Bundling and Competition for Slots*
(Preliminary: Please do not circulate)

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Abstract

We study competition among upstream firms when each of them sells a portfolio of distinct products and the downstream firm has limited slots (or shelf space). In this situation, we study how bundling affects competition for slots. When the downstream has k number of slots, social efficiency requires that it purchases the best k products among all upstream firms' products. We find that under bundling, the outcome is always socially efficient but under individual sale, the outcome is not necessarily efficient. Under individual sale, each upstream firm faces a trade-off between quantity and rent extraction due to the coexistence of the internal competition (i.e. competition among its own products) and the external competition (i.e. competition from other firms' products), which can create inefficiency. On the contrary, bundling removes the internal competition and the external competition among bundles makes it optimal for each upstream firm to sell only the products belonging to the best k. This unambiguous welfare-enhancing effect of bundling is novel and robust.

Key words: Bundling, Competition among Portfolios, Limited Slots (or Shelf Space)

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1 Introduction

In vertical relations, very often each upstream firm sells a portfolio of distinct products which compete for limited slots (or shelf space) of downstream firms. In this situation, upstream firms may employ bundling as a strategy to win over the competition for the limited slots. The practice of bundling has been a major antitrust issue and a subject of intensive research in the past. However, to the best of our knowledge, the theoretical Industrial Organization literature seems to have paid little attention to competition among portfolios of distinct products and, in particular, no paper has studied how bundling affects competition among portfolios for limited slots. In this paper, we attempt to provide a new perspective on bundling by analyzing how it affects competition among portfolios of distinct products for limited slots, and social welfare.

Examples of the situation we described above are abundant. For instance, in the movie industry, each movie distributor has a portfolio of distinct movies and buyers (either movie theaters or TV stations) have limited slots. More precisely, the number of movies that can be projected in a season (or in a year) by a theater is constrained by time and the number of projection rooms. Likewise, the number of movies that a TV station can show at prime time during a year is also limited. Actually, allocation of slots in movie theaters has been one of the main issues of the last presidential election in France regarding the movie industry[1]. Furthermore, bundling in the movie industry (known as block booking[2]) was declared illegal in two supreme court decisions in U.S.: *Paramount Pictures*, where blocks of films were rented for theatrical exhibition, and *Loew's*, where blocks of films were rented for television exhibition. In addition, recently in *MCA Television Ltd. v. Public Interest Corp.* (11th Circuit, April 1999), the court of appeals reaffirmed the *per se* illegal status of block booking.

Another example we have in mind is that of manufacturers' competition for each retailer's shelf space. Typically, each manufacturer produces a range of different products (for instance, think about all the products sold under the brand name of Nestle) and manufacturers compete for each retailer's limited shelf space. In this context, manufacturers having a large portfolio of products may practice bundling (often called full-line forcing)

[1] Cahiers du Cinema (April, 2007) proposes to limits the number copies per film since certain movies by saturating screens limits other films' access to screens and asks the presidential candidates' opinions about the policy proposal.

[2] Block booking refers to "the practice of licensing, or offering for license, one feature or group of features on the condition that the exhibitor will also license another feature or group of features released by distributors during a given period" (Unites States v. Paramount Pictures, Inc., 334 U.S. 131, 156 (1948)).

for their advantage and there has been antitrust cases related to this practice: Procter & Gamble / Gillette[3] and Société des Caves de Roquefort.[4]

Moreover, since we consider a general model in which a firm can bundle any number of products, our setup can be applied to bundling a large number of information goods, a common practice in the Digital era (for instance, bundling of electronic academic journals).

In our model, we assume away any asymmetric information or any uncertainty about values of products. This will allow us to depart from the existing literature on block booking or bundling (see the review of the related literature later on in this section) and to identify what seems to us a first-order effect of bundling associated with the downstream firm's slot constraint. Actually, in the case of movie industry, Kenney and Klein (1983) point out that price discrimination explanation is inconsistent with the facts of *Paramount* and *Loew's* since the prices of the blocks varied a great deal across markets. Furthermore, in the Digital era, the prices are more and more tailored to buyers' characteristics as can be seen in the pricing of academic journals.

More precisely, we consider competition among n upstream firms when the downstream firm has $k(>0)$ number of slots. Each firm i has a portfolio of n_i distinct products. In our setting, a product needs to occupy a slot to generate some value (i.e. a profit) to the downstream firm. The upstream firms' products are heterogenous in terms of the value that each of them generates to the downstream firm. Therefore, social efficiency requires the downstream firm's slots to be allocated to the best k products among all products owned by n upstream firms. We focus on studying how bundling affects the set of the products that occupy the limited slots.

As the main result, we find that under bundling the outcome of competition is always socially efficient, while this is not necessarily the case under individual sale. Under individual sale, each upstream firm faces a trade-off between quantity and rent extraction due to the coexistence of the internal competition (i.e. competition among its own products) and the external competition (i.e. competition from other firms' products): as a firm increases the number of products it induces the downstream firm to buy, it should abandon more rent for each product it sells. This trade-off can make the outcome inefficient. On the contrary, bundling removes the internal competition and the external competition among bundles makes it optimal for each upstream firm to sell only the products that belong to the best k.

We think that this unambiguous welfare-enhancing effect of bundling is pretty novel. Furthermore, we show that the efficiency property of bundling is very robust in that it

[3]DG Competition case COMP/M.3732

[4]Conseil de la Concurrence, Decision 04-D-13, 8th April 2004.

holds regardless of whether we consider a sequential or simultaneous game or whether or not we allow firms to contract directly on exclusive use of slots. Our result has strong policy implications that go beyond the rule of reason supported by the existing literature analyzing bundling in a second-degree price discrimination framework.

There are only a few papers on block booking. According to the leverage theory, on which the Supreme Court's decisions were based, block booking allows a distributor to extend its monopoly power in a desirable movie to an undesirable one. This theory was criticized by Stigler since the distributor is better off by selling only the desirable movie at a higher price. As an alternative, Stigler (1968) proposed a theory based on second-degree price discrimination. However, Kenney and Klein (1983) point out that simple price discrimination explanation is inconsistent with the facts of *Paramount* and *Loew's* since the prices of the blocks varied a great deal across markets. Instead, they argue that block booking mainly prevents exhibitors from oversearching, (i.e. from rejecting films revealed ex post to be of below-average value). Their hypothesis is empirically tested in a recent paper by Hanssen (2000) but the author finds little support for the hypothesis.[5]

Most papers on bundling study bundling of two (physical) goods in the context of second-degree price discrimination and focus on either surplus extraction (Schmalensee, 1984, McAfee et al. 1989, Salinger 1995 and Armstrong 1996, 1999) in a monopoly setting or entry deterrence (Whinston 1990 and Nalebuff 2004) in a duopoly setting. Bakos and Brynjolfsson (1999, 2000)'s papers are an exception, in that they study bundling of a large number of information goods, but they maintain the second-degree price discrimination framework. Their first paper shows that bundling allows a monopolist to extract more surplus (since it reduces the variance of average valuations by the law of large numbers) and thereby unambiguously increases social welfare;[6] the second paper applies this insight to entry deterrence (we do not address the entry deterrence issue). Since we assume complete information and hence full surplus extraction is possible under the monopoly setting, the rent extraction issue does not arise in our framework and there is no use in applying the law of large number.

In Jeon-Menicucci (2006), we took a framework similar to the one in this paper to study bundling electronic academic journals. More precisely, publishers owning portfolios of distinct journals compete to sell them to a library who has a fixed budget to allocate between journals and books. Publishers are assumed to have complete information about the value that the library obtains from each journal (and about the budget). We found that bundling is a profitable strategy both in terms of surplus extraction and entry deterrence.

[5] But Kenny and Klein (2000) do not agree with Hanseen's analysis.

[6] See also Armstrong (1999).

Conventional wisdom says that bundling has no effect in such a setting and this is true in the absence of the budget constraint. However, when the budget constraint binds, we found that each firm has a strict incentive to adopt bundling but bundling reduces social welfare by reducing the library's consumption of journals and books. In this paper, instead of focusing on the budget constraint of the buyer, we focus on his slot constraint. Another difference is that Jeon-Menicucci (2006) focus on products (journals) of homogeneous value while in this paper we consider products of heterogenous value. In spite of similarities of the frameworks, the result we obtain here is completely opposite to the one in the previous paper since we find that the allocation under bundling is always socially efficient while the allocation under individual sale is not necessarily efficient.

Finally, to our knowledge, Shaffer (1991) is the only paper that explicitly models the downstream firm's limited shelf space.[7] He considers an upstream monopolist selling two substitutable products with variable quantity. He finds that brand specific two-part tariffs alone do not allow the monopolist to capture the maximum rent from the downstream firm but full-line forcing (equivalent to bundling) does. We consider products of independent values and hence the rent extraction issue Shaffer considers does not arise. In a general setup of competition in which seller i has n_i number of products and the buyer has $k(<\sum_i n_i)$ number slots, we study how bundling affects the set of the products that occupy the slots. Although products have independent values, competition arises in our setup because of the slot constraint.

In what follows, section 2 reviews the Chicago School Criticism of leverage theory with a simple model and related it to our model. Section 3 illustrates the key results with a simple example. Section 4 presents the model. Section 5 presents the main results: efficiency of bundling in the simultaneous pricing game. Section 6 characterizes the duopoly sequential pricing game without bundling in order to point out the trade-off between quantity and rent extraction. Section 7 shows as a robustness check that bundling gives efficient outcome in the case of sequential pricing as well. Section 8 shows that firms have an incentive to practice bundling. Section 9 drives implications on merger. Section 10 concludes. Most proofs are gathered in the Appendix.

2 Chicago School Criticism of Leverage Theory[8]

According to the leverage theory of tying (or bundling), a multiproduct firm with monopoly power in one market can monopolize a second market using the leverage provided by its

[7] See also Vergé (2001) who performs the social welfare analysis in the setup of Shaffer (1991).

[8] This section more or less follows section 2 of Choi (2004).

monopoly power in the first market. The theory, however, was largely discredited as a result of criticisms originating in the Chicago School (see e.g. Bowman 1957, Posner 1976, Bork 1978). In this section, we review the Chicago School Criticism of leverage theory with a simple model and explains our framework and contribution with respect to it.

Consider two independent products 1 and 2 and two firms A and B. Firm A is the monopolist of product 1 and A and B compete in the market for product 2. There is a single buyer who has a unit demand for each product. The buyer's willingness to pay for product 1 is $u_A^1 > 0$: the buyer's willingness to pay for product 2 produced by A (or B) is $u_A^2 > 0$ ($u_B^2 > 0$). Assume that the cost of production is zero for all products. In addition, we assume that $u_A^1 + u_A^2 > u_B^2$, which implies that by bundling, A can force the buyer to buy both products from A.

In the absence of bundling, firm $i (= A, B)$ simultaneously chooses a price for product $j (= 1, 2)$ $p_i^j \in \mathbf{R}_+$. In equilibrium, A always sells product 1 at $p_A^1 = u_A^1$ and sells product 2 at $p_A^2 = \max\{0, u_A^2 - u_B^2\}$ if and only if $u_A^2 \geq u_B^2$. Hence, A's profit without bundling is given by $p_A^1 + p_A^2 = u_A^1 + \max\{0, u_A^2 - u_B^2\}$. Note that under individual sale, the outcome is always socially efficient.

Suppose now that A bundles both products and charge P_A. Then, in equilibrium, A succeeds in selling both products at $P_A = u_A^1 + u_A^2 - u_B^2$. Note that under bundling, the outcome is socially inefficient if $u_A^2 < u_B^2$.

Comparing A's profit without bundling with its profit with bundling shows that bundling does not affect the profit if A is more efficient than B in product 2 (i.e. $u_A^2 \geq u_B^2$) and decreases it otherwise. This shows that A never has the incentive to practice bundling for the purpose of monopolizing the tied product market. Furthermore, a laissez-faire policy always achieves social efficiency since firm A's private incentive to practice bundling is aligned with social incentive.

However, we notice that Chicago School's criticism is a *weak argument* in the double sense: a social planner never has any strict incentive to favor bundling (since outcome is always socially efficient without bundling but it can be inefficient with bundling) and firms never have any strict incentive to practice bundling (since a firm can never strictly increase its profit with bundling).

In what follows, we consider a general model of competition among any number of firms when each of them sells a portfolio of any number of distinct products to a downstream firm (i.e. a buyer), called D. We assume that all products are independent: the value that D obtains from a product does not depend on the set of other products that D buys. Instead, we assume that D has a limited number of slots available and this creates competition among products. In this setting, we find a strict argument for laissez-faire regarding bundling. In particular, we show that the outcome of competition among portfolios is always efficient under bundling but can be inefficient without bundling. In addition, we show that each firm can strictly prefer bundling to no bundling since bundling allows a firm to avoid cannibalization due to the internal competition among its own products. In the next section, we illustrate these results through a simple example.

3 Illustration with a simple example

There are two upstream firms, called A and B. A has two products of value $(u_A^1, u_A^2) = (4, 3)$ and B has one product of value $u_B^1 = 2$: u_i^j means the value that the downstream firm D obtains from the j-th best product among firm i's products. Note that now all three products are distinct and hence there is no direct competition among the products. However, D has only two slots, which generates competition among them. The production cost is zero for all products. We note that social efficiency requires that the two slots be occupied by only A's products.

3.1 Without bundling

3.1.1 Equilibrium non-existence under simultaneous pricing

Consider first a simultaneous game of pricing without bundling: firm $i(= A, B)$ simultaneously chooses a price for product $j(= 1, 2)$ $p_i^j \in \mathbf{R}_+$. We show below that this game has no equilibrium in pure strategy. We assume as a tie-breaking rule that if D is indifferent among several products, D buys the products with highest (gross) values. Without loss of generality, we can assume that A chooses prices such that $4 - p_A^1 \geq \max\{0, 3 - p_A^2\}$: the net profit that D makes from buying A's best product is positive and larger than the one it makes from buying A's second best product.

