CONSTRAINED EFFICIENT TAXATION OF CAPITAL

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Abstract. In this paper I show that, in the overlapping generations economy with production of Diamond (1965), the laissez-faire competitive equilibrium steady state is constrained inefficient and can be Pareto-improved upon by a government following an active fiscal policy that taxes or subsidizes linearly the returns to savings while balancing the budget period by period through a lump-sum transfer or tax, respectively, on second period income. Each period tax/subsidy rate and lump-sum transfer/tax is determined as a function of past saving decisions. The constrained inefficiency is not due to any precautionary over-saving since there is no uncertainty in the model. Nor does this policy intend to finance any public spending, since there is none in the model either. The only purpose of the intervention is to make possible to implement in a decentralized way as a competitive equilibrium the constrained efficient steady state the maximizes the utility of the representative agent among all the steady state allocation attainable through the market.

1. Introduction

Whether the taxation of capital returns is a good or bad idea is a recurrent issue in the economic literature.[1] Arguments for and against it are put forward by, respectively, those who would like to see taxes on capital income eliminated (or at least reduced) because of the inefficiencies they may introduce in the allocation of resources (as in Chamley (1986), Judd (1987)), and those who think that taxes on capital income serve a purpose if only because they may help undo some inefficiencies due to the incompleteness of markets (e.g. oversaving as a self-insurance against uninsurable risks, as in Aiyagari (1995), Chamley (2001)). Actually, the conclusions depend crucially on the framework in which the question is addressed, namely the neoclassical growth model or the overlapping generations model. In effect, in the ideal case of absence of uncertainty, while in the neoclassical growth model the taxation of capital returns induces a distortion of the competitive equilibrium factor prices that only can create inefficiencies (not surprisingly, given the efficiency of the laissez-faire competitive equilibrium allocations of this model), it turns out that the breakdown of the First Welfare Theorem in the overlapping generations model prevents to replicate this kind of argument in that setup.

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[1]See, for instance, Atkeson, Chari, and Kehoe (1999) and Conesa, Kitao, and Krueger (2007)

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It was nonetheless argued in Diamond (1970) that, in the overlapping generations economy with production of Diamond (1965), an increase in the tax rate of capital returns decreases the steady state utility of the representative agent, so that a reduction or outright elimination of taxes on capital returns would be Pareto-improving. This idea has made its way in the literature, as surveyed in Furstenberg and Malkiel (1977). It should be noted, nonetheless, that the negative impact of the taxation of capital returns on the steady state representative agent's utility found in Diamond (1970) is guaranteed, more specifically, only in the case in which the after-tax returns to savings exceed the rate of growth of the population.

As a matter of fact, Diamond (1970) is just a small part of the answer to a more general question that can be addressed in the overlapping generations model with production in Diamond (1965). This question is whether the laissez-faire steady state competitive equilibrium is constrained inefficient in this model or not (on top of being inefficient as a result of the breakdown of the First Welfare Theorem in this setup), that is to say, whether among the steady state allocations attainable through the market it is suboptimal or not.[2] The answer to this question is yes: even among just the feasible steady states in which factors are priced by their marginal productivities and without transfers across agents, the laissez-faire competitive equilibrium steady state is inefficient.[3] Indeed, all the agents could be better-off if they just decided to save differently from what they choose to save at the laissez-faire competitive equilibrium steady state, even if their labor and savings are priced by their marginal productivities.

But then there is also the question of whether the constrained efficient level of savings can be attained in a decentralized way as a competitive equilibrium after some well-chosen government intervention. The answer to this second question is again yes, and the right intervention just requires (in case the second-best constrained efficient steady state requires overaccumulating capital with respect to the first-best efficient steady state) to tax linearly each generation's capital returns and, simultaneously, to give back to the same generation the amount raised in this way as a second period lump-sum transfer.[4] In doing so, no resources are redistributed across agents or generations and the government does never incur in any deficit or superavit. Actually this is the kind of policy considered in Diamond (1970) in order to asses the impact of taxes on capital returns in this setup, but it turns out that the tax rate that implements the constrained efficient level of per capita savings as a competitive equilibrium is such that the after-tax return to savings *does not* exceed the rate of growth of the population, so that the result in Diamond (1970)

[2] The notion of constrained efficiency has its origins in the literature on general equilibrium with incomplete markets, where the issue was addressed of whether a general equilibrium allocation that is inefficient because some markets are missing could still be efficient *among those attainable through the existing markets*. Geanakoplos and Polemarchakis (1986) established the generic constrained inefficiency of the general equilibria of finite horizon exchange economies with uncertainty whenever the structure of financial assets is incomplete. Geanakoplos, Magill, Quinzii and Drèze (1990) established this result when there is production.

[3] Except for the special case in which the competitive equilibrium steady state is already optimal among *all* steady states, i.e. only among those remunerating factors by their marginal productivities. In other words, the laissez-faire competitive equilibrium steady state is inefficient except when it happens to coincide with the golden rule allocation.

