rho \in (0, 1)$, let $\{s^1, \ldots, s^n\}$ be a set of equilibrium proposals (that depend on ρ). First, note that these n equilibrium proposals are linearly independent since the determinant of (11) is

$$\begin{vmatrix} s_1^1 & \rho^{n-1}s_1^1 & \cdots & \rho s_1^1 \\ \rho s_2^2 & s_2^2 & & \rho^2 s_2^2 \\ \vdots & & \ddots & \vdots \\ \rho^{n-1}s_n^n & \rho^{n-2}s_n^n & \cdots & s_n^n \end{vmatrix} = \prod_{i=1}^n s_i^i \cdot \begin{vmatrix} 1 & \rho^{n-1} & \cdots & \rho \\ \rho & 1 & & \rho^2 \\ \vdots & & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{vmatrix}$$

(multiply the last row by ρ^k and substract it from the k-th row)

$$= \prod_{i=1}^n s_i^i \cdot \begin{vmatrix} 1-\rho^n & 0 & \cdots & 0 \\ \rho & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{vmatrix}$$

$$= (1-\rho^n)^n \prod_{i=1}^n s_i^i \neq 0,$$

due to the second part of Proposition 3. Given that $\{s^1, \ldots, s^n\}$ are linearly independent in R^n, they span a unique hyperplane. Denote the normalized norm vector the such a unique hyperplane by $p \in \Delta$. Since the unit simplex Δ is compact, without loss of generality, assume that $p \to p^* \in \Delta$ for (any sequence) $\rho \to 1$. By construction, we have that for all $i \in \Delta$, $s^i \to s^*$ as $\rho \to 1$. Since both s^i and s^{i+1} are on the hyperplane with norm vector p, we have

$$\begin{aligned} p \cdot (s^i - s^{i+1}) &= (1-\rho^{n-1})p_i s_i^i + (\rho - 1)\sum_{j \neq i} p_j s_j^{i+1} = 0 \\ &\Rightarrow \frac{1-\rho^{n-1}}{1-\rho} p_i s_i^i = \sum_{j\neq i} p_j s_j^{i+1}. \end{aligned} \tag{12}$$

As $\rho \to 1$, by the L'hospital rule, (12) becomes

$$(n-1)p_i^* s_i^* = \sum_{j \neq i} p_j^* s_j^* \Rightarrow p_i s_i^i = \sum_{j=1}^n p_j^* s_j^*.$$

Therefore, $p_i^* s_i^* = p_j^* s_j^*$ for all $i \neq j$. ■

5 Concluding Remarks

To conclude, we provide an example where the SSPE correspondence $\varphi(\cdot)$ is not left lower hemi continuous at $\rho = 1$. Consider a normalized rationing problem (S, c), where

$$S = \left\{ (s_1, s_2) \in R_-^2 : s_1 \leq -1 - 2s_2, s_2 \leq -1 - \frac{3}{2}s_1 \right\}.$$

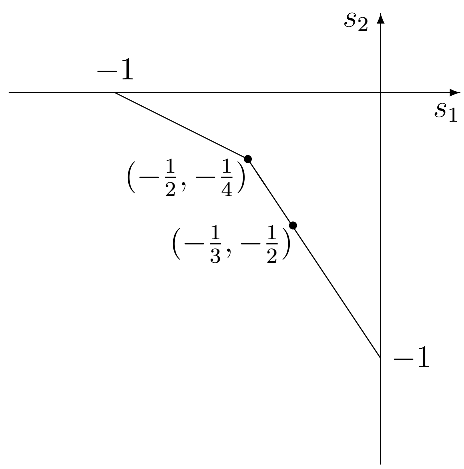

Figure 2: A rationing problem with two NRSs and one SSPE.

Note that there are two Nash rationing solutions: $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ and $\left(-\frac{1}{3}, -\frac{1}{2}\right)$. This rationing problem is illustrated in Figure 2.

For all $\rho \in (0, 1)$, the bargaining mechanism has a unique NRS. Depending on the value of ρ, equilibrium proposals may be on the two linear segments of the Pareto frontier of S, or the same linear segment. Recall that $s_1^1 < \rho s_1^1 = s_1^2$ and $s_2^2 < \rho s_2^2 = s_2^1$, there are three cases to consider.

Case 1: Suppose $\{s^1, s^2\}$ is a SSPE, where s^1 and s^2 are on the two linear segments. Then we must have

$$s_1^1 = -1 - 2\rho s_2^2 \quad \text{and} \quad s_2^2 = -1 - \frac{3}{2}\rho s_1^2,$$

which yield that $s_1^1 = \frac{1-2\rho}{3\rho^2-1}$ and $s_2^2 = \frac{2-3\rho}{6\rho^2-2}$. Note that s^1 and s^2 are on the two linear segments if and only if

$$\frac{1-2\rho}{3\rho^2-1} \leq -\frac{1}{2} \quad \text{and} \quad \frac{2-3\rho}{6\rho^2-2} \leq -\frac{1}{4},$$

which are true if and only if $\rho \in \left(0, \frac{1}{3}\right]$.