First, there is no equilibrium in which A sells only its best product (i.e. there is no equilibrium with $p_A^2 > 1$). Suppose first that A charges $p_A^2 > 3$. Then, B's best response is $p_B^1 = 2$. A charges $p_A^1 = 4$ and hence achieves a profit equal to 4. This cannot be an equilibrium since A can deviate and charge for instance $p_A^{2\prime} = 3$ and $p_A^{1\prime} = 4$. Then A sells both products and realizes a profit equal to 7. Suppose now that A charges $p_A^2 \in (1, 3]$. Then, B can sell its product by charging $p_B^1 = p_A^2 - 1 - \varepsilon$ with $\varepsilon(> 0)$ small enough. $4 - p_A^1 \geq \max\{0, 3 - p_A^2\}$ implies that A charges $p_A^1 = 1 + p_A^2$ and hence A's profit is $1 + p_A^2$. Consider now A's deviation in which A charges $p_A^{2\prime} = p_A^2 - \varepsilon$ and $p_A^{1\prime} = p_A^1 - \varepsilon$. Then A sells both products and realizes a profit equal to $1 + 2(p_A^2 - \varepsilon)$, which is larger than $1 + p_A^2$.

Second, there is no equilibrium in which A sells both products (i.e. there is no equilibrium with $p_A^2 \le 1$). Note first that $p_A^2 \le 1$ together with $4 - p_A^1 \ge \max\{0, 3 - p_A^2\}$ implies that $p_A^1 \le 1 + p_A^2$ and therefore A's profit cannot be larger than 3 ($\ge 2 + p_A^2$). However, A can realize a profit equal to 4 by choosing $p_A^1 = 4$ and $p_A^2 = 3$ regardless of B's strategy.

The above example illustrates well some problems that A faces. On the one hand, there is a commitment issue: if A can commit not to sell its second best product, then A avoids competition from B and realizes a profit equal to 4 by extracting D's whole surplus from A's best product. However, A cannot commit to this policy: since B in turn responds by charging a monopoly price extracting D's whole surplus, A is tempted to sell its second best product as well by undercutting p_B^1. On the other hand, starting from a situation in which A sells only the best product at $p_A^1 = 4$, if A wants to sell the second best product as well, A suffers from internal competition. In other words, matching $u_B^1 - p_B^1 (> 0)$ by charging a price $p_A^2 \le u_A^2 - (u_B^1 - p_B^1)$ requires A also to reduce p_A^1 in order to maintain $u_A^1 - p_A^1 = u_A^2 - p_A^2$. Since any $p_A^2 > 1$ will be undercut by B, the Sup of the profit that A can achieve by selling both products is 3, which is lower than the profit A can achieve by selling only the best product (by charging for instance $p_A^1 = 4$ and $p_A^2 = 3$). Therefore, we have a circular argument and this is why the equilibrium does not exist.

3.1.2 Inefficiency in the sequential pricing

Consider now a sequential game of pricing in which A first chooses (p_A^1, p_A^2) and then B chooses p_B^1 after observing (p_A^1, p_A^2). Under the sequential game, A can commit to its pricing strategy and hence the equilibrium in pure strategy exists. However, we below show that the outcome is not socially efficient.

As we have seen before, if A wants to sell only its best product, it can charge for instance $p_A^1 = 4$ and $p_A^2 = 3$ and realizes a profit equal to 4. By contrast, if A wants to sell both products, because of the internal competition, it must charge $p_A^1 = 2$ and $p_A^2 = 1$. Then, its profit is 3. Therefore, without bundling, A prefers selling only its best product and the outcome is inefficient from social point of view.

3.2 Bundling

Consider now that A sells a bundle of both products and charges a price $P_A \in \mathbf{R}_+$. For notational consistency, let $P_B \in \mathbf{R}_+$ denote the price that B charges for its product. According to our theorem in section 5, regardless of whether the pricing game is simultaneous or sequential, there is a unique equilibrium outcome and it is efficient. Furthermore, we can show that A's profit under bundling is higher than its profit without bundling.

For instance in the simultaneous game of pricing, the unique equilibrium is $P_A = 5$ and $P_B = 0$. In the equilibrium, D buys A's bundle and hence the outcome is socially efficient. It is easy to see why this is an equilibrium. A has no incentive to charge a higher price; then D prefers buying B's product instead of A's bundle. Given that B's profit is zero, $P_B = 0$ is one of B's best responses.

The intuition for why bundling restores efficiency is the following: A's bundling gets rid of the internal competition between A's own products and makes the external competition with respect to B's product efficient. To explain this, let us consider an imaginary situation in which A sells a bundle composed of only its best product and charges $P_A = 4$. Then, D will buy it for sure and also buy B's product. Suppose now that A includes the second best product into the bundle and charges $P'_A = 4 + \varepsilon$ with $\varepsilon(> 0)$ small enough: i.e. after adding the second best product, A increases the price of the bundle by $\varepsilon(> 0)$ small enough. Given that A's second-best product is superior to B's product by 1, D will buy A's bundle and replace B's product with A's second best product. as long as $P'_A - P_A$ is smaller than $u^2_A - u^1_B = 1$. In a general model in which D has k number of slots, we show that a firm has a strict incentive to add any product that belongs to the k best among all products in the industry since adding such a product allows it to charge a strictly higher price. This is why the equilibrium under bundling is efficient. By contrast, without bundling, if A charges $p^1_A = 4$ and $p^2_A = \varepsilon$ with $\varepsilon(> 0)$ small enough, there will be cannibalization between A's two products and D will switch from A's first-best product to the second-best product.

Since A does not suffer from the internal competition under bundling, A's profit is higher under bundling than without bundling in our example. This is certainly true under the sequential game of pricing. Even under the simultaneous game of pricing, we know that without bundling, an upper bound of A's profit conditional on selling only the best product is 4 and an upper bound of A's profit conditional on selling both products is 3. Therefore, A has a strict incentive to practice bundling.

Therefore, our simple example illustrates well our strict our argument for laissez-faire with respect to bundling: both the social and the private incentives are strictly aligned.

4 The setting

4.1 Model

There are n upstream firms, denoted by $i = 1, ..., n$, and a downstream firm, denoted by D. Each firm i has a portfolio of $n_i(\geq 1)$ products, and all products are distinct. Firm D has a limited number of slots (or shelf space) to distribute the upstream firms' products: the number of slots is given by $k(\geq 1)$. We assume for simplicity that the cost of producing each product is zero for each firm, and the cost of distributing each product is zero for D.

D's distribution of a product requires one unit of slot. Therefore, D can distribute at most k number of products. In this setup, we consider products of heterogenous value and study how bundling affects the set of the products occupying the limited slots. More precisely, we are interested in knowing when D distributes the best k number of products. Let u_i^j denote the gross profit that D obtains from distributing the j-th best product of i; thus $u_i^1 \geq u_i^2 \geq ... \geq u_i^{n_i} \geq 0$ for $i = 1, ..., n$. In the case that $n_i > k$, it is obvious that only the k best products of firm i matter in our setting. In the case of $n_i < k$, we define $u_i^{n_i+1} = ... = u_i^k = 0$. In this way we can, without loss of generality, think that each firm's portfolio consists of k products. Let u^j denote the gross profit (or surplus) that D obtains from the j-th best product among all products owned by n upstream firms, thus $u^1 \geq u^2 \geq ... \geq u^{nk}$. We assume $u^k > u^{k+1}$. Let $U \equiv u^1 + ... + u^k$. Let q_i^{fb} denote the number of i's products in the set of the k best products among all products owned by both upstream firms: by definition, $q_1^{fb} + ... + q_n^{fb} = k$.

4.2 Bundling

Under bundling, let $B_i(q_i)$ denote the bundle of firm i which consists of the q_i best products in firm i's portfolio for $i = 1, ..., n$. Firm $i$chooses a price $P_i(q_i)$ for $B_i(q_i)$, for $q_i = 1, ..., k$. We consider a game in which each firm i offers a menu of bundles $\{P_i(q_i), B_i(q_i)\}$ and D can build its portfolio by choosing its most preferred bundles. We let $U(q_1, ..., q_n)$ denote the gross profit of D from buying the bundles $B_1(q_1) \cup ... \cup B_n(q_n)$; its net profit is then $U(q_1, ..., q_n) - P_1(q_1) - ... - P_n(q_n)$. Obviously, the function U takes into account the slot constraint, which implies that – if $q_1 + ... + q_n > k$ – only k product among the ones in $B_1(q_1) \cup ... \cup B_n(q_n)$ can be distributed by D. Hence, D will distribute the k best products among the ones he purchases. For instance, $U(k, ..., k) = U(q_1^{fb}, ..., q_n^{fb}) = U$. Let $U_{-i}(q_1, ..., q_{i-1}, q_{i+1}, ...q_n)$ denote the gross profit of D from buying the bundles $B_1(q_1) \cup ... \cup B_{i-1}(q_{i-1}) \cup B_{i+1}(q_{i+1}) \cup ... \cup B_n(q_n)$. Finally, we use q_i^* to denote the quantity D buys from firms i in equilibrium; this implies that D pays $P_i(q_i^*)$ to firm i.

Under bundling, we distinguish two different settings in terms of slot contracting: free disposal and exclusive slot. In the case of free disposal, upstream firms do not impose any constraint on the allocation of slots. More precisely, when D buys more than k number of products, D can choose the best k among them to occupy the slots and freely dispose of the rest. By contrast, in the case of exclusive slot, if D buys q_i number of products from i, D must allocate q_i number of slots exclusively on i's products: hence, D can buy at most k

number of products. We show later on that to some extent, this difference in terms of slot contracting matters for the outcome of competition.

We introduce a tie-breaking rule:

T1: If D is indifferent among different bundles (or products), D buys the bundle (product) that generates the highest gross value.

This is a standard tie-breaking rule. We will keep this for the case of individual sale (i.e. the case without bundling) as well.

4.3 Individual sale

Under individual sale (i.e. without bundling), firm i chooses $p_i^j \geq 0$ for its product with value u_i^j, and we define $w_i^j \equiv u_i^j - p_i^j$ as the net profit that D obtains from buying this product. Let $\mathbf{p}_i \equiv (p_i^1, p_i^2, ..., p_i^k)$ and $\mathbf{w}_i \equiv (w_i^1, w_i^2, ..., w_i^k)$ denote the vectors of prices and of net profits for the products of firm i, respectively. It is clear that there is a one-to-one correspondence between $\mathbf{p}_i$ and $\mathbf{w}_i$, and therefore we can equivalently express firm i's decision problem in terms of either $\mathbf{p}_i$ or $\mathbf{w}_i$. However, when we use $\mathbf{w}_i$ we need to recall that $w_i^j \leq u_i^j$ for $i = 1, ..., n$ and $j = 1, ..., n_i$. In particular, we will sometimes refer to the condition

$$w_1^1 \leq u_1^1, \quad w_1^2 \leq u_1^2, \quad ..., \quad w_1^k \leq u_1^k \tag{1}$$

for firm 1.

We have seen above that in the simultaneous pricing game of individual sale, there is no NE. Thus, we consider in section 6 the following sequential game in a duopoly setting:

Stage 1. Firm 1 chooses $\mathbf{p}_1$;

Stage 2. After observing $\mathbf{p}_1$, firm 2 chooses $\mathbf{p}_2$;

Stage 3. D makes its purchase decision.

We use the concept of subgame perfect Nash equilibrium (SPNE) to determine the outcome of this sequential game. Thus we start with D's purchases at stage three: D chooses the k products yielding the highest non-negative net profits. However, it is necessary to specify how D deals with ties, i.e. with products which have the same net profit. Therefore we introduce one more tie-breaking rule:

T2: If D is indifferent between buying a product from 1 and a product from 2 (both with non-negative net profits), and cannot buy both of them, then D buys 2's product.

T2 is motivated by the fact that in our sequential game, given the price of 1's product in question, 2, as the follower, can always lower by $\varepsilon > 0$ its price to break D's indifference. Formally, in some cases 2 has no best reply without this assumption.

5 Bundling

In this section, we study a simultaneous game with bundling. We first study competition with pure bundle in which each firm i offers only a single bundle of all its products and charges a price P_i. And then we study competition with menu of bundles that we introduced in section 4.2.

5.1 Competition with pure bundle

Consider competition in which each firm i offers only a single bundle of all its products $B_i(k)$ and charges a price P_i. We assume free disposal. This setting makes some sense in the case of Digital good. For instance, consider an American broadcasting company which sells its TV programs to other countries. Then, it can offer the same package and charge different prices depending on the demand in each country. Each buyer, a broadcasting company in each different country, is free to decide which programs to broadcast.

Consider the following price

$$P_i^* = U - U_{-i}(k, ..., k).$$

When firm i chooses its price, it assumes that all the other bundles are bought and charges the price equal to the extra value that D can get from buying its bundle. Note that if firm i has at least a product which belongs to the k best products among all products (i.e. $q_i^{fb} > 0$), $P_i^* > 0$: otherwise, $P_i^* = 0$. It is easy to see that this $\{P_i^*\}$ is an equilibrium. Obviously, firm i has no interest in reducing the price. firm i has no incentive to increase its price either: since D is indifferent between buying $B_i(k)$ and not, any increase in price above P_i^* induces D to stop buying it.

The following proposition shows that this is indeed a unique equilibrium.

Proposition 1 *(pure bundle and free disposal) Consider the simultaneous game of pricing in which each firm i sells a unique bundle including all its products. Assume free disposal. There is a unique equilibrium in which firm i charges*

$$P_i^* = U - U_{-i}(k, ..., k).$$

The equilibrium is efficient.

Proof. The proof of uniqueness should be written. ■

Even though the equilibrium in this pure bundling is pretty simple, the intuition for which this generates efficiency is very important. In this case of free disposal, starting from a bundle B_i with price P_i, adding an additional product into the bundle never makes the bundle less attractive and therefore i can command at least the same price. In other words, firm i does not need to worry about internal competition among its own products under bundling. Therefore, selling a bundle that includes all firm i's products weakly dominates selling a bundle that does not include all i's products. In equilibrium, all firms end up selling their bundles and therefore D can choose the best k products to occupy the slots, which makes the equilibrium efficient. In the next section, we consider the more general case of competition with menu of bundles.