[4] Symmetrically, in case of second-best underaccumulation of capital the returns to savings need rather to be subsidized and a second period lump-sum tax needs to be raised.

does not apply.

More specifically, I consider the setup in Diamond (1965) in which generations of identical agents living for two periods consume and receive a labor income when young, part of which they carry over to the next period as savings that they spend —along with their returns— on consumption when old. The consumption good is produced every period out of the aggregate savings and the labor supplied by means of a constant returns to scale Cobb-Douglas production function. Note that with this technology a higher steady state level of per capita savings remunerated by the marginal productivity of aggregate capital implies always a higher level of consumption when old, while the consumption when young first increases (as the increase of marginal productivity of labor dominates over the impact of a bigger share of savings) and then decreases (as the additional savings more than eat up the gains in labor productivity). Thus the constrained efficient steady state level of per capita savings is reached when the discounted marginal utility from an increase in consumption when old due to additional savings is exactly offset by the corresponding marginal disutility from the fall in consumption when young induced by the increase in savings. Since when choosing their savings, price-taking agents do not take into account the impact of their decisions on the factors' marginal productivities, it turns out that at the laissez-faire competitive equilibrium steady state there is still room for improving the representative agent's utility.

It is well worth to remind that the agents face no uncertainty about their labor income, so that any excess aggregate savings in this setup from the efficient level is not due to any precautionary motive in the face of uninsurable risks.

It should be reminded also, however, that the constrained efficient steady state is not a laissez-faire outcome. Under perfect competition no agent can exploit on his own the impact that the saving decisions have on the rate of return on capital and the productivity of labor, since his only deviation would have a negligible impact on the aggregate. The agents lack a coordinating device that would allow them to attain a steady state allocation that Pareto-improves over the laissez-faire competitive equilibrium steady state allocation. I show in what follows that the active fiscal policy described above supply the agents with such a coordinating device. This policy plays no other role than allowing to Pareto-improve upon the laissez-faire steady state, in particular it does no finance any public spending since there is none. More specifically, this policy consists of, firstly, taxing or subsidizing linearly the return to savings depending, respectively, on whether the constrained efficient steady state over- or under-accumulates capital compared to the first-best efficient level of per capita savings; and secondly, making a lump-sum transfer or imposing a lump-sum tax, respectively, on second period income. The tax/subsidy rate and transfer/lump-sum tax at each period are determined as a function (through the marginal productivities of capital and labor) of the level of savings chosen in the previous period. At the steady state the lump-sum transfer/tax equals the amount raised/payed by the linear tax/subsidy period by period, so that the government incurs in no deficit or superavit, and the allocation is feasible. Interestingly enough, labor income needs neither to be taxed nor to be subsidized to attain the constrained efficient steady state.

The rest of the paper proceeds as follows. Section 2 Characterizes the competitive equilibria of the economy. Section 3 establishes the constrained inefficiency of any inefficient competitive equilibrium steady state (Proposition 1). Section 4 characterizes the constrained efficient steady state (Proposition 2) and establishes that it is a competitive equilibrium outcome only if it coincides with the efficient steady state (Proposition 3). As a matter of fact, the efficient, constrained efficient, and competitive equilibrium steady states are either all identical or all distinct (Proposition 4). Section 5 characterizes the fiscal policy that allows to decentralize the constrained efficient steady state as a competitive equilibrium (Proposition 5). The policy requires taxing or subsidizing the returns on savings depending on whether the second best steady state overaccumulates capital with respect to the first best (Proposition 6). Section 6 uses the previous results to establish a relation between the agents' impatience, the population growth rate, and the capital share of income implying that savings returns should be taxed if agents are patient enough or the population grows fast enough.

2. Competitive equilibria of the OG economy

Consider an agent living for two periods, t and $t+1$, in which he is, say, young and old respectively. When young he can work l hours four a real hourly wage of w_t. His real income when young $w_t l$ can then either be consumed immediately or saved for consumption when old. Let c_t^t denote the share of his income $w_t l$ that he consumes when young, and let a^t be his savings for consumption when old. He can lend his savings for a rate of return of R_{t+1}, and then consume the returns $R_{t+1}a^t$ when old. Let c_{t+1}^t denote his consumption when old. The agent evaluates consumption by a utility function u and discounts future utilities by a discount factor $0 < \beta \leq 1$. This agent faces then the problem of deciding how much to save of his income when young. Formally, the agent's problem is

$$\begin{aligned} \max_{0 \leq c_t^t, c_{t+1}^t, a^t} \quad & u(c_t^t) + \beta u(c_{t+1}^t) \\ c_t^t + a^t &= w_t l \\ c_{t+1}^t &= R_{t+1} a^t \end{aligned} \tag{1}$$

for given values of w_t and R_{t+1} (without loss of generality, capital is supposed to depreciate entirely in one period). Under standard assumptions the agent's optimal saving a^t is then completely characterized to be a function of w_t and R_{t+1} through the condition

$$\frac{1}{\beta} \frac{u'(w_t l - a^t)}{u'(R_{t+1} a^t)} = R_{t+1} \tag{2}$$

while his optimal consumption when young and old are determined by a^t through his budget constraints above.