Case 2: Suppose $\{s^1, s^2\}$ is a SSPE, where s^1 and s^2 are on the first linear segment. Then we must have

$$s_1^1 = -1 - 2\rho s_2^2 \quad \text{and} \quad \rho s_1^1 = -1 - 2s_2^2,$$

which yield that $s_1^1 = -\frac{1}{\rho+1}$ and $s_2^2 = -\frac{1}{2\rho+2}$. Note that s^1 and s^2 are on the first linear segment if and only

$$-\frac{1}{\rho+1} \leq -\frac{1}{2} \quad \text{and} \quad -\frac{1}{2\rho+2} \geq -\frac{1}{4}.$$

However, the second inequality is no true for all $\rho \in (0,1)$, which means that Case 2 is not possible.

Case 3: Suppose $\{s^1, s^2\}$ is a SSPE, where s^1 and s^2 are on the second linear segment. Then we must have

$$s_2^2 = -1 - \frac{3}{2}\rho s_1^1 \quad \text{and} \quad \rho s_2^2 = -1 - \frac{3}{2}s_1^1,$$

which yield that $s_1^1 = -\frac{2}{3\rho+3}$ and $s_2^2 = -\frac{1}{\rho+1}$. Note that s^1 and s^2 are on the second linear segment if and only

$$-\frac{2}{3\rho+3} \geq -\frac{1}{2} \quad \text{and} \quad -\frac{1}{\rho+1} \leq -\frac{1}{4},$$

which are true if and only if $\rho \in \left[\frac{1}{3}, 1\right)$.

To summarize, the bargaining mechanism for the ratting problem in this example has a unique SSPE outcome, and as $\rho \to 1$, such a unique SSPE outcome converge to the NRS $\left(-\frac{1}{3}, -\frac{1}{2}\right)$. Although $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ is also a NRS of this problem, it cannot be the limit of SSPE outcomes of the bargaining mechanism. In other words, the reverse of Proposition 4 does not hold in general.

References

[1] Binmore, K.G., Rubinstein, A., Wolinsky, A. (1986): "The Nash Bargaining Solution in Economic Modelling," *Rand Journal of Economics* 17, 176-188.

[2] Chae, S., Yang, J.-A. (1994): "An N-Person Pure Bargaining Game," *Journal of Economic Theory* 62, 86-102.

[3] Chae, S., Yang, J.-A. (1990): "An N-Person Bargaining Game with Alternating Demands," *Seoul Journal of Economics* 3, 255-261.

[4] Chae, S., Yang, J.-A. (1988): "The Unique Perfect Equilibrium of an N-Person Bargaining Game," *Economics Letters* 28, 221-223.

[5] Haller, H. (1986): "Non-cooperative Bargaining Games of $N \geq 3$ Players," *Economics Letters* 22, 11-13.

[6] Herrero, M. (1985): "N-Player Bargaining and Involuntary Unemployment," Ph.D. dissertation, London School of Economics.

[7] Herrero, M. (1989): "The Nash Program: Non-convex Bargaining Problems," *Journal of Economic Theory* 49, 266-277.

[8] Huang, C.-Y. (2002): "Multilateral Bargaining: Conditional and Unconditional Offers," *Economic Theory* 20, 401-412.

[9] Jun, B.H. (1985): "A Structure Consideration on 3-Player Bargaining," Ph.D. dissertation, University of Pennsylvania.

[10] Krishna, V., Serrano, R. (1996): "Multilateral Bargaining," *Reviews of Economic Studies* 63, 61-80.

[11] Miyagawa, E. (2002): "Subgame-perfect Implementation of Bargaining Solutions," *Games and Economic Behavior* 41, 292-308.

[12] Mariotti, M., Villar, A. (2005): "The Nash Rationing Problem," *International Journal of Game Theory* 33, 367-377.

[13] Mariotti, M., Villar, A. (2006): "The Nash Rationing Problem Revisited: Individual Rationality Constraints and Duality," *Working Paper*, Queen Mary, University of London.

[14] Osborne, M., Rubinstein, A. (1990): *Bargaining and Markets*, Academic Press, San Diego, U.S.A.

[15] Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica* 50, 97-109.

[16] Suh, S.-C., Wen, Q. (2006): "Multi-Agent Bilateral Bargaining and Nash Bargaining Solution," *Journal of Mathematical Economics* 42, 61-73.