5.2 Competition with menu of bundles

In this section, we study competition with menu of bundles that we introduced in section 4.2. We need to introduce a further piece of notation. For each firm i, let u_{-i}^j denote the value of the j-th best product among the ones in the portfolios of all firms except from i, for $j = 1, ..., k$.[9] Then we show below that regardless of whether we consider free disposal or exclusive slot, an equilibrium exists such that each firm i chooses prices as follows for its bundles:

$$\begin{aligned}
P_i^*(1) &= \max\{0, u_i^1 - u_{-i}^k\}, \\
P_i^*(2) &= \max\{0, u_i^1 - u_{-i}^k\} + \max\{0, u_i^2 - u_{-i}^{k-1}\} = P_i^*(1) + \max\{0, u_i^2 - u_{-i}^{k-1}\}, \\
P_i^*(3) &= P_i^*(2) + \max\{0, u_i^3 - u_{-i}^{k-2}\}, \\
&\quad ..., \\
P_i^*(k) &= P_i^*(k-1) + \max\{0, u_i^k - u_{-i}^1\}
\end{aligned}$$

5.2.1 Free disposal

In this section, we consider the case of free disposal. In order to explain the above price schedule, we focus on firm i's pricing. Because of free disposal, without loss of generality, we can restrict attention to non-decreasing price schedules: $P_i(1) \le ... \le P_i(k)$. In case i does not sell any product, i expects that D will constitute a portfolio composed of the best k products from all firms except i. Now, suppose $q_i^{fb} > 1$ and i sells its best product. Then, D will replace the worst product in the portfolio with i's best product. Therefore, the maximum that i can charge for $B_i(1)$ is equal to the value of its best product u_i^1 minus the value of the worst product in D's portfolio u_{-i}^k. This explains $P_i^*(1)$. Suppose now that in addition to offering $(B_i(1), P_i^*(1))$, i offers a bundle of its best two products. Then,

[9] Needless to say, we may define u_{-i}^j for $j = 1, ..., (n-1)k$, but for our purposes it suffices to consider the values $u_{-i}^1, ..., u_{-i}^k$.

it matters whether the value of i's second best product u_i^2 is larger than the value of the worst product in D's new portfolio that includes i's best product, u_{-i}^{k-1}. D will modify this portfolio by including i's second best product if and only if $u_i^2 > u_{-i}^{k-1}$ and in this case the maximum price that i can charge for $B_i(2)$ is $P_i^*(1) + u_i^2 - u_{-i}^{k-1}$. By contrast, if $u_i^2 \leq u_{-i}^{k-1}$, $B_i(2)$ does not add any value with respect to $B_i(1)$ since D will not include i's second best product into the portfolio that occupies the slots. Therefore, in this case, $P_i^*(2)$ must be equal to $P_i^*(1)$. This also implies that $P_i^*(3) = ... = P_i^*(k) = P_i^*(2)$ if $q_i^{fb} = 2$.

When each firm i offers a menu of bundles $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^k$ defined above, D chooses to buy $B_i(q_i^{fb})$ (i.e. D buys exactly the efficient units) or $B_i(q_i)$ with $q_i > q_i^{fb}$, for $i = 1, ..., n$. Notice, however, that even though D buys $B_i(q_i)$ with $q_i > q_i^{fb}$, D ends up using only the q_i^{fb} efficient products in $B_i(q_i)$ to occupy the slots. Hence, the outcome is efficient. Furthermore, $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^k$ is an equilibrium. For instance, i's equilibrium profit is $P_i^*(q_i^{fb}) = \sum_{j=1}^{q_i^{fb}} (u_i^j - u_{-i}^{k+1-j}) \equiv P_i^e$. Given other firms' strategy, by construction, $P_i^*(q_i)$ is the highest profit that i can achieve conditional on selling $B_i(q_i)$: at $P_i^*(q_i)$, D is indifferent between buying $B_i(q_i)$ or not and hence any increase in price above $P_i^*(q_i)$ induces D to stop buying $B_i(q_i)$. Furthermore, $P_i^*(q_i)$ strictly increases till $q_i = q_i^{fb}$ and then remains constant for $q_i > q_i^{fb}$. Therefore, i cannot achieve a profit larger than $P_i^*(q_i^{fb})$, which proves that $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^k$ is an equilibrium. The next proposition establishes that the outcome of this equilibrium is the unique outcome in any NE of the game: although there can be multiple equilibria, all have the same outcome in that it is efficient (i.e. the best k products occupy the slots) and each firms makes the same profit $P_i^e \equiv P_i^*(q_i^{fb})$.

Theorem 1 *(menu of bundle and free disposal) In the simultaneous pricing game with free disposal in which each upstream firm offers a menu of bundles, any NE of the game has the same unique outcome: for $i = 1, ..., n$, D buys $B_i(q_i^*)$ with $q_i^* \geq q_i^{fb}$ and $P_i^*(q_i^*) = P_i^e$.*

Theorem 1 shows that the result of Proposition 1 that we obtained in the case of competition with pure bundle remains robust when we consider competition with menu of bundle. We already pointed out, right after Proposition 1, that under free disposal, bundling removes any internal competition among products belonging to the same firm. In the case of pure bundles, efficiency of the external competition among bundles is rather trivially obtained since D ends up buying all products. The analysis of competition with menu of bundle reveals that when we endogenize the composition of each firm i's bundle, the competitive position of i's bundle strictly improves until its bundle includes its q_i^{fb} number of best products and then adding more product into the bundle does not affect its competitive position. This induces each firm i to sell at least its q_i^{fb} number of best products and hence the outcome is socially efficient.

5.2.2 Exclusive slots

Consider now the case of exclusive slots. In the previous case of free disposal, it is impossible for i to induce D to allocate more than q_i^{fb} slots on i. In contrast, under exclusive slots, firm i may be tempted to impose D to buy more than q_i^{fb} number of products; in the extreme case, i may try D to occupy all slots with i's products by proposing a contract $P_i(1) = ... = P_i(k-1) = \infty$ and $P_i(k) < \infty$. However, we can show that $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^k$ is an equilibrium in the case of exclusive slots.

Suppose that each firm i proposes $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^k$. Note first that under exclusive slots, D will buy exactly k number of products. Then, it is obvious that it is optimal for D to buy exactly q_i^{fb} best products from i for $i = 1, ..., n$.[10] This implies that i's equilibrium profit is $P_i^*(q_i^{fb}) = \sum_{j=1}^{q_i^{fb}}(u_i^j - u_{-i}^{k+1-j}) \equiv P_i^e$.

Now we study whether 1 has any incentive to deviate. First, conditional on selling a bundle of $q_1(\leq q_1^{fb})$ best products, the maximal price that 1 can charge is given by $P_1^*(q_1)$. When we computed $P_i^*(q_i)$.for $q_i \leq q_i^{fb}$, we already took into account the fact that q_i best products of i would be included into D's portfolio that occupies the slots. Therefore, requiring D to allocate q_1 slots exclusively on 1's products is redundant and hence does not affect the price of the bundle that 1 can command.

Therefore, the question is to know whether 1 has an incentive to sell more than q_1^{fb}. To answer the question, suppose that 1 decides to sell a bundle of $q_1^{fb} + 1$ number of products. Under exclusive slots, buying $q_1^{fb} + 1$ products implies that the $q_1^{fb} + 1$ th best product of 1 replaces a product that belongs to the k best. In order to explain which product D stops buying, we notice that from the definition of $P_i^*(q_i)$, the net profit that D obtains from the q_ith product of i is $u_{-i}^{k+1-q_i}$ for $q_i \leq q_i^{fb}$. Since under the menu of bundle, buying q_i^{fb}th product without buying $q_i^{fb} - 1$the product from i is impossible, the change in D's profit after 1's deviation is:

$$u_1^{q_1^{fb}+1} - u_{-i}^{k+1-q_i^{fb}} \text{ for some } i \neq 1$$

Note that $u_{-i}^{k+1-q_i^{fb}}$ is the value from the best product among all firms different from i excluding those belonging to the k best. In other words, we have

$$u_{-i}^{k+1-q_i^{fb}} = \max_{j \neq i} \left\{ u_j^{q_j^{fb}+1} \right\}.$$

Therefore, we have

$$u_1^{q_1^{fb}+1} - \max_{j \neq i} \left\{ u_j^{q_j^{fb}+1} \right\} \leq 0.$$

[10] The reason is that (i) given $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^k$, D's payoff cannot be higher under exclusive slots than under free disposal (ii) buying exactly q_i^{fb} best products from i allows D to achieve its maximal payoff under free disposal.

This implies that 1 has no incentive to force D to buy $q_1^{fb}+1$ number of products and for the same reason 1 has no incentive to force D to buy more than q_1^{fb} number of products. This proves that $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^{k}$ is an equilibrium.

Proposition 2 *(menu of bundle and exclusive slots) In the simultaneous pricing game with exclusive slots in which each upstream firm offers a menu of bundles, there exists a NE in which each firm i offers $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^{k}$. In the equilibrium, for $i=1,...,n$, D buys exactly $B_i(q_i^{fb})$ and $P_i^*(q_i^*)=P_i^e$.*

Proposition 2 shows that the efficient equilibrium always exists under exclusive slots. This might suggest that there is no difference between the setting with free disposal and the setting with exclusive slots. However, in what follows, we will show that under exclusive slots, some inefficient equilibria may arise due to coordination failure among firms.

Note that under free disposal, no coordination among firms is needed. For instance, the worst case for firm i is that D already bought all products that belong to the rival firms. Even in this case, as long as $q_i^{fb}>0$, i can induce D to buy $B_i(q_i^{fb})$ by charging $P_i^*(q_i^*)=P_i^e$. By contrast, under exclusive clots, whether i can successfully sell its q_i^{fb} best products depends on the menu of bundles offered by rival firms. We below illustrate this issue through an example.

Consider the following example. D has three slots. There are three firms (1, 2, 3) and each of them has three products.

$$\begin{aligned}(u_1^1, u_1^2, u_1^3) &= (10, 8, 6);\\ (u_2^1, u_2^2, u_2^3) &= (9, 7, 1);\\ (u_3^1, u_3^2, u_3^3) &= (9, 7, 1)\end{aligned}$$

,

In this example, social efficiency requires that each firm occupies only one slot. This is what happens in the equilibrium with $\{B_i(q_i), P_i^*(q_i)\}_{q_i=1}^{k}$. Then, 1's profit is equal to 3. However, we can also find an inefficient equilibrium. More precisely, in this equilibrium, each firm offers a pure bundle including all three products and engages in Bertrand competition among the bundles with equilibrium price $(P_1, P_2, P_3) = (7, 0, 0)$. This is an equilibrium and in the equilibrium, firm 1 occupies all three slots and realizes a profit larger than the profit in the efficient equilibrium. The reason for having this inefficient equilibrium is that firm 2 and firm 3 fail to coordinate by not offering menu of bundles. For instance, if 2 offers a bundle of the two best products at price equal to 0.4 and 3 offers the best product at price equal to 0.4, D will reject 1's offer even if 1 charges zero price.

One extreme case in which there is no such coordination issue is duopoly. In this case, we can show that the game with exclusive slots is equivalent to the game with free disposal in terms of outcome.

Therefore, we have:

Proposition 3 *Consider the simultaneous pricing game in which each upstream firm offers a menu of bundles.*

(i) In the case of duopoly, the game with free disposal is outcome equivalent to the game with exclusive slots

(ii) If there is more than two firms, under exclusive slots, there can be inefficient equilibria due coordination failure among firms.

6 Sequential pricing without bundling

In this section, we consider the sequential pricing game without bundling that we introduced in section 4.3.

6.1 A preliminary result

Recall that we have set $w_i^j = u_i^j - p_i^j$ for $i = A, B$ and $j = 1, ..., n_i$, and $\mathbf{w}_i = (w_i^1, w_i^2, ..., w_i^{n_i})$ for $i = A, B$. In $\hat{\mathbf{w}}_i \equiv (w_i^{(1)}, w_i^{(2)}, ..., w_i^{(n_i)})$ we order instead the net profits in a decreasing way, which means that $w_i^{(1)} \geq w_i^{(2)} \geq ... \geq w_i^{(n_i)}$. We now prove a simple and intuitive result: there is no loss of generality in assuming that $w_i^{(j)} = w_i^j$ for all j.

Lemma 1 *Without loss of generality, we can restrict our attention to the case in which $w_i^1 \geq w_i^2 \geq ... \geq w_i^{n_i}$ (i.e., $\mathbf{w}_i = \hat{\mathbf{w}}_i$) for i=A,B.*

In particular, lemma 1 implies the following monotonicity condition for firm A, which we will use repeatedly in the remaining of the paper:

$$w_A^1 \geq w_A^2 \geq ... \geq w_A^k \tag{2}$$

The lemma also implies that when firm i is selling m_i number of products, i is actually selling its products with the m_i highest gross values.

6.2 Stage two

Now we apply backwards induction to firm B, by examining his decision at stage two. Precisely, we take $\mathbf{w}_A$ as given and consider the following questions: given $m \in \{1, ..., n_B\}$, is it feasible for B to sell m units? If so, what is the highest profit B can make by selling m units?

Lemma 2 *Given $\mathbf{w}_A$ and $m \in \{1, ..., n_B\}$, it is feasible for B to sell m units if and only if $u_B^m > w_A^{k-m+1}$. In this case, the highest profit B can earn by selling m products is $u_B^1 + ... + u_B^m - m \max\left\{w_A^{k-m+1}, 0\right\}$.*

The basic idea of the lemma is that D buys m units of B if and only if m products of B are among the k products with the highest net profits. For instance, consider the case of $m = 1$. If $w_A^k \geq u_B^1$, then B cannot sell any product because the inequality $w_A^k > w_B^1$ necessarily holds and therefore D will buy k products from A and none from B. If instead $w_A^k < u_B^1$, B succeeds in selling his best product by charging a sufficiently low price p_B^1 such that $w_A^k < u_B^1 - p_B^1$ and p_B^j large enough for $j \geq 2$. Precisely, from T1, the highest price which induces D to buy B's best product is $p_B^1 = u_B^1 - \max\left\{w_A^k, 0\right\}$. In words, B can sell his best product only if the k-th best product of A gives D a net profit that is smaller than the gross profit of the best product of B. In short, it must be possible for B to push out the k-th best product of A by pricing aggressively enough his own best product.