Suppose now this agent is one of the many members of the generation born in period t of an economy whose population grows at a rate $1+n > 0$, and whose many firms produce consumption good out of capital and labor through a constant returns to scale Cobb-Douglas production function net of capital depreciation $F(K, L) = K^\alpha L^{1-\alpha}$, with $0 < \alpha < 1$. Under perfect competition the wage and the rental rate of capital are determined, at each period t, by the marginal productivities of labor and capital respectively. Since the capital K_t available at any given period t consists of the previous period aggregate savings $(1+n)^{t-1}a^{t-1}$, and aggregate labor L_t is $(1+n)^t l$, then the wage and rental rate that the agents living in periods t and $t+1$ face are actually

$$\begin{aligned} R_{t+1} =& F_K(\frac{a^t}{1+n}, l) \\ w_t =& F_L(\frac{a^{t-1}}{1+n}, l) \end{aligned} \tag{3}$$

(given the homogeneity of degree 1 of the production function) where and a^{t-1} is the amount that each of the agents born in the previous period $t-1$ decided to save. Note that the satisfaction of the agents' budget constraints guarantees the feasibility of the allocation of resources, since adding up, at any date t, the budget constraints of the $(1+n)^{t-1}$ old agents

$$c_t^{t-1} = R_t a^{t-1} \tag{4}$$

and the $(1+n)^t$ contemporaneous young agents

$$c_t^t + a^t = w_t l \tag{5}$$

it follows that

$$\begin{aligned} c_t^t + \frac{1}{1+n} c_t^{t-1} + a^t &= \\ R_t \frac{a^{t-1}}{1+n} + w_t l &= F_K(\frac{a^{t-1}}{1+n}, l)\frac{a^{t-1}}{1+n} + F_L(\frac{a^{t-1}}{1+n}, l)l \\ &= F(\frac{a^{t-1}}{1+n}, l). \end{aligned} \tag{6}$$

(from the homogeneity of degree 1 of the production function). Thus, the competitive equilibrium allocations are completely characterized by the following per capita savings dynamics that results from each agent's utility maximization and the determination of the factors prices by their marginal productivities

$$\frac{1}{\beta} \frac{u'(F_L(\frac{a^{t-1}}{1+n}, l)l - a^t)}{u'(F_K(\frac{a^t}{1+n}, l)a^t)} = F_K(\frac{a^t}{1+n}, l). \tag{7}$$

3. Constrained inefficiency of the competitive equilibrium steady state

A competitive equilibrium steady state of this overlapping generations economy is characterized by a constant sequence of per capita savings a^c satisfying the dynamics (7) above, i.e. solving the equation

$$\frac{1}{\beta} \frac{u'(F_L(\frac{a^c}{1+n}, l)l - a^c)}{u'(F_K(\frac{a^c}{1+n}, l)a^c)} = F_K(\frac{a^c}{1+n}, l). \tag{8}$$

It is well known that the steady state competitive equilibrium level of per capita savings a^c needs not be the one that maximizes the utility of the representative agents among all feasible steady states, that is to say, a^c is typically distinct from the efficient per capita savings a^g that solves

$$\begin{aligned} &\max\ u(c_1) + \beta u(c_2) \\ c_1 + &\frac{c_2}{1+n} + a = F(\frac{a}{1+n}, l) \end{aligned} \tag{9}$$

i.e. characterized by satisfying the condition

$$F_K(\frac{a}{1+n}, l) = 1+n. \tag{10}$$

Whenever $a^g < a^c$ the laissez-faire market allocation overaccumulates capital with respect to the efficient steady state. On the contrary, if $a^c < a^g$ holds, then the free market inefficiently under accumulates capital.

The next proposition establishes that whenever the competitive equilibrium steady state a^c of this economy is not efficient among all feasible steady states (i.e. whenever it is distinct from the efficient steady state a^g), then it is not even constrained efficient, i.e. it is not optimal among all feasible steady states remunerating factors by their marginal productivities. This is interesting because it indicates that there is room for Pareto-improving upon the laissez-faire allocation even without interfering with the working of markets or without having to resort to redistributing income across agents, i.e. across generations here, as attaining the efficient steady state would require.

Proposition 1. *The competitive equilibrium steady state allocation of resources is constrained inefficient whenever it is not efficient.*

Proof. The utility u^c that every agent obtains at the competitive equilibrium steady state with per capita savings a^c is $u(F_L(\frac{a^c}{1+n}, l)l - a^c) + \beta u(F_K(\frac{a^c}{1+n}, l)a^c)$, so that (a^c, u^c) is in the graph of the function ϕ defined by

$$\phi(a) = u(F_L(\frac{a}{1+n}, l)l - a) + \beta u(F_K(\frac{a}{1+n}, l)a). \tag{11}$$