For an arbitrary value of m in $\{1, ..., n_B\}$, the same argument shows that the inequality $w_A^{k-m+1} < u_B^m$ is necessary, i.e. it must be possible for B to block out the $(k-m+1)$-th best product of A by pricing suitably his own m best products. Otherwise, $w_A^{k-m+1} > w_B^m$ and therefore D will buy at least $k - m + 1$ units from 1, and at most $k - (k - m + 1) = m - 1$ from B. When $w_A^{k-m+1} < u_B^m$, B succeeds in selling m products by charging prices $p_B^1, ..., p_B^m$ such that $w_B^1 = ... = w_B^m = \max\left\{w_A^{k-m+1}, 0\right\}$ (again, recall T1), or equivalently $p_B^j = u_B^j - \max\left\{w_A^{k-m+1}, 0\right\}$ for $j = 1, ..., m$ and p_B^j large for $j = m+1, ..., n^B$; the resulting profit for B is $u_B^1 + ... + u_B^m - m \max\left\{w_A^{k-m+1}, 0\right\}$.

In view of lemma 2 we define as follows the profit B can make by selling m units, for $m \in \{1, ..., n_B\}$:[11]

$$\pi_B(m) \equiv \begin{cases} u_B^1 + ... + u_B^m - m \max\left\{w_A^{k-m+1}, 0\right\} & \text{if } u_B^m > w_A^{k-m+1}; \\ 0, & \text{otherwise.} \end{cases}$$

In order to examine how π_B depends on m, we begin by noticing that the higher is m, the more restrictive is the inequality $u_B^m > w_A^{k-m+1}$. Thus, if B is unable to sell m units because $u_B^m \leq w_A^{k-m+1}$, he is *a fortiori* unable to sell $\tilde{m} > m$ units.

[11] This profit depends also on $\mathbf{w}_A$, even though we do not emphasize this fact in the notation.

Now we consider a case in which $u_B^{m+1} > w_A^{k-m} > 0$, so that B is able to sell $m+1$ products (and also fewer than $m+1$) and we below examine how increasing his sale by one more product affects B's profit. When B sells m units, we have seen that he earns a profit of $u_B^1 + ... + u_B^m - mw_A^{k-m+1}$ by charging prices $p_B^j = u_B^j - w_A^{k-m+1}$ for $j = 1, ..., m$; these prices are determined by the fact that B needs to block out the $(k-m+1)$-th best product of A. If instead he sells $m+1$ units, B needs to push out the $(k-m)$-th best product of A, which is more valuable than the $(k-m+1)$-th. Prices are then $\hat{p}_B^j = u_B^j - w_A^{k-m}$ for $j = 1, ..., m+1$, and $\hat{p}_B^j < p_B^j$ for $j = 1, ..., m$. This generates a loss for B, on his m best units, equal to $m(w_A^{k-m} - w_A^{k-m+1})$. However, now B gains $\hat{p}_B^{m+1} = u_B^{m+1} - w_A^{k-m} > 0$ from the sale of the $(m+1)$-th unit. Whether B prefers selling $m+1$ units to m units depends on the comparison between the loss $m(w_A^{k-m} - w_A^{k-m+1})$ and the gain $u_B^{m+1} - w_A^{k-m}$. In other words, (2) makes B face *a trade-off between quantity and (per unit) rent extraction*: as B increases the number of products he sells, he must leave more surplus per unit to D.

6.3 Stage one

We first study the optimal pricing conditional on that A sells $k-m$ units. And then, we study the optimal m that maximizes A's profit.

6.3.1 A's profit when he sells $k-m$ units

Now we consider the first stage of the game in order to determine the profit A can make as a function of the number of products he sells. Hence, suppose that A wants to sell $k-m$ units for $m \in \{0, 1, ..., n_B\}$. Then, we inquire whether (i) there exists $\mathbf{w}_A$ such that, taking into account the best response by B, induces D to buy $k-m$ units from A; (ii) within the set of $\mathbf{w}_A$ which allow A to sell $k-m$ units, we identify the vector that maximizes A's profit.

Formally, the conditions that allow A to sell $k-m$ products can be stated by using the following incentive constraints:

$$(\text{IC}_{m,m'}) \qquad \pi_B(m) \geq \pi_B(m') \quad \text{for} \quad \text{any} \quad m' \neq m \quad \text{and} \quad m' \in \{1, ..., n_B\} \qquad (3)$$

Condition (3) means that B prefers to sell m units rather than $m' \neq m$. In particular, (3) implies that B is not going to push out the $(k-m)$-th best unit of A (nor any better product of A),[12] and therefore D will buy $k-m$ number of products from A. Then A's

[12]Actually, it suffices to satisfy the constraints $(\text{IC}_{m,m'})$ for all $m' > m$. However, it turns out that it is costless for A to satisfy also the constraints $(\text{IC}_{m,m'})$ for $m' < m$ (see the proof of Proposition 4).

profit is given by:

$$\pi_A(k-m) \equiv \sum_{j=1}^{k-m}(u_A^j - w_A^j)\mathbf{1}_{[w_A^j \geq 0]}$$

which, we note, is not affected by $(w_A^{k-m+1}, ..., w_A^k)$. We investigate below whether there is a set of $\mathbf{w}_A$ which satisfy (3) and, if so, we maximize $\pi_A(k-m)$ in this set.

We start by observing that it is certainly possible for A to sell $k - n_B$ units, and that he can do so without leaving any surplus to D on these products. In order to show the details, suppose that A chooses $p_A^j = u_A^j$ for $j = 1, ..., k - n_B$ and p_A^j high enough for $j = k - n_B + 1, ..., k$. In this way, A's n_B worst products are not competing with B's products while A's best $k - n_B$ products give D zero surplus. Then, B will reply by charging $p_B^j = u_B^j$ for $j = 1, ..., n_B$, and D will buy $k - n_B$ products from A and n_B products from B, earning no profit.

When A's objective is to induce B to sell only $m(< n_B)$ products, as it will become clear later on, B has two strategies: *accommodation* or *fight*. "Accommodation" means that B contents himself with occupying m slots. "Fight" means that B tries to occupy more than m slots by blocking out some extra units of A. Obviously, to achieve his goal, A must choose prices such that B prefers accommodation to fight, which is equivalent to the property that $(\text{IC}_{m,m'})$ is satisfied for all $m' > m$. What makes the case of $m = n_B$ straightforward is that B sells *all* his n_B units by accommodating, and thus he will not fight.

The next proposition characterizes the condition under which A is able to sell $k - m$ units and the profit maximizing vector $\mathbf{w}_A$ (hence, the optimal prices) conditional on selling $k - m$ units. For expositional facility, we introduce the following notation. Given $m \in \{0, 1, ..., n_B - 1\}$, let

$$\mu_m^{k+1-m''} \equiv \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''}) \qquad \text{for} \quad m'' = m+1, ..., n_B \tag{4}$$

Proposition 4 *For a given $m \in \{0, 1, ..., n_B - 1\}$,*

(i) a. A can find $\mathbf{w}_A$ that induces D to buy $k - m$ units from A if and only if

$$u_A^{k+1-m''} > \mu_m^{k+1-m''} \qquad \textit{for} \quad m'' = m+1, ..., n_B \tag{5}$$

b. Let $\hat{m} \in \{0, 1, ..., n_B - 1\}$ denote the smallest m for which (5) is satisfied [we set $\hat{m} = n_B$ if (5) fails to hold for any $m \in \{0, 1, ..., n_B - 1\}$]. Then, (5) is satisfied also for $m = \hat{m} + 1, ..., n_B$.

(ii) If $m \geq \hat{m}$, the profit maximizing $\mathbf{w}_A$ for A is as follows:

a. when $m = 0$, $w_A^1 = ... = w_A^k = u_B^1$;

b. when $m \in \{1, ..., n_B - 1\}$,

$$\begin{aligned} w_A^{k-m+1} &= w_A^{k-m+2} = ... = w_A^k = 0 && (6) \\ w_A^{k-m''+1} &= \max\{w_A^{k-m''+2}, \mu_m^{k-m''+1}\} \qquad \textit{for} \quad m'' = m+1, ..., n_B && (7) \\ w_A^1 &= w_A^2 = ... = w_A^{k-n_B} = w_A^{k-n_B+1} && (8) \end{aligned}$$

We below give the intuition of the results in Proposition 4; we focus on explaining the profit maximizing $\mathbf{w}_A$ conditional on selling $k-m$ units for $m \geq 1$, described in Proposition 4(ii)b.[13] Given A's objective to sell $k-m$ units, A should structure his prices for the best $k-m$ products (the ones to sell) very differently from the prices for the m worst units (the ones not to sell). On the one hand, regarding the m worst products, it is optimal to charge very high prices (higher than their values) so that B does not face any competition from them; precisely, (6) reveals that choosing $w_A^{k-m+1} = w_A^{k-m+2} = ... = w_A^k = 0$ is optimal. The reason is that this pricing maximizes B's profit from accommodation and hence reduces B's temptation to fight. In fact, the pricing allows B to extract full surplus $u_B^1 + ... + u_B^{m'}$ from his best m' products if he wants to sell only $m' \leq m$ products. Then, obviously, B strictly prefers selling m units to selling less than m, and hence downward incentive constraints (i.e. ($\mathrm{IC}_{m,m'}$) for $m' < m$) are trivially satisfied. On the other hand, regarding the best $k-m$ units to sell, the prices should be competitive enough to make it unprofitable for B to sell more than m units. In particular, A cannot extract full surplus from these products since if he attempts to do that, B can sell all of his n_B products by leaving no surplus per product to D, given T1.

To explain the optimal pricing of the best $k-m$ units, suppose that B wants to sell $m+1$ units instead of m units. Lemma 2 shows that B can sell $m+1$ products only if $w_A^{k-m} < u_B^{m+1}$. In this case, B makes a profit equal to $\pi_B(m+1) = u_B^1 + ... + u_B^{m+1} - (m+1)w_A^{k-m}$ and we have

$$\pi_B(m+1) - \pi_B(m) = u_B^{m+1} - (m+1)w_A^{k-m}$$

As we discussed after Lemma 2, $u_B^{m+1} - (m+1)w_A^{k-m}$ is composed of the loss $-mw_A^{k-m}$ on B's best m units (with respect to selling them at full prices) plus the gain $u_B^{m+1} - w_A^{k-m}$ from selling the $(m+1)$-th unit. Therefore, $w_A^{k-m} \geq \frac{u_B^{m+1}}{m+1} = \mu_m^{k-m}$ allows to satisfy $\pi_B(m) \geq \pi_B(m+1)$: note that it is less restrictive than $w_A^{k-m} \geq u_B^{m+1}$. Hence, the smallest value of w_A^{k-m} satisfying ($\mathrm{IC}_{m,m+1}$) is $w_A^{k-m} = \mu_m^{k-m}$, as described in (7). In order to deter B from selling $m+2$ units, we can argue as before. A sufficient condition is $w_A^{k-m-1} \geq u_B^{m+2}$, but

[13] Proposition 4(ii)a is straightforward, as the best way for A to sell k products is to set $w_A^1 = ... = w_A^k$ equal to the value of B's best product, u_B^1, provided that $u_A^k > u_B^1$.

when $w_A^{k-m-1} < u_B^{m+2}$ we must have:

$$\pi(m+2) - \pi(m) = u_B^{m+1} + u_B^{m+2} - (m+2)w_A^{k-m-1} \leq 0,$$

which is equivalent to $w_A^{k-m-1} \geq \mu_m^{k-m-1} = \frac{1}{m+2}(u_B^{m+1} + u_B^{m+2})$. Therefore, ($IC_{m,m+2}$) is satisfied if $w_A^{k-m-1} \geq \min\{\mu_m^{k-m-1}, u_B^{m+2}\}$. However, w_A^{k-m-1} should also satisfy the monotonicity condition (2) (in particular, $w_A^{k-m-1} \geq w_A^{k-m}$). From $w_A^{k-m-1} \geq \min\{\mu_m^{k-m-1}, u_B^{m+2}\}$ and $w_A^{k-m-1} \geq w_A^{k-m}$, we find that the smallest value of w_A^{k-m-1} satisfying ($IC_{m,m+2}$) is $w_A^{k-m-1} = \max\{w_A^{k-m}, \mu_m^{k-m-1}\}$, as described in (7).[14] By iterating the argument we obtain the smallest values of w_A^{k-m}, $w_A^{k-m-1}, ..., w_A^{k-n_B+1}$ which satisfy (3), as described in (7). This explains the pricing of the worst $n_B - m$ units of A among the $k - m$ units to sell. Finally, regarding the pricing of the best $k - n_B$ units to sell, we observe that the variables in $(w_A^1, ..., w_A^{k-n_B})$ do not affect (3) and thus each of them can be set equal to $w_A^{k-n_B+1}$ to satisfy the monotonicity condition (2), as described in (8). In this way we have found the smallest values of $w_A^1, ..., w_A^k$ which satisfy (2) and (3).

As we mentioned in section 2, the values in $\mathbf{w}_A$ are feasible only if they satisfy (1) since otherwise there exist no prices $p_A^1 > 0, ..., p_A^k > 0$ such that $w_A^j = u_A^j - p_A^j$ for $j = 1, ..., k$. Hence, u_A^j must be larger than the profit-maximizing w_A^j characterized in Proposition 4(ii). This is why (5) is necessary and sufficient for A to be able to sell $k - m$ units. Notice that Proposition 4(i)b implies that there is an $\hat{m}$ between 0 and n_B such that A is able to sell any number of units between 0 and $k - \hat{m}$, but out arguments above imply that A will always sell at least $k - n_B$ units, if $k > n_B$.

6.3.2 Maximizing A's profit with respect to m

Since Proposition 4 allows to compute $\pi_A(k - m)$ for any $m \geq \hat{m}$, the profit-maximizing m can be found by comparing $\pi_A(k - n_B)$, $\pi_A(k - n_B + 1)$, ..., $\pi_A(k - \hat{m})$. Before seeing a few examples and a useful property of π_A, we can improve our understanding of the problem of A by comparing $\pi_A(k - m)$ with $\pi_A(k - m + 1)$, in order to examine the incentives of A to increase his supply. Let us use here $w_A^1(m), ..., w_A^{k-m}(m)$ to denote D's net profits from buying A's products, as determined by (7)-(8), when A sells $k - m$ products.

Then we find

$$\begin{aligned} w_A^{k-m}(m) &= \mu_m^{k-m}, \quad w_A^{k-m-1}(m) = \max\{\mu_m^{k-m}, \mu_m^{k-m-1}\}, ..., \\ w_A^{k-n_B+1}(m) &= \max\{\mu_m^{k-m}, \mu_m^{k-m-1}, ..., \mu_m^{k-n_B+1}\} = w_A^1(m) = ... = w_A^{k-n_B}(m). \end{aligned}$$

[14] Actually, w_A^{k-m-1} must be equal to the highest between w_A^{k-m} and $\min\{\mu_m^{k-m-1}, u_B^{m+2}\}$, but (i) when $\min\{\mu_m^{k-m-1}, u_B^{m+2}\} = \mu_m^{k-m-1}$, we can write $w_A^{k-m-1} = \max\{\mu_m^{k-m}, \mu_m^{k-m-1}\}$; (ii) when $\min\{\mu_m^{k-m-1}, u_B^{m+2}\} = u_B^{m+2}$, $w_A^{k-m-1} = \max\{\mu_m^{k-m}, \mu_m^{k-m-1}\}$ still holds because u_B^{m+2} is much smaller than u_B^{m+1} and it turns out that this implies that $w_A^{k-m} = \mu_m^{k-m}$ is larger than both u_B^{m+2} and μ_m^{k-m-1}.

When instead A sells $k-m+1$ products, we have:

$$\begin{aligned} w_A^{k-m+1}(m-1) &= \mu_{m-1}^{k-m+1}, \quad w_A^{k-m}(m-1) = \max\{\mu_{m-1}^{k-m+1}, \mu_{m-1}^{k-m}\}, ..., \\ w_A^{k-n_B+1}(m-1) &= \max\{\mu_{m-1}^{k-m+1}, \mu_{m-1}^{k-m}, ..., \mu_{m-1}^{k-n_B+1}\} = w_A^1(m-1) = ... = w_A^{k-n_B}(m-1). \end{aligned}$$

It is straightforward to see from (4) that $\mu_{m-1}^{k+1-m''} > \mu_m^{k+1-m''}$ for any $m'' \in \{m+1, ..., n_B\}$, thus we have $w_A^{k+1-m''}(m-1) > w_A^{k+1-m''}(m)$ for any $m'' \in \{m+1, ..., k\}$.

The latter inequality is very intuitive: in order to sell one extra unit, (i.e. $k-m+1$ rather than $k-m$ units). A must increase the rent it abandons to D for all the $k-m$ initial units. Thus, when we compare $\pi_A(k-m+1) = \sum_{j=1}^{k-m+1}(u_A^j - w_A^j(m-1))$ with $\pi_A(k-m) = \sum_{j=1}^{k-m}(u_A^j - w_A^j(m))$, we see that $\pi_A(k-m+1)$ contains the additional term $u_A^{k-m+1} - w_A^{k-m+1}(m-1) > 0$, which is A's profit on the $(k-m+1)$-th unit sold, but A's profit on each of his first $k-m$ units is reduced from $u_A^j - w_A^j(m)$ to $u_A^j - w_A^j(m-1)$, as we just proved that $w_A^j(m-1) > w_A^j(m)$ for $j \in \{1, ..., k-m\}$. In words, as it is the case with B, A also faces a *trade off between quantity and rent extraction*: as A sells more units, it should leave more surplus per unit to D. Precisely, as A increases its sales from $k-m$ to $k-m+1$, inducing B to accommodate becomes more difficult for two reasons. First, B's ability to fight is now stronger since he can use his m-th best unit, with value u_B^m, which was previously sold. Second, B has now less to lose by trying to push out a product of A, since selling $m-1$ products makes the profit from accommodation (described just after Lemma 2) smaller than when selling m. Therefore, when A sells one extra unit, in order to induce B not to fight, A should make his units more competitive by leaving D a higher surplus for each unit.

We now present a result which simplifies the task of finding the optimal m. Precisely, we prove a concavity-like property of π_A which states that the marginal profit for A from selling one extra unit is decreasing: the profit increase from selling $k-m+2$ products instead of $k-m+1$ is smaller than the profit increase from selling $k-m+1$ products instead of $k-m$.

Proposition 5 *(i) Suppose that it is feasible for A to sell $k-m+2$ units (i.e. $m-2 \geq \hat{m}$). Then $\pi_A(k-m+2) - \pi_A(k-m+1) \leq \pi_A(k-m+1) - \pi_A(k-m)$.*
(ii) The optimal m for A, denoted by m_A^{**}, is characterized as follows: $\pi_A(m_A^{**}) \geq \max\{\pi_A(m_A^{**}-1), \pi_A(m_A^{**}+1)\}$ if $k-n_B+1 \leq m_A^{**} \leq k-\hat{m}-1$, $\pi_A(m_A^{**}) \geq \pi_A(m_A^{**}-1)$ if $m_A^{**} = k-\hat{m}$, $\pi_A(m_A^{**}) \geq \pi_A(m_A^{**}+1)$ if $m_A^{**} = k-n_B$.

Notice that the concavity-like property of π_A described in Proposition 5(i) implies immediately Proposition 5(ii): in order to test the optimality of m_A^{**}, it suffices to compare the profit as the number of products to sell for A is decreased by one unit or increased by one unit. In what follows, to give further insight, we study some specific settings.

6.3.3 When only the local incentive constraint ($\text{IC}_{m,m+1}$) matters

Let us present first the simple case in which only the local incentive constraint ($\text{IC}_{m,m+1}$) matters. We saw that when A wants to sell $k-m$ units, downward incentive constraint are trivially satisfied but satisfying upward constraints requires A to abandon some surplus to D. We below present a special case in which satisfying only ($\text{IC}_{m,m+1}$) is sufficient to satisfy (3), and this makes it straightforward to derive $\pi_A(k-m)$.

Corollary 1 *Given m such that $\hat{m} \le m \le n_B - 2$, if $u_B^{m+2} \le \frac{1}{m+1}u_B^{m+1}$ then (5) is equivalent to $u_A^{k-m} > \frac{1}{m+1}u_B^{m+1}$. When this condition is satisfied, (6)-(8) imply $w_A^1 = ... = w_A^{k-m} = \frac{1}{m+1}u_B^{m+1} > 0 = w_A^{k-m+1} = ... = w_A^k$; thus $\pi_A(k-m) = u_A^1 + ... + u_A^{k-m} - \frac{k-m}{m+1}u_B^{m+1}$.*

Precisely, if u_B^{m+2} is sufficiently smaller than u_B^{m+1}, it turns out that $\mu_m^{k-m} \ge \mu_m^{k-m-1} \ge ... \ge \mu_m^{k-n_B+1}$ and then (5) is satisfied if and only if $u_A^{k+1-m''} > \mu_m^{k+1-m''}$ holds for $m'' = m+1$, or equivalently $u_A^{k-m} > \frac{1}{m+1}u_B^{m+1}$. If this condition is satisfied, then the optimal prices for A are such that the products he wants to sell give a constant net profit to D equal to $\frac{1}{m+1}u_B^{m+1}$, the profit satisfying ($\text{IC}_{m,m+1}$) with equality. If the condition $u_B^{m+2} \le \frac{1}{m+1}u_B^{m+1}$ holds *for every* $m \in \{\hat{m}, ..., n_B - 2\}$, then we have

$$\pi_A(k-m+1) - \pi_A(k-m) = u_A^{k-m+1} - \frac{1}{m}u_B^m - (k-m)(\frac{1}{m}u_B^m - \frac{1}{m+1}u_B^{m+1}).$$

Note however that the conditions $\frac{1}{\hat{m}+1}u_B^{\hat{m}+1} \ge u_B^{\hat{m}+2}$, $\frac{1}{\hat{m}+2}u_B^{\hat{m}+2} \ge u_B^{\hat{m}+3}$, ..., $\frac{1}{n_B-1}u_B^{n_B-1} \ge u_B^{n_B}$ are somewhat restrictive, since they imply that the values of B's products decrease quite quickly. This also suggests that in general more than one upward incentive constraints matter, as in the examples below.

6.3.4 Example 1: When $n_B = 3$

Suppose that $n_B = 3$. In order to sell $k-3$ units, A sets

$$p_A^1 = u_A^1, \quad p_A^2 = u_A^2, \quad ..., \quad p_A^{k-3} = u_A^{k-3}, \quad p_A^{k-2} \ge u_A^{k-2}, \quad p_A^{k-1} \ge u_A^{k-1}, \quad p_A^k \ge u_A^k.$$

and then B chooses $p_B^1 = u_B^1$, $p_B^2 = u_B^2$, $p_B^3 = u_B^3$. Hence, $\pi_A(k-3) = u_A^1 + u_A^2 + ... + u_A^{k-3}$. In order to find $\pi_A(k-2)$ we have to consider ($\text{IC}_{2,3}$), which is given by

$$(\text{IC}_{2,3}) \quad w_A^{k-2} \ge \frac{1}{3}u_B^3.$$

Therefore, A chooses

$$p_A^1 = u_A^1 - \frac{1}{3}u_B^3, \quad p_A^2 = u_A^2 - \frac{1}{3}u_B^3, \quad ... \quad , p_A^{k-2} = u_A^{k-2} - \frac{1}{3}u_B^3, \quad p_A^{k-1} \ge u_A^{k-1}, \quad p_A^k \ge u_A^k.$$

which is feasible only if $u_A^{k-2} > \frac{1}{3}u_B^3$. Then, B plays $p_B^1 = u_B^1$, $p_B^2 = u_B^2$. Hence, $\pi_A(k-2) = u_A^1 + u_A^2 + ... + u_A^{k-2} - \frac{k-2}{3}u_B^3$.
In order to find $\pi_A(k-1)$ we need to consider both ($\text{IC}_{1,2}$) and ($\text{IC}_{1,3}$), which are given by:

$$(\text{IC}_{1,2}) \quad w_A^{k-1} \geq \frac{1}{2}u_B^2.$$

$$(\text{IC}_{1,3}) \quad w_A^{k-2} \geq \max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\}.$$

Hence, satisfying the incentive constraints is feasible if $u_A^{k-1} > \frac{1}{2}u_B^2$ and $u_A^{k-2} > \max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\}$. Then, A chooses

$$\begin{aligned} p_A^j &= u_A^j - \max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\} \text{ for } j = 1, ..., k-2; \\ p_A^{k-1} &= u_A^{k-1} - \frac{1}{2}u_B^2, \quad p_A^k \geq u_A^k. \end{aligned}$$

Then $\pi_A(k-1) = u_A^1 + u_A^2 + ... + u_A^{k-2} + u_A^{k-1} - (k-2)\max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\} - \frac{1}{2}u_B^2$.

Finally, A is able to sell k units if and only if $u_A^k > ku_B^1$, and then $\pi_A(k) = u_A^1 + u_A^2 + ... + u_A^{k-2} + u_A^{k-1} + u_A^k - ku_B^1$.

In order to fix the ideas, suppose that $u_B^2 > 2u_B^3$, so that $\max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\} = \frac{1}{2}u_B^2$. Then, from Proposition 5(ii), we see for instance that it is optimal for A to sell $k-2$ products if $\pi_A(k-2) \geq \max\{\pi_A(k-1), \pi_A(k-3)\}$, which is equivalent to $u_A^{k-2} \geq \frac{k-2}{3}u_B^3$ and $u_A^{k-1} \leq \frac{k-2}{2}(u_B^2 - u_B^3) + \frac{1}{2}u_B^2$. The first inequality implies that the gain on the $(k-2)$-th sold by A, $u_A^{k-2} - \frac{1}{3}u_B^3$, is larger than his loss on the $k-3$ units, $\frac{k-3}{3}u_B^3$, with respect to selling them at full prices. The second inequality means that selling the $(k-1)$-th unit yields a profit of $u_A^{k-1} - \frac{1}{2}u_B^2$ but results in a loss of $\frac{k-2}{2}(u_B^2 - u_B^3)$, which is larger than $u_A^{k-1} - \frac{1}{2}u_B^2$.

6.3.5 Example 2: When all B's products have the same value

Suppose that $u_B^1 = u_B^2 = ... = u_B^{n_B} \equiv u_B > 0$. In this case, for $m(= 1, ..., n_B - 1)$ and $m''(= m+1, ..., n_B)$, we find that $\mu_m^{k+1-m''} = \frac{m''-m}{m''}u_B$. Thus $\mu_m^{k+1-m''}$ is increasing in m''. Given m, the profit-maximizing $w_A^1, ..., w_A^{k-m}$, determined by (7)-(8), are

$$\begin{aligned} w_A^{k-m} &= \frac{1}{m+1}u_B, w_A^{k-m-1} = \frac{2}{m+2}u_B, ..., w_m^{k-n_B+2} = \frac{n_B - m - 1}{n_B - 1}u_B, \\ w_m^{k-n_B+1} &= \frac{n_B - m}{n_B}u_B = w_A^1 = ... = w_A^{n_B}. \end{aligned}$$

If $m \geq \hat{m}$, we have that $\pi_A(k-m) = u_A^1 + ... + u_A^{k-m} - [\frac{1}{m+1} + \frac{2}{m+2} + ... + \frac{n_B-m-1}{n_B-1} + \frac{n_B-m}{n_B}(k - n_B + 1)]u_B$.