Note however that, for a constant returns to scale neoclassical production function the derivative at the competitive equilibrium steady state level of per capita savings a^c of ϕ is strictly negative (respectively positive) whenever at this steady state there is overaccumulation (resp. underaccumulation) of capital with respect to the representative agent utility maximizing steady state level of per capita savings a^g characterized by

$$F_K(\frac{a^g}{1+n}, l) = 1+n. \tag{12}$$

In effect, note that

$$\begin{aligned} \phi'(a^c) = u'(c_0^c)[F_{LK}(\frac{a^c}{1+n}, l)\frac{l}{1+n} - 1] \\ + \beta u'(c_1^c) \cdot [F_K(\frac{a^c}{1+n}, l) + F_{KK}(\frac{a^c}{1+n}, l)\frac{a^c}{1+n}] \end{aligned} \tag{13}$$

where $c_0^c = F_L(\frac{a^c}{1+n}, l)l - a^c$ and $c_1^c = F_K(\frac{a^c}{1+n}, l)a^c$ are the competitive equilibrium steady state consumptions when young and old respectively. But at the competitive equilibrium steady state it holds that

$$-u'(c_0^c) + \beta u'(c_1^c) F_K(\frac{a^c}{1+n}, l) = 0 \tag{14}$$

so that the derivative $\phi'(a^c)$ simplifies to

$$\phi'(a^c) = u'(c_0^c) F_{LK}(\frac{a^c}{1+n}, l)\frac{l}{1+n} + \beta u'(c_1^c) F_{KK}(\frac{a^c}{1+n}, l)\frac{a^c}{1+n}. \tag{15}$$

Therefore, $\phi'(a^c) < (>) 0$ holds if, and only if,

$$\frac{1}{\beta}\frac{u'(c_0^c)}{u'(c_1^c)}\frac{1}{1+n} < (>) - \frac{F_{KK}(\frac{a^c}{1+n}, l)\frac{a^c}{1+n}}{F_{LK}(\frac{a^c}{1+n}, l)l} \tag{16}$$

or, equivalently —since the right-hand side is 1 because of the homogeneity of degree 1 of the neoclassical production function F, and the marginal rate of substitution $\frac{1}{\beta}\frac{u'(c_0^c)}{u'(c_1^c)}$ in the left-hand side is $F_K(\frac{a^c}{1+n}, l)$ at the competitive steady state levels of consumption— if, and only if,

$$F_K(\frac{a^c}{1+n}, l) < (>)\, 1 + n = F_K(\frac{a^g}{1+n}, l). \tag{17}$$

That is to say, $\phi'(a^c) < (>) 0$ holds if, and only if,

$$a^c > (<)\, a^g \tag{18}$$

because of the decreasing marginal productivity of capital. Q.E.D.

4. Constrained efficient steady state

If the laissez-faire steady state market allocation can be Pareto improved without redistributing income or interfering with the functioning of markets, but rather (as established in Proposition 1 above) saving less in case the competitive equilibrium steady state overaccumulates or saving more in case it underaccumulates, what is then the steady state level of per capita savings that, if chosen by the agents, would make everyone strictly better off? There is indeed a feasible steady state level of per capita savings a^* that provides the highest level of utility to all agents among those steady states compatible with the factors being remunerated by their marginal productivities. The next proposition gives a characterization of this constrained efficient steady state.

Proposition 2. *The constrained efficient steady state level of per capita savings a^* is characterized by the condition*

$$\frac{1}{\beta}\frac{u'(F_L(\frac{a^*}{1+n}, l)l - a^*)}{u'(F_K(\frac{a^*}{1+n}, l)a^*)} = (1+n)\frac{F_K(\frac{a^*}{1+n}, l) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}} \tag{19}$$

as long as $0 < a^* < F_L(\frac{a^*}{1+n}, l)l$.[5]

Proof. In effect, firstly ϕ is everywhere strictly concave because

$$\begin{aligned}\phi''(a) =& u''(c_0)[F_{LK}(\frac{a}{1+n}, l)l - 1]^2 + u'(c_0)F_{LKK}(\frac{a}{1+n}, l)\frac{l}{(1+n)^2} \\ &+ \beta u''(c_1)[F_K(\frac{a}{1+n}, l) + F_{KK}(\frac{a}{1+n}, l)\frac{a}{1+n}]^2 \\ &+ \beta u'(c_1)[\frac{2}{1+n}F_{KK}(\frac{a}{1+n}, l) + F_{KKK}(\frac{a}{1+n}, l)\frac{a}{(1+n)^2}] < 0\end{aligned} \tag{20}$$

—where $c_0 = F_L(\frac{a}{1+n}, l)l - a$ and $c_1 = F_K(\frac{a}{1+n}, l)a$— given that all the terms are negative since

$$F_{LKK}(K, L) = (1-\alpha)\alpha(\alpha-1)K^{\alpha-2}L^{-\alpha} < 0 \tag{21}$$

and

$$2F_{KK}(K, L) + F_{KKK}(K, L)K = \alpha^2(\alpha-1)K^{\alpha-2}L^{1-\alpha} < 0. \tag{22}$$