In order to find the optimal m, we exploit lemma 5. Thus, $m = n_B$ is optimal if $\pi_A(k - n_B) \geq \pi_A(k - n_B + 1)$, i.e. if $\frac{u_A^{k-n_B+1}}{u_B} \leq \frac{k-n_B+1}{n_B}$. Finally, for m between 1 and $n_B - 1$, m is optimal if $\pi(k-m) - \pi(k-m-1) \geq 0$ and $\pi(k-m+1) \leq \pi(k-m)$, i.e.

$$\frac{1}{m} + \frac{1}{m+1} + ... + \frac{1}{n_B - 1} + \frac{k - n_B + 1}{n_B} \geq \frac{u_A^{k-m+1}}{u_B} \qquad \text{and}$$

$$\frac{u_A^{k-m}}{u_B} \geq \frac{1}{m+1} + \frac{1}{m+2} + ... + \frac{1}{n_B - 1} + \frac{k - n_B + 1}{n_B}$$

6.4 Inefficiency of individual sale

Under individual sale, there is no particular reason for competition to lead to the efficient outcome. The previous analysis shows that each firm faces a trade-off between quantity and rent extraction. In our sequential game, the profile of products effectively consumed by D is determined by the first mover, firm A. However, there is no reason that the trade-off for firm A induces him to sell m_A^* number of products. Although we completely characterized each firm's strategy in equilibrium, having a general characterization of the condition under which competition under individual sale leads to the efficient outcome is messy since it depends on the vectors $(u_A^1, ..., u_A^{n_A})$ and $(u_B^1, ..., u_B^{n_B})$. In order to provide the intuition for why the competition under individual sale does not necessarily leads to the efficient allocation, we below provide two simple examples. Let m_i^{**} denote the number of products sold by firm i under individual sale. Obviously, we have $m_A^{**} + m_B^{**} = k$. In what follows, we give an example of $m_A^{**} < m_A^*$ and another example of $m_A^{**} > m_A^*$.

Example of $m_A^{**} < m_A^*$

Suppose $n_B = 1$ and $u_A^k > u_B^1$, so that $m_B^* = 0$. We have

$$\pi_A(k-1) = U_A(k-1);$$

$$\pi_A(k) = U_A(k-1) + u_A^k - k u_B^1.$$

Therefore,

$$\pi_A(k-1) - \pi_A(k) > 0 \quad \text{if and only if} \quad u_A^k < k u_B^1.$$

Hence, if $u_A^k < k u_B^1$, we have $m_A^{**} = k - 1 < m_A^* = k$. The intuition for this result is simple. If A sells only $k - 1 = k - n_B$ products, he can extract full surplus from his $k - 1$ best products since B will not fight. However, if A chooses to sell all k products, it has to leave D a net surplus equal to u_B^1 for each of his product in order to block out B's product. This trade-off between quantity and rent extraction makes A sell only his $k - 1$ best products when u_B^1 is not too smaller than u_A^k.

Example of $m_A^{**} > m_A^*$

Consider the setting of example 2 in section 3: we have $u_A^1 > ... > u_A^{m_A^*} > u_B > u_A^{m_A^*+1} > ... > u_A^k$, $u_B^1 = ... = u_B^{m_B^*} = u_B$. Suppose $m_B^* = n_B$. We have:

$$\pi_A(k - m_B^*) = U_A(k - m_B^*);$$

$$\pi_A(k - m_B^* + 1) = U_A(k - m_B^*) + u_A^{m_A^*+1} - (k - m_B^* + 1)\frac{u_B}{m_B^* + 1}.$$

Therefore,

$$\pi_A(k - m_B^* + 1) - \pi_A(k - m_B^*) > 0 \quad \text{iff} \quad u_A^{m_A^*+1} > \frac{(k - m_B^* + 1)}{m_B^* + 1} u_B.$$

For instance, if $u_A^{m_A^*+1}$ is close to u_B^1, we have $\pi_A(k-m_B^*+1)-\pi_A(k-m_B^*) > 0$ if $m_B^* > k/2$.

To sharpen the intuition, suppose $m_A^* = 0$, $m_B^* = k \geq 2$, $u_A^1 \simeq u_B$ Then, $\pi_A(k-m_B^*) = 0$. Hence, A has to sell at least one product to generate a profit. Suppose that A charges $p_A^1 = \varepsilon(> 0)$ very small and very high prices on the other products. If B accommodates A's product, B's profit is $(k - 1)u_B$. Instead, if B blocks A's product out, B's profit is $ku_B - k(u_A^1 - \varepsilon) \simeq k\varepsilon$. Therefore, B prefers accommodation and hence A can sell his inferior product. This example is symmetric to the previous example: A takes advantage of B's trade-off between quantity and rent extraction in order to sell his inferior product.

Proposition 6 *Consider the sequential pricing game without bundling. Then, the outcome is not necessarily efficient.*

7 Robustness: Sequential pricing with bundling

In this section, we study the sequential pricing game with bundling. We here consider pure but endogenous bundling with the tie-breaking rule that each firm does not include into its bundle any product that does not strictly increase its profit. We analyze both free disposal and exclusive slots and show that the equilibrium is unique and efficient in both cases.

7.1 Free disposal

Consider first the case of free disposal. At stage one firm A chooses q_A of his products to include into his bundle B_A, and a price P_A for B_A.

At stage two, after observing the move of A, B chooses q_B of his products to include into his bundle B_B and a price P_B for B_B. In order to specify the value of a bundle for D, it is not enough to specify the number of units it contains, but it is necessary to know the precise products in the bundle. However, it is intuitive that if i inserts q_i of his products in M_i, it is optimal for him to choose the best q_i products among the ones he can sell. For $i = A, B$, let $U_i(q_i)$ denote the gross value of M_i for D if it includes the q_i best products of i: $U_i(q_i) = \sum_{j=1}^{q_i} u_i^j$. Let $U_{AB}(q_{A,}q_B)$ denote D's gross profit from buying both bundles, taking into account the capacity constraint of D; thus $U_{AB}(q_A, q_B) \leq U_A(q_A) + U_B(q_B)$, with equality if and only if $q_A + q_B \leq k$. Therefore, the net profit of D from buying only M_i is $U_i(q_i) - P_i$, while D's profit from buying B_A and B_B is $U_{AB}(q_A, q_B) - P_A - P_B$.

As we did for the game with individual sales, we apply backwards induction starting with stage three. Clearly, D determines his purchase by maximizing his own payoff. About tie-breaking, we assume that D buys both bundles if $U_{AB}(q_A, q_B) - P_A - P_B \geq \max\{U_A(q_A) - P_A, U_B(q_B) - P_B\}$, while he buys B_B if $U_B(q_B) - P_B = U_A(q_A) - P_A > U_{AB}(q_A, q_B) - P_A - P_B$ (this is consistent with T1).

At stage two, given (q_A, P_A), B wants to choose q_B and (the maximal) P_B such that D decides to buy B_B. In order to achieve this objective, B can choose between two strategies as under individual sale: accommodation and fight. B can try to induce D to buy both bundles or try to induce D to buy only B_B (and block out B_A). Recall from section 2 that m_i^* is the number of firm i's products of among the k best products overall, thus $m_A^* + m_B^* = k$. Before starting the analysis, it is useful to introduce the function

$$\bar{q}(q_A) = \begin{cases} \min\{n_B, k - q_A\} & \text{if } q_A < m_A^* \\ m_B^* & \text{if } q_A \geq m_A^* \end{cases}$$

The interpretation of $\bar{q}(q_A)$ is as follows: if D has purchased B_A which includes A's best q_A units, then $\bar{q}(q_A)$ is the maximal number of products of B which D would effectively distribute under the slot constraint in case D buys B_B as well. The next lemma characterizes B's optimal strategy at stage 2.

Lemma 3 *At stage two, given a pair* (q_A, P_A),
(i) B fights by choosing $q_B = n_B$ *and* $P_B = U_B(n_B) - U_A(q_A) + P_A$ *if* $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$;
(ii) B accommodates by choosing $q_B = \bar{q}(q_A)$ *and* $P_B = U_{AB}[q_A, \bar{q}(q_A)] - U_A(q_A)$ *if* $P_A \leq U_{AB}(q_A, n_B) - U_B(n_B)$.

Not surprisingly, B ends up fighting (accommodating) when P_A is large (small). Precisely, in order to fight, B includes all its products into B_B since this decreases the relative value to D of buying both bundles against buying only B_B, and at the same times maximizes the value of B_B. Then it is feasible for B to block B_A out when D's profit from buying only B_A is smaller than D's profit from buying only B_B, which is equivalent to $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$.

Suppose now that B accommodates B_A. Notice that for any q_A, B finds it optimal to induce D to buy and distribute at least his m_B^* best units since $\bar{q}(q_A) \geq m_B^*$. Furthermore, if $q_A < m_A^*$, it is optimal for B to sell more than m_B^* units (if $n_B > m_B^*$) since his profit is equal to $U_{AB}(q_A, q_B) - U_A(q_A)$. However, if $q_A \geq m_A^*$, it is optimal for B to sell only m_B^* units since including more than m_B^* units does not affect his profit. Notice also that as long as $q_A \geq m_A^*$ and B accommodates B_A, D buys and distributes at least m_A^* best products of A.

Corollary 2 *Suppose that B accommodates B_A. Then,*

(i) For any q_A, B can induce D to buy and distribute at least his m_B^* best units; hence A can never induce D to buy and distribute more than his m_A^* best units.

(ii) Suppose $q_A \geq m_A^*$. D always buys and distributes at least the m_A^* best units of A; hence B can never induce D to buy and distribute more than his m_B^* best units.

The next proposition describes the equilibrium under bundling and shows that each firm i chooses $q_i = m_i^*$ along the equilibrium path.

Proposition 7 *In the sequential pricing game with pure endogenous bundling and with free disposal, there exists a unique SPNE and equilibrium strategies are as follows:*
(i) A chooses $q_A = m_A^*$ and $P_A = U_{AB}(m_A^*, n_B) - U_B(n_B)$;
(ii) B plays as described by Lemma 3, and along the equilibrium path chooses $q_B = m_B^*$ and $P_B = U_{AB}(m_A^*, m_B^*) - U_A(m_A^*) = U_B(m_B^*)$.
(iii) D buys both bundles and hence consumes the k best among both firms' products.

Proof. At stage one, firm A will choose (q_A, P_A) in a way which induces B to accommodate (otherwise A makes no profit), hence $P_A = U_{AB}(q_A, n_B) - U_B(n_B)$ and q_A is selected in order to maximize $U_{AB}(q_A, n_B)$. Then it follows that $q_A = m_A^*$ because $q_A < m_A^*$ implies $U_{AB}(q_A, n_B) < U_{AB}(m_A^*, n_B)$, while $q_A > m_A^*$ implies that some units are included in B_B but do not increase the profit of A. Given $q_A = m_A^*$ and $P_A = U_{AB}(m_A^*, n_B) - U_B(n_B)$, B will choose as described by Lemma 3(ii); thus $q_B = \bar{q}(m_A^*) = m_B^*$ and $P_B = U_{AB}(m_A^*, m_B^*) - U_A(m_A^*) = U_B(m_B^*)$. ■

Given B's best response described in Lemma 3, A chooses the largest P_A which induces B to accommodate (i.e. $P_A = U_{AB}(q_A, n_B) - U_B(n_B)$) and sets $q_A = m_A^*$, since $U_{AB}(q_A, n_B)$ increases with q_A up to $q_A = m_A^*$ and then becomes constant. Then, the highest P_B at which B induces D to buy both bundles is $P_B = U_{AB}(m_A^*, q_B) - U_A(m_A^*)$, from Lemma 3(ii), and this is maximized at $q_B = m_B^*$ since $U_{AB}(m_A^*, q_B)$ increases with q_B up to $q_B = m_B^*$ and then becomes constant. Therefore, D ends up consuming the k best among both firms' products.[15]

7.2 Exclusive slots

The next proposition shows that the equilibrium outcome under bundling is the same regardless of whether firms use exclusive slots or not.[16]

Proposition 8 *In the sequential pricing game with pure endogenous bundling and with exclusive slots, there exists a unique SPNE and equilibrium strategies are as follows, in which* $\hat{q}_B(q_A) \equiv \min\{k - q_A, n_B\}$*:*
(i) A chooses $q_A = m_A^*$ *and* $P_A = U_{AB}(m_A^*, n_B) - U_B(n_B)$*;*
(ii) Given a pair (q_A, P_A)*, B blocks* B_A *out by playing* $q_B = n_B$ *and* $P_B = U_B(n_B) - U_A(q_A) + P_A$ *when* $P_A > U_A(q_A) + U_B[\hat{q}_B(q_A)] - U_B(n_B)$*; conversely, B accommodates by playing* $q_B = \hat{q}_B(q_A)$ *and* $P_B = U_{AB}[q_A, \hat{q}_B(q_A)] - U_A(q_A) = U_B[\hat{q}_B(q_A)]$ *if* $P_A \leq U_A(q_A) + U_B[\hat{q}_B(q_A)] - U_B(n_B)$*.*
(iii) Along the equilibrium path, (q_A, P_A, q_B, P_B) *are like in the SPNE described in Proposition 7, and thus D still buys both bundles and consumes the k best products among both firms' products.*

In order to provide an intuition for our result, we consider what happens if A includes $m_A^* + 1$ units instead of m_A^* units into his bundle. Remember that without exclusive contracts, this does not affect the set of the products that will be effectively distributed by D, which implies that (i) this does affect B's response and (ii) the price that A charges for the bundle remains the same as well. Hence, under T3, A prefers including only m_A^* units into his bundle.

Consider now exclusive deals. Then, if A includes one more unit into his bundle, this affects B's choice between accommodation and flight since if B accommodates B_A, B can sell only $m_B^* - 1$ units and obtains profit equal to $U_B(m_B^* - 1)$, which is smaller than $U_B(m_B^*)$. If B blocks B_A out, he chooses $q_B = n_B$ and P_B such that

$$U_B(n_B) - P_B \geq U_A(m_A^* + 1) - P_A.$$

This implies that in order to induce B to accommodate B_A, A must choose P_A such that

$$U_B(m_B^* - 1) \geq U_B(n_B) - U_A(m_A^* + 1) + P_A.$$

[15] A higher P_B would make D buy only M_A.

[16] Notice however that B's best response is different from the one described in lemma 3.

Hence, A's profit is $U_B(m_B^* - 1) + U_A(m_A^* + 1) - U_B(n_B)$, which is smaller than his profit when A sells only m_A^* units ($U_B(m_B^*) + U_A(m_A^*) - U_B(n_B)$). It is interesting to notice that the difference in A's profits is exactly equal to

$$u_A^{m_A^*+1} - u_B^{m_B^*} < 0.$$

If A sells $m_A^* + 1$ units through exclusive contracts, it induces D to replace the m_B^*-th best product of B with the $m_A^* + 1$-th best product of A, which is inferior to the former. Hence, A should compensate D for the reduction in D's surplus by reducing its price by $u_B^{m_B^*} - u_A^{m_A^*+1}$. Therefore, A finds optimal to sell only m_A^* units.

8 Incentive to bundle in the sequential game

Since the equilibrium does not exist in the simultaneous pricing game without bundling, we here study the inventive to bundle in the sequential pricing game. We have examined above the two different regimes of no bundling and bundling. Now we inquire which regime will endogenously emerge when each seller can choose between bundling and no bundling. In short, we find that bundling is weakly dominant for firm B and, given that B bundles, also for A it is weakly dominant to practice bundling.

Proposition 9 *(i) If firm B can make a profit by pricing his products independently, then B can make at least the same profit by bundling.*
(ii) Given that B chooses to bundle, if firm A can make a profit by pricing his products independently, then A can make at least the same profit by bundling.

While Proposition 9(i) suggests that B never loses from bundling Proposition 9(ii) establishes the same result given that B bundles, as established by Proposition 9(i).

Thus, bundling emerges endogenously when it is not forbidden. In order to improve our understanding of this fact, it might be useful to examine the benefits of B from bundling when A uses individual sales. In the case in which B also uses individual sales and wants to sell m products (this objective is attainable if and only if $u_B^m > w_A^{k-m+1}$) we know from Lemma 2 that his profit is $u_B^1 + ... + u_B^m - mw_A^{k-m+1}$. On the other hand, we show in the proof of Proposition 9 that he can make $U_B(m) - (w_A^{k-m+1} + w_A^{k-m+2} + ... + w_A^k)$ by bundling. Therefore, with individual sales he leaves D a net profit equal to w_A^{k-m+1} on each unit of the m units he sells,[17] for a combined value of mw_A^{k-m+1}. With bundling, instead, he needs to leave to D the net value of the m worst products of A, $w_A^{k-m+1} + w_A^{k-m+2} + ... + w_A^k$, which

[17] w_A^{k-m+1} is the value for D of the best product of A he does not purchase.

is (weakly) smaller than mw_A^{k-m+1}. The reason for the result is that with individual sales, D can replace each single product of B with the $k-m+1$-th product of A if the product of B yields D a profit smaller than w_A^{k-m+1}. On the other hand, D has less flexibility when B bundles as he and can substitute B_B only with the m worst units of A. This gives an edge to B and allows him to extract a (weakly) higher price from D.

9 Implication on Merger

to be written

10 Conclusion

We studied how bundling affects competition for limited slots in a general setting in which each upstream firm owns a portfolio of distinct products. We found that the outcome under bundling is socially efficient in that only the best products occupy the limited slots. We also proved that this results is quite robust; the results holds regardless of whether we consider a simultaneous or a sequential game, regardless of the order of the players if we consider a sequential game, regardless of whether or not we allow firms to sign exclusive contracts on the use of slots.

On the contrary, we showed that under individual sale, the outcome is not necessarily efficient. Under simultaneous game, there is no equilibrium in pure strategy. Under sequential game, the number of products that each upstream firm sells is determined by a trade-off between quantity and rent extraction such that there is no particular reason to expect that this number coincides with the socially efficient one.

This unambiguous welfare-enhancing result under competing portfolios is quite novel and has strong policy implications which go beyond the rule of reason standard based on the existing literature on bundling.

11 Appendix

11.1 Proof of Theorem 1

We first prove that the profile of schedules in which $P_i(q_i) = P_i(q_i-1)+\max\{0, u_i^{q_i} - u_{-i}^{k+1-q_i}\}$ for all i is a NE. In order to see why, consider D's purchases from firm 1. Whatever bundles D purchases from firms 2, 3, ..., n, D maximizes his profit by buying $B_1(q_1^{fb})$ from 1. Indeed, in this way D's profit increases by $u_{-1}^{k-q_1^{fb}+1} + ... + u_{-1}^k$ minus the units of firms different

from 1 with value between $u_{-1}^{k-q_1^{fb}+1}$ and u_{-1}^{k} which are included in the bundles D buys from 2,3,...,n. This difference is zero or positive, but in the case of zero the tie breaking rule which favors choosing products with the highest gross values applies to make D willing to buy $B_1(q_1^{fb})$. Actually, however, D achieves the same result by purchasing $B_1(q_1)$ with $q_1 > q_1^{fb}$.
Now consider the possibility of deviation for firm 1. The argument given above suggests that D will buy $B_2(q_2) \cup ... \cup B_n(q_n)$ (with $q_2 \geq q_2^{fb}$, ..., $q_n \geq q_n^{fb}$) from firms 2,3, ..., n. Then it is impossible for 1 to induce D to consume more than q_1^{fb} units of 1, and the maximal revenue he can make by selling q_1^{fb} units to 1 is P_1^e: if 1 chooses $P_1(q_1^{fb}) > P_1^e$ then $B_2(k) \cup ... \cup B_n(k)$ gives D a higher payoff than buying $B_1(q_1^{fb})$.

Now we prove that in any other NE D buys $B_i(q_i^*)$ with $q_i^* \geq q_i^{fb}$ and $P_i(q_i^*) = P_i^e$ for all i.

Step 1: $q_i^* \geq q_i^{fb}$ for $i = 1, ..., n$

Suppose that there is a NE in which D $q_1^* < q_1^{fb}$, and firm 1 makes profit $P_1(q_1^*)$. We can prove that the following is a profitable deviation for firm 1: 1 charges a high price for $B_1(q_1)$, for $q_1 \neq q_1^{fb}$, and price $P_1(q_1^*) + \varepsilon$ (with $\varepsilon > 0$ and small) for $B_1(q_1^{fb})$, that a price slightly higher than 1's profit in the supposed NE. We now show that D buys $B_1(q_1^{fb})$, which makes 1's profit higher than $P_1(q_1^*)$. Notice that buying $B_1(q_1^{fb}) \cup B_2(q_2^*) \cup ... \cup B_n(q_n^*)$ yields D payoff $U(q_1^{fb}, q_2^*, ..., q_n^*) - P_1(q_1^*) - \varepsilon - \sum_{j=2}^{n} P_j(q_j^*)$, which is larger than $U(q_1^*, q_2^*, ..., q_n^*) - P_1(q_1^*) - \sum_{j=2}^{n} P_j(q_j^*)$, D's payoff in the supposed NE, because $U(q_1^{fb}, q_2^*, ..., q_n^*) > U(q_1^*, q_2^*, ..., q_n^*) + \varepsilon$. If instead D does not buy $B_1(q_1^{fb})$, then it buys only from firms different from 1, and this cannot yield a payoff larger than $U(q_1^*, q_2^*, ..., q_n^*) - P_1(q_1^*) - \sum_{j=2}^{n} P_j(q_j^*)$, otherwise D would not be purchasing optimally in the supposed NE.

Therefore, $q_1^* \geq q_1^{fb}$ in any NE and the same property holds for any other firm.

Step 2: $P_i(q_i^*) \geq P_i^e$ for $i = 1, ..., n$

Suppose that there exists a NE such that $P_1(q_1^*) < P_1^e$ (the case of a firm $i \neq 1$ is analogous). Then we prove that the following is a profitable deviation for firm 1: 1 charges a high price for $B_1(q_1)$, for $q_1 \neq q_1^*$, and price P_1^e for $B_1(q_1^*)$. We now show that D buys $B_1(q_1^*)$, which makes 1's profit higher than $P_1(q_1^*)$. Notice that the payoff from buying $B_2(q_2) \cup ... \cup B_n(q_n)$ (i.e., from buying only bundles of firms different from 1) is $U(0, q_2, ..., q_n) - \sum_{j=2}^{n} P_j(q_j)$, for any $(q_2, ..., q_n)$. When considering whether to buy also $B_1(q_1^*)$ or not, D compares $U(q_1^*, q_2, ..., q_n) - P_1^e - \sum_{j=2}^{n} P_j(q_j)$ with $U(0, q_2, ..., q_n) - \sum_{j=2}^{n} P_j(q_j)$. Then we see that the inequality $U(q_1^*, q_2, ..., q_n) - P_1^e - \sum_{j=2}^{n} P_j(q_j) \geq U(0, q_2, ..., q_n) - \sum_{j=2}^{n} P_j(q_j)$ holds because P_1^e is such that $U(q_1^*, q_2, ..., q_n) - P_1^e = U(0, q_2, ..., q_n)$ if $B_2(q_2) \cup ... \cup B_n(q_n)$ includes the k best units which are owned by the firms different from 1, otherwise $U(q_1^*, q_2, ..., q_n) - P_1^e > U(0, q_2, ..., q_n)$ holds. In the case of equality we conclude that D buys also $B_1(q_1^*)$

because of the tie breaking rule according to which D maximizes the gross value of units he buys in case of indifference.

Step 3: In any NE, $P_i(q_i^*) \leq P_i^e$ for $i = 1, ..., n$

Suppose that there exists a NE such that D buys $B_1(q_1^*) \cup B_2(q_2^*) \cup ... \cup B_n(q_n^*)$ with $q_i^* \geq q_i^{fb}$ and prices $P_i(q_i^*)$ for $i = 1, ..., n$, and that $P_1(q_1^*) = P_1^e + \delta_1$ with $\delta_1 > 0$. We show that there is some firm $i \neq 1$ for which a profitable deviation exists. First we investigate some properties of δ_1, and to this end we let Z denote the set of the units of firms different from 1 with value between $u_{-1}^{k-q_1^*+1}$ and u_{-1}^k which are *not* included in $B_2(q_2^*) \cup ... \cup B_n(q_n^*)$ – we use u_{-1}^Z to represent the aggregate value of these units. Notice that after buying $B_2(q_2^*) \cup ... \cup B_n(q_n^*)$, D's increase in profit from buying $B_1(q_1^*)$ is $u_{-1}^Z - \delta_1$, thus it is necessary that $Z \neq \emptyset$ and $\delta_1 \leq u_{-1}^Z$. However, it is also necessary that D is unable to make a profit higher than the equilibrium profit by buying only from firms different from 1.[18] Thus we introduce $\mu \geq 0$ with the interpretation that the maximum profit D can make by buying only from firms different from 1 is by $\mu \geq 0$ higher[19] than the payoff from buying $B_2(q_2^*) \cup ... \cup B_n(q_n^*)$. Thus, in order for D to buy $B_1(q_1^*)$ it is necessary that $u_{-1}^Z - \delta_1 \geq \mu$, that is $u_{-1}^Z - \mu \geq \delta_1$. But then, if $\delta_1 < u_{-1}^Z - \mu$ it is possible for 1 to increase his profit by increasing $P_1(q_1^*)$ until $P_1^e + u_{-1}^Z - \mu$ because D cannot do better by buying only from other firms; thus $\delta_1 = u_{-1}^Z - \mu$. This equality reveals that $\mu < u_{-1}^Z$, otherwise the condition $\delta_1 > 0$ is violated. In particular, $\mu < u_{-1}^Z$ implies that if D attempts to buy the units in Z, it is necessary to pay to some firm $i \neq 1$ a price P_i higher than P_i^e. Then take precisely a firm $i \neq 1$ with one (to fix the ideas) unit in Z which charges for $B_i(q_i^* + 1)$ a price $P_i > P_i^e$ price. Let it deviate by requiring price $P_i^e + \varepsilon$ ($\varepsilon > 0$ and small) for $B_i(q_i^* + 1)$. It follows that by purchasing only from firms different from 1, D's payoff is larger than μ plus the payoff from buying $B_2(q_2^*) \cup ... \cup B_n(q_n^*)$ because $P_i^e + \varepsilon < P_i$. Thus, D's payoff from buying from firms different from 1 is larger than the NE payoff and D cannot improve his payoff without buying $B_i(q_i^* + 1)$. This establishes the existence of a profitable deviation for i, since i gains $P_i^e + \varepsilon$ instead of P_i^e.

11.2 Proof of Lemma 1

What matters for D's purchases (hence for A's and B's profits) are the vectors $\hat{\mathbf{w}}_A$ and $\hat{\mathbf{w}}_B$. Given $(\hat{\mathbf{w}}_A, \hat{\mathbf{w}}_B)$, suppose that $\mathbf{w}_B \neq \hat{\mathbf{w}}_B$ and let m_B denote the number of products which D purchases from B; this means that D buys from B the products with net profits $w_B^{(1)}, w_B^{(2)}, ..., w_B^{(m_B)}$. Let $u_B^{(j)}$ represent D's gross profit of the product with the net profit

[18] For instance, how much would it cost, in excess to $P_2^e + ... + P_n^e$, to buy the units in E? Say it cost γ. Then it is necessary that $\delta_1 \leq \gamma$.

[19] $\mu \geq u_{-1}^Z - \gamma$.