So that a^* such that $\phi'(a^*) = 0$ maximizes the representative agent market-constrained steady state utility $\phi(a)$. Then the constrained optimal steady state level of capital a^* is characterized by the condition $\phi'(a^*) = 0$, or equivalently by (19) above as long as $0 < a^* < F_L(\frac{a^*}{1+n}, l)l$, given that

$$\begin{aligned}(1+n) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n} = \\ (1+n) + \alpha(\alpha-1)\left(\frac{a^*}{(1+n)l}\right)^{\alpha-1} > 0\end{aligned} \tag{23}$$

In effect, since a^* maximizes

$$\phi(a) = u(F_L(\frac{a}{1+n}, l)l - a) + \beta u(F_K(\frac{a}{1+n}, l)a) \tag{24}$$

and

$$\frac{d}{da}[F_L(\frac{a}{1+n}, l)l - a] = 0 \tag{25}$$

for $\frac{a}{(1+n)l} = \left(\frac{\alpha(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}}$ while

$$\frac{d}{da}[F_K(\frac{a}{1+n}, l)a] = \alpha^2\left(\frac{a}{(1+n)l}\right)^{\alpha-1} > 0, \tag{26}$$

then $\phi'(a) > 0$ for $\frac{a}{(1+n)l} = \left(\frac{\alpha(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}}$, that is to say, necessarily,

$$\frac{a^*}{(1+n)l} > \left(\frac{\alpha(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}} \tag{27}$$

[5]This is guaranteed if the utility function is not additively separable and has the boundary behavior implied by the assumption that they need to consume a positive amount in each period in order to stay alive. Additive separability has been assumed only for notational convenience.

as stated above.

Finally, the steady state allocation giving a consumption $F_L(\frac{a^*}{1+n}, l)l - a^*$ when young and a consumption $F_K(\frac{a^*}{1+n}, l))a^*$ when old is feasible since

$$F_L(\frac{a^*}{1+n}, l)l - a^* + \frac{1}{1+n} F_K(\frac{a^*}{1+n}, l)a^* + a^* = F(\frac{a^*}{1+n}, l). \tag{28}$$

Q.E.D.

Note however that the constrained efficient steady state level of per capita savings a^* is not a laissez-faire competitive equilibrium outcome, unless it is actually efficient among all feasible steady states, as the next proposition establishes.

Proposition 3. *The constrained efficient steady state level of per capita savings a^* is a laissez-faire competitive equilibrium outcome if, and only if, it coincides with the efficient level of per capita savings a^g.*

Proof. In effect, a^* does not satisfy

$$\frac{1}{\beta} \frac{u'(F_L(\frac{a}{1+n}, l)l - a)}{u'(F_K(\frac{a}{1+n}, l)a)} = F_K(\frac{a}{1+n}, l) \tag{29}$$

unless

$$F_K(\frac{a^*}{1+n}, l) = (1+n) \frac{F_K(\frac{a^*}{1+n}, l) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}} \tag{30}$$

i.e. unless

$$F_K(\frac{a^*}{1+n}, l) = 1+n \tag{31}$$

that is to say, unless a^* equals the efficient level of per capita savings a^g. Q.E.D.

But whenever the constrained efficient level of per capita savings a^* is distinct from the first best level a^g, then both are distinct from the competitive equilibrium level a^c as well, as established in Proposition 4 below. Thus, in this case the laissez-faire steady state market allocation is both inefficient and constrained inefficient.

Proposition 4. *Either all the efficient, constrained efficient, and competitive equilibrium steady states a^g, a^*, and a^c coincide, or they all are distinct, i.e.*

$$a^c = a^g \Leftrightarrow a^g = a^* \Leftrightarrow a^* = a^c. \tag{32}$$

Proof. Assume $a^c = a^g$. Then

$$\begin{aligned} \frac{1}{\beta} \frac{u'(F_L(\frac{a^g}{1+n}, l)l - a^g)}{u'(F_K(\frac{a^g}{1+n}, l)a^g)} &= \\ F_K(\frac{a^g}{1+n}, l) &= 1+n \\ &= (1+n) \frac{F_K(\frac{a^g}{1+n}, l) + F_{KK}(\frac{a^g}{1+n}, l)\frac{a^g}{1+n}}{(1+n) + F_{KK}(\frac{a^g}{1+n}, l)\frac{a^g}{1+n}} \end{aligned} \tag{33}$$

so that $a^g = a^*$.

Assume $a^g = a^*$. Then

$$\frac{1}{\beta}\frac{u'(F_L(\frac{a^*}{1+n},l)l - a^*)}{u'(F_K(\frac{a^*}{1+n},l)a^*)} =$$

$$\begin{aligned} 1+n &= F_K(\frac{a^g}{1+n},l) \\ &= F_K(\frac{a^*}{1+n},l) \end{aligned} \tag{34}$$

so that $a^* = a^c$ and $a^c = a^g$, since $a^g = a^*$.

Assume $a^* = a^c$. Then

$$\begin{aligned} F_K(\frac{a^*}{1+n},l) &= \\ F_K(\frac{a^c}{1+n},l) &= \frac{1}{\beta}\frac{u'(F_L(\frac{a^c}{1+n},l)l - a^c)}{u'(F_K(\frac{a^c}{1+n},l)a^c)} \\ &= (1+n)\frac{F_K(\frac{a^c}{1+n},l) + F_{KK}(\frac{a^c}{1+n},l)\frac{a^c}{1+n}}{(1+n) + F_{KK}(\frac{a^c}{1+n},l)\frac{a^c}{1+n}} \\ &= (1+n)\frac{F_K(\frac{a^*}{1+n},l) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n}} \end{aligned} \tag{35}$$

from which

$$F_K(\frac{a^*}{1+n},l) = 1+n \tag{36}$$

so that $a^g = a^*$. Q.E.D.

The question now is therefore whether some government intervention can make of the constrained efficient steady state level of per capita savings a^* a competitive equilibrium outcome whenever it is distinct from the efficient one. This question is addressed in the next section.

5. Decentralization of the constrained efficient steady state

Consider now a government with the ability to tax or subsidize linearly the agents capital income as well as to raise a lump-sum tax or to distribute them a lump-sum transfer when old. In particular, suppose the government taxes the returns from savings that the agent living at t and $t+1$ gets when old at a rate $\tau_{t+1} < 1$ (it is a subsidy if negative), while distributing to him at the same time a lump-sum transfer T_{t+1} (a lump-sum tax if negative). Then that agent's problem is

$$\begin{aligned} &\max_{0 \le c_t^t, c_{t+1}^t, a^t} u(c_t^t) + \beta u(c_{t+1}^t) \\ &c_t^t + a^t = w_t l \\ &\quad c_{t+1}^t = (1-\tau_{t+1})R_{t+1}a^t + T_{t+1} \end{aligned} \tag{37}$$

and his optimal saving a^t is characterized by the condition

$$\frac{1}{\beta}\frac{u'(w_t l - a^t)}{u'((1-\tau_{t+1})R_{t+1}a^t + T_{t+1})} = (1-\tau_{t+1})R_{t+1} \tag{38}$$

for a given real wage w_t, a return to savings R_{t+1}, a capital income tax (or subsidy) rate τ_{t+1}, and a lump-sum transfer (or tax) T_{t+1}.

Proposition 5. *If the government taxes linearly the return to savings of each generation t at a rate $\tau_{t+1} < 1$ (a subsidy if $\tau_{t+1} < 0$) and subsidizes the second period income by a lump-sum transfer T_{t+1} (a lump-sum tax if negative), determined as functions of the previous generation per capita savings a^{t-1} according to*

$$\begin{aligned}\tau_{t+1} &= 1 - \frac{1+n}{F_K(\frac{a^{t-1}}{1+n}, l)} \cdot \frac{F_K(\frac{a^{t-1}}{1+n}, l) + F_{KK}(\frac{a^{t-1}}{1+n}, l)\frac{a^{t-1}}{1+n}}{(1+n) + F_{KK}(\frac{a^{t-1}}{1+n}, l)\frac{a^{t-1}}{1+n}} \\ T_{t+1} &= \tau_{t+1} F_K(\frac{a^{t-1}}{1+n}, l)a^{t-1}\end{aligned} \tag{39}$$

then the steady state competitive equilibrium becomes constrained efficient, while the government keeps a period by period balanced budget.

Proof. If the government determines in each period the tax rate and the lump-sum transfer as a function of the current level of capital (saved by the previous generation) according to (39) above, then the competitive equilibrium dynamics for the level of capital is given by the equation

$$\begin{aligned}&\frac{1}{\beta}\frac{u'(F_L(\frac{a^{t-1}}{1+n}, l)l - a^t)}{u'(F_K(\frac{a^{t-1}}{1+n}, l)a^{t-1} + (1-\tau_{t+1})[F_K(\frac{a^t}{1+n}, l)a^t - F_K(\frac{a^{t-1}}{1+n}, l)a^{t-1}])} \\ &\qquad = \frac{1+n}{F_K(\frac{a^{t-1}}{1+n}, l)} \cdot \frac{F_K(\frac{a^{t-1}}{1+n}, l) + F_{KK}(\frac{a^{t-1}}{1+n}, l)\frac{a^{t-1}}{1+n}}{(1+n) + F_{KK}(\frac{a^{t-1}}{1+n}, l)\frac{a^{t-1}}{1+n}} F_K(\frac{a^t}{1+n}, l)\end{aligned} \tag{40}$$

whose steady state is characterized precisely by the steady state constrained efficiency condition

$$\frac{1}{\beta}\frac{u'(F_L(\frac{a^*}{1+n}, l)l - a^*)}{u'(F_K(\frac{a^*}{1+n}, l)a^*)} = (1+n) \cdot \frac{F_K(\frac{a^*}{1+n}, l) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}} \tag{41}$$

Note also that the steady state tax/subsidy rate satisfies $\tau^* < 1$ indeed, which guarantees that the agents' problem is well defined (in particular that their budget set is compact). In effect, the solution to the steady state constrained efficiency condition above satisfies

$$\begin{aligned}1 - \tau^* &= \frac{1+n}{F_K(\frac{a^*}{1+n}, l)} \cdot \frac{F_K(\frac{a^*}{1+n}, l) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}} \\ &= \frac{1+n}{F_K(\frac{a^*}{1+n}, l)} \cdot \frac{1}{\beta}\frac{u'(F_L(\frac{a^*}{1+n}, l)l - a^*)}{u'(F_K(\frac{a^*}{1+n}, l)a^*)} \\ &> 0\end{aligned} \tag{42}$$

as requested.

From the definitions of the second period lump sum transfer T_{t+1} follows trivially that, at the steady state, what the government withdraws from each generation in a distortionary way is exactly offset by the resources it injects to that same generation in a non-distortionary way, so that the government's budget is balanced period by period. Q.E.D.

Note, incidentally, that for the computation at any given period t of the distortionary and lump-sum taxes and subsidies, the fiscal authority uses only information already known at the time t of choosing τ_{t+1} and T_{t+1}.

Note also that at the constrained efficient steady state level of per capita savings a^*, the net of tax interest rate $(1-\tau^*)F_K(\frac{a^*}{1+n},l)$ *does not* exceed the population growth rate $1+n$ when $a^g < a^*$. In effect, since

$$\tau^* = 1 - \frac{1+n}{F_K(\frac{a^*}{1+n},l)} \cdot \frac{F_K(\frac{a^*}{1+n},l) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n}} \tag{43}$$

then

$$\begin{aligned}(1-\tau^*)F_K(\frac{a^*}{1+n},l) &= (1+n) \cdot \frac{F_K(\frac{a^*}{1+n},l) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n}} \\ &\leq 1+n\end{aligned} \tag{44}$$

because $F_K(\frac{a^*}{1+n},l) \leq 1+n$ when $a^* \geq a^g$ and $(1+n) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n} > 0$ as established in Proposition 2. As a consequence, the result in Diamond (1970) does not apply to the constrained efficient steady state level of per capita savings a^* whenever $a^* < a^g$.

6. Tax or subsidize savings?

Proposition 5 above established that the constrained efficient steady state can be attained in a decentralized way with the appropriate tax rate (positive or negative) on savings returns coupled with some lump-sum transfer or tax. But when exactly attaining constrained efficiency requires taxing and when subsidizing savings?

As the next proposition establishes, if the constrained efficient steady state level of per capita savings a^* requires overaccumulating capital with respect to the efficient steady state level of per capita savings a^g, then a^* can only be attained taxing savings returns and distributing second period lump-sum transfers. Conversely, if the constrained efficient steady state requires underaccumulating capital with respect to the efficient steady state, it can only be attained subsidizing savings returns and raising a second period lump-sum tax.

Proposition 6. *The decentralization of the constrained efficient steady state level of per capita savings a^* as a competitive equilibrium requires taxing (resp. subsidizing) linearly the capital revenue, coupled with a second period lump-sum transfer (resp. tax) if, and only if the constrained efficient steady state level of per capita savings a^* is bigger (resp. smaller) than the efficient steady state level of per capita savings a^g, i.e.*

$$0 < (>)\tau^* \Leftrightarrow a^g < (>)a^*. \tag{45}$$

Proof. Note that, at the constrained efficient steady state level of per capita savings a^*, capital revenue is taxed (resp. subsidized) if the constant rate

$$\tau^* = 1 - \frac{1+n}{F_K(\frac{a^*}{1+n},l)} \frac{F_K(\frac{a^*}{1+n},l) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n},l)\frac{a^*}{1+n}} \tag{46}$$

is positive (resp. negative), i.e. if, and only if,

$$1 > (<) \frac{1+n}{F_K(\frac{a^*}{1+n}, l)} \cdot \frac{F_K(\frac{a^*}{1+n}, l) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}} \tag{47}$$

that is to say, if and only if,

$$\frac{F_K(\frac{a^*}{1+n}, l)}{1+n} > (<) \frac{F_K(\frac{a^*}{1+n}, l) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}}{(1+n) + F_{KK}(\frac{a^*}{1+n}, l)\frac{a^*}{1+n}} \tag{48}$$

which (given that the denominator in the right-hand side is positive as established in Proposition 2) holds if, and only if,

$$F_K(\frac{a^*}{1+n}, l) < (>) 1+n = F_K(\frac{a^g}{1+n}, l) \tag{49}$$

i.e. if, and only if,

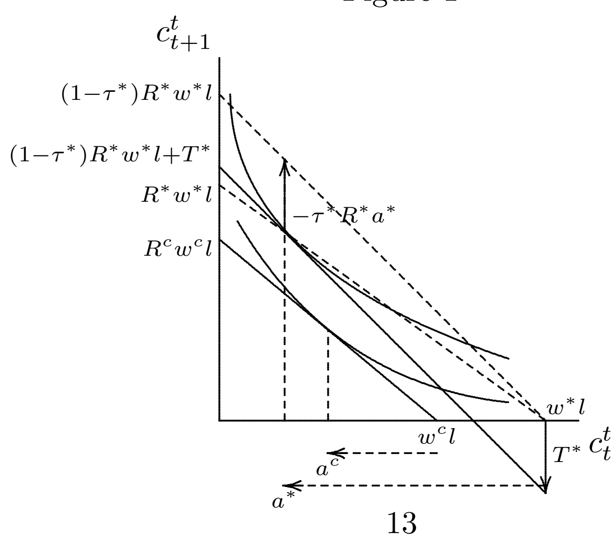
$$a^* > (<) a^g. \tag{50}$$

Q.E.D.

In case it may seem counterintuitive that taxing in a distortionary may be Pareto-improving, note that what the taxation of capital income aims at is the reduction (but not elimination) of the overaccumulation of capital (with respect to the first best) from the laissez-faire competitive equilibrium level of per capita savings a^c to a smaller level a^*. Reducing per capita savings further below a^* is not efficient if factors are to be remunerated by their marginal productivities and no redistribution takes place. Similarly, subsidizing savings returns can Pareto improve in case of excessive underaccumulation under laissez-faire.

Figure 1 below illustrates qualitatively (for the case $\tau^* < 0$) how the constrained efficient steady state improves upon the laissez-faire competitive equilibrium steady state from the representative agent's viewpoint. Note that the higher level of per capita savings a^* raises the labor income up to w^*l from its competitive value w^cl. It also deteriorates the return to savings R^* from its competitive value R^c. The subsidy $-\tau^*R^*a^*$ of this return is compensated exactly by the lump-sum tax T^* on second period income.

Figure 1



It is worth noticing that in the steady state of the competitive equilibrium dynamics with the active fiscal policy described above the government does never incur in any deficit or superavit and that the allocation is feasible. Notwithstanding, for the non-stationary paths the government budget constraint will not be balanced and requires an injection or withdrawal of resources (which in the later case could be devoted to public spending).

Note also that no generation can manipulate the determination of the government's policy, since it is determined by what the previous generation did. Nonetheless, it turns out that at the steady state each generation chooses to do exactly what the previous generation did.

7. Impatience, demographics, and taxes

As Proposition 7 below establishes, when the utility function of the representative agent specifically takes the logarithmic form, one obtains a more precise picture of when should the returns to capital be taxed and when subsidized depending on the agents' degree of impatience, the capital share of income, and the population growth rate. It turns out that returns to capital should be taxed unless the rate of growth of the population of the economy falls within an interval determined by the representative agent degree of impatience and the factors' shares of income. More specifically, capital returns should be subsidized only if the population growth rate $1+n$ falls within the interval $(\frac{\alpha}{1-\alpha}, \frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha})$, where α is capital's share of income and $1-\alpha$ is labor's share. Since $\frac{1+\beta}{\beta}$ varies from 2 to $+\infty$ as the representative agent becomes increasingly impatient and, typically, the ratio of capital to labor share is around 1/2 or at any rate below 1, for a patient enough *growing* population capital returns should be taxed in order to Pareto improve upon laissez-faire allocation.

Proposition 7. *If the discount factor is β, $u(c) = \ln c$, and $F(K, L) = K^\alpha L^{1-\alpha}$, then capital revenue should be taxed unless the population growth rate falls in $(\frac{\alpha}{1-\alpha}, \frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha})$, i.e.*

$$1 + n \in (\frac{\alpha}{1-\alpha}, \frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha}) \Leftrightarrow \tau^* < 0. \tag{51}$$

Proof. This is a consequence of the fact that if $u(c) = \ln c$, then

$$a^g < a^* \Leftrightarrow 1 + n \notin (\frac{\alpha}{1-\alpha}, \frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha}). \tag{52}$$

In effect, $a^g < a^*$ if, and only if,

$$\frac{1}{\beta}\frac{u'(F_L(\frac{a^g}{1+n}, l)l - a^g)}{u'(F_K(\frac{a^g}{1+n}, l)a^g)} < 1 + n \tag{53}$$

that is to say, if, and only if,

$$\frac{1}{\beta}\frac{\alpha\left(\frac{a^g}{(1+n)l}\right)^{\alpha-1}}{(1-\alpha)\left(\frac{a^g}{(1+n)l}\right)^{\alpha-1} - 1} < 1 + n \tag{54}$$

or equivalently, since

$$\alpha\left(\frac{a^g}{(1+n)l}\right)^{\alpha-1} = F_K(\frac{a^g}{1+n}, l) = 1+n, \tag{55}$$

if, and only if,

$$\frac{1}{\beta}\frac{1}{(1-\alpha)\frac{1+n}{\alpha}-1} < 1 \tag{56}$$

that is to say, if, and only if,

$$\frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha} < 1+n \tag{57}$$

and $\frac{\alpha}{1-\alpha} < 1+n$ or

$$\frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha} > 1+n \tag{58}$$

and $\frac{\alpha}{1-\alpha} > 1+n$, i.e. if, and only if,

$$\frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha} < 1+n \tag{59}$$

or

$$\frac{\alpha}{1-\alpha} > 1+n. \tag{60}$$

Q.E.D.

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