$w_B^{(j)}$. Then, B's profit is given by

$$\pi_B = \sum_{j=1}^{m_B} \left[u_B^{(j)} - w_B^{(j)} \right].$$

Now suppose that B choose prices $\tilde{p}_B^j = u_B^j - w_B^{(j)}$ for $j = 1, ..., n_B$, and denote by $\tilde{w}_B^j$ the resulting net profits for D. Then the same vector $\hat{\mathbf{w}}_B$ as before is obtained and $\tilde{w}_B^1 = w_B^{(1)} \geq \tilde{w}_B^2 = w_B^{(2)} \geq ... \geq \tilde{w}_B^{n_B} = w_B^{(n_B)}$. Thus, T1 and T2 imply that D will still purchase m_B number of products from B, and now B's profit is

$$\tilde{\pi}_B = \sum_{j=1}^{m_B} (u_B^j - \tilde{w}_B^j)$$

By definition of u_B^j, $\tilde{\pi}_B$ is at least as large as π_B and, in particular, $\tilde{\pi}_B > \pi_B$ if $\sum_{j=1}^{m_B} u_B^j > \sum_{j=1}^{m_B} u_B^{(j)}$, that is if the products sold initially by B are different from B's m_B products with the highest net profits.

The above argument applies to firm B since it chooses $\mathbf{p}_B$ after observing $\mathbf{p}_A$, and thus can take $\mathbf{w}_A$ as given. Conversely, firm A cannot take $\mathbf{w}_B$ as given and the argument must be slightly augmented as follows. If, given $\mathbf{w}_A$, it is optimal for B to choose prices such that a certain $\mathbf{w}_B$ is obtained, any $\mathbf{p}_A$ which leaves unaltered $\mathbf{w}_A$ leaves unaffected the incentives for firm B, and also his best reply prices. This allows to argue as above for B: in case that $\mathbf{w}_A \neq \hat{\mathbf{w}}_A$, let A choose $\tilde{p}_A^j = u_A^j - w_A^{(j)}$ for $j = 1, ..., k$ so that $\tilde{w}_B^j = w_B^{(j)}$ for $j = 1, ..., k$ and the same vector $\hat{\mathbf{w}}_A$ as before is obtained. Then, with respect to the initial situation, (i) B will not change his reply; (ii) D will still buy m_A products of A; (iii) A's profit will not decrease.

11.3 Proof of Proposition 4

Proof of (i)a There exists $\mathbf{w}_A$ such that D will buy $k - m$ units from A if and only if there exists $\mathbf{w}_A$ which satisfies (1), (2) and (3). Thus, since it is more likely that (1) is satisfied the smaller are $w_A^1, ..., w_A^k$, in order to prove (i)a we first find the smallest values of $w_A^1, ..., w_A^k$ which satisfy (2) and (3), and then we show that these values satisfy (1) if and only if (5) holds.

By Lemma 2, there exists $\mathbf{p}_B$ such that D buys m' units from B if and only if $u_B^{m'} > w_A^{k-m'+1}$. In particular, it is feasible for B to sell $m \in \{1, ..., n_B - 1\}$ units if and only if $u_B^m > w_A^{k-m+1}$. If firm A chooses w_A^{k-m+1} such that $w_A^{k-m+1} \geq u_B^m$, then it would actually sell at least $k - m + 1$ units; thus it must be the case that $u_B^m > w_A^{k-m+1}$. This inequality implies $u_B^{m'} > w_A^{k-m'+1}$ for $m' = 1, ..., m-1$. Therefore, for $m' < m$, (IC$_{m,m'}$) is equivalent to

$$\pi_B(m) - \pi_B(m') = u_B^{m'+1} + ... + u_B^m - m \max\left\{w_A^{k-m+1}, 0\right\} + m' \max\left\{w_A^{k-m'+1}, 0\right\} \geq 0. \quad (9)$$

For $m'' > m$, instead, $u_B^m > w_A^{k-m+1}$ does not imply $u_B^{m''} > w_A^{k-m''+1}$. In case that $u_B^{m''} \leq w_A^{k-m''+1}$, we have $\pi_B(m'') = 0$ and then (IC$_{m,m''}$) is trivially satisfied. In case that $u_B^{m''} > w_A^{k-m''+1}$, then (IC$_{m,m''}$) is equivalent to

$$\pi_B(m'') - \pi_B(m) = u_B^{m+1} + ... + u_B^{m''} - m'' \max\left\{w_A^{k-m''+1}, 0\right\} + m \max\left\{w_A^{k-m+1}, 0\right\} \leq 0. \quad (10)$$

Therefore, (3) reduces to (9) for $m' = 1, ..., m-1$, and to $u_B^{m''} \leq w_A^{k-m''+1}$ and/or (10) for $m'' = m+1, ..., n_B$.

We first prove that it is convenient to choose $w_A^{k-m+1} = w_A^{k-m+2} = ... = w_A^k = 0$. For $m'' = m+1, ..., n_B$, the value of w_A^{k-m+1} which most relaxes (10) is $w_A^{k-m+1} = 0$, and this [together with (2)], implies $w_A^{k-m+2} = ... = w_A^k = 0$; these values of $(w_A^{k-m+2}, ..., w_A^k)$ satisfy (9) for any $m' \in \{1, ..., m-1\}$ and do not affect (IC$_{m,m''}$) for $m'' > m$. Thus, with $w_A^{k-m+1} = w_A^{k-m+2} = ... = w_A^k = 0$ we have taken care of (9). We now turn our attention to (10).

Given $w_A^{k-m+1} = 0$, (10) is equivalent to $w_A^{k-m''+1} \geq \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$. In particular, for $m'' = m+1$ we find

$$w_A^{k-m} \geq \frac{1}{m+1} u_B^{m+1} \quad (11)$$

This condition is less restrictive than $w_A^{k-m} \geq u_B^{m+1}$, the other way to satisfy (IC$_{m,m+1}$), and therefore (IC$_{m,m+1}$) is satisfied if and only if (11) holds – notice that the right hand side of (11) is μ_m^{k-m}. For $m'' = m+2$, (IC$_{m,m+2}$) is satisfied if and only if

$$w_A^{k-m-1} \geq \min\left\{\frac{1}{m+2}(u_B^{m+1} + u_B^{m+2}), u_B^{m+2}\right\} \quad (12)$$

and since $u_B^{m+1} \geq u_B^{m+2}$, either one can be the minimum in the right hand side of (12). Likewise, for $m'' = m+3, ..., n_B$, (IC$_{m,m''}$) is satisfied if and only if

$$w_A^{k-m''+1} \geq \min\left\{\frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''}), u_B^{m''}\right\} = \min\{\mu_m^{k-m''+1}, u_B^{m''}\}$$

In general, however, we cannot set $w_A^{k-m''+1} = \min\{\mu_m^{k-m''+1}, u_B^{m''}\}$ for $m'' = m+1, ..., n_B$ because (2) may be violated. The lowest values for $w_A^{k-m}, w_A^{k-m-1}, ..., w_A^{k-n_B+1}$ which satisfy (IC$_{m,m''}$) and (2) are given by $w_A^{k-m''+1} = \max\left\{w_A^{k-m''+2}, \min\{\mu_m^{k-m''+1}, u_B^{m''}\}\right\}$ for $m'' = m+1, ..., n_B$, but we can actually prove that this is equivalent to setting

$w_A^{k-m''+1} = \max\{w_A^{k-m''+2}, \mu_m^{k-m''+1}\}$ for $m'' = m+1, ..., n_B$, or equivalently $w_A^{k-m''+1} = \max\{\mu_m^{k-m}, ..., \mu_m^{k-m''+1}\}$ for $m'' = m+1, ..., n_B$. Precisely, if $\min\{\mu_m^{k-m''+1}, u_B^{m''}\} = u_B^{m''}$ then $\max\left\{w_A^{k-m''+2}, \min\{\mu_m^{k-m''+1}, u_B^{m''}\}\right\} = w_A^{k-m''+2} = \max\{w_A^{k-m''+2}, \mu_m^{k-m''+1}\}$. In order to see this fact, suppose that $\min\{\mu_m^{k-m''+1}, u_B^{m''}\} = u_B^{m''}$ for some $m'' \in \{m+2, m+3, ..., n_B\}$, and that this is the smallest m'' with this property. Then $u_B^{m''} \leq \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$, or equivalently $u_B^{m''} \leq \frac{1}{m''-1}(u_B^{m+1} + ... + u_B^{m''-1}) = \mu_m^{k-m''+2}$. On the other hand, $\min\{\mu_m^{k-m''+2}, u_B^{m''-1}\} = \mu_m^{k-m''+2}$ by definition of m'', thus $w_A^{k-m''+2} \geq \mu_m^{k-m''+2}$ and $w_A^{k-m''+1} = \max\{w_A^{k-m''+2}, u_B^{m''}\} = w_A^{k-m''+2}$. Furthermore $\mu_m^{k-m''+2} \geq \mu_m^{k-m''+1}$ is true because it is equivalent to $u_B^{m''} \leq \mu_m^{k-m''+2}$, which we know to be true. Thus, $w_A^{k-m''+1}$ can be written as $\max\{w_A^{k-m''+2}, \mu_m^{k-m''+1}\}$, both when $\mu_m^{k+1-m''} < u_B^{m''}$ (this is obvious) and when $\mu_m^{k+1-m''} \geq u_B^{m''}$ (as we just proved).

Finally, we observe that no incentive constraint imposes any restriction on $w_A^1, w_A^2, ..., w_A^{k-n_B}$; thus we can pick $w_A^1 = w_A^2 = ... = w_A^{k-n_B} = w_A^{k-n_B+1}$ to satisfy (2).

In this way we have identified the lowest values of $w_A^1, ..., w_A^k$ which satisfy (2) and (3), and they are described by (6)-(8). However, these values are feasible if and only if they satisfy (1). Clearly, the conditions $w_A^j < u_A^j$ for $j \in \{m+1, ..., k\}$ are satisfied given (6). For $j \in \{k-n_B+1, ..., k-m\}$ we have $w_A^j = \max\{w_A^{j+1}, \mu_m^j\}$, and thus $w_A^j < u_A^j$ for $j \in \{k-n_B+1, ..., k-m\}$ if and only if (5) is satisfied. Finally, from $u_A^{k-n_B+1} > w_A^{k-n_B+1}$ it follows that $u_A^j > w_A^j = w_A^{k-n_B+1}$ for $j = 1, ..., k-n_B$. This establishes that A is able to sell $k-m$ units if and only if (5) is satisfied.

Proof of (i)b Now we suppose that (5) is satisfied for a certain $m^* \in \{0, 1, ..., n_B-2\}$, and show that (5) is satisfied also for $m = m^*+1$. If A wants to sell $k-m^*-1$ units, (5) reduces to $u_A^{k+1-m''} > \mu_{m+1}^{k+1-m''}$ for $m'' = m^*+2, ..., n_B$. This condition holds, as long as (5) is satisfied, because it involves a subset of the inequalities which appear in (5) and $\mu_{m+1}^{k+1-m''} < \mu_m^{k+1-m''}$ for $m'' = m^*+2, ..., n_B$.

Proof of (ii) If we assume that (5) is satisfied for a certain m, then it is straightforward to see that the values of $w_A^1, ..., w_A^k$ determined by (6)-(8) maximize the profit of A. Indeed, (6)-(8) identify the smallest values of $w_A^1, ..., w_A^k$ which satisfy (2) and (3), and $\pi_A(k-m)$ is decreasing in $w_A^1, ..., w_A^k$.

11.4 Proof of Proposition 5

Since $\pi_A(k-m)=\sum_{j=1}^{k-m}[u_A^j - w_A^j(m)]$, we find

$$\begin{aligned}
\pi_A(k-m+1)-\pi_A(k-m) &= u_A^{k-m+1} - w_A^{k-m+1}(m-1) - \sum_{j=1}^{k-m}[w^j_{A(m-1)} - w_A^j(m)] \\
&= u_A^{k-m+1} - w_A^{k-m+1}(m-1) - [w_A^{k-m}(m-1) - w_A^{k-m}(m)] \\
&\quad -[w_A^{k-m-1}(m-1) - w_A^{k-m-1}(m)] - ... - [w_A^1(m-1) - w_A^1(m)]
\end{aligned}$$

and

$$\begin{aligned}
\pi_A(k-m+2)-\pi_A(k-m+1) &= u_A^{k-m+2} - w_A^{k-m+2}(m-2) - \sum_{j=1}^{k-m+1}[w_A^j(m-2) - w_A^j(m-1)] \\
&= u_A^{k-m+2} - w_A^{k-m+2}(m-2) - [w_A^{k-m+1}(m-2) - w_A^{k-m+1}(m-1)] \\
&\quad -[w_A^{k-m}(m-2) - w_A^{k-m}(m-1)] \\
&\quad -[w_A^{k-m-1}(m-2) - w_A^{k-m-1}(m-1)] - ... \\
&\quad -[w_A^1(m-2) - w_A^1(m-1)]
\end{aligned}$$

In order to prove that $\pi_A(k-m+2)-\pi_A(k-m+1) \leq \pi_A(k-m+1)-\pi_A(k-m)$ it suffices to show that

$$w_A^{k-m+1}(m-1)+[w_A^{k-m}(m-1)-w_A^{k-m}(m)]+[w_A^{k-m-1}(m-1)-w_A^{k-m-1}(m)]+...+[w_A^1(m-1)-w_A^1(m)]$$

is smaller (or equal) than

$$\begin{aligned}
&w_A^{k-m+2}(m-2) + [w_A^{k-m+1}(m-2) - w_A^{k-m+1}(m-1)] + [w_A^{k-m}(m-2) - w_A^{k-m}(m-1)] \\
&+[w_A^{k-m-1}(m-2) - w_A^{k-m-1}(m-1)] + ... + [w_A^1(m-2) - w_A^1(m-1)]
\end{aligned}$$

since $u_A^{k-m+2} \leq u_A^{k-m+1}$. In order to accomplish this task, we first prove that

$$w_A^{k-m+1}(m-1) \leq w_A^{k-m+2}(m-2) + w_A^{k-m+1}(m-2) - w_A^{k-m+1}(m-1) \tag{13}$$

and then we show that

$$w_A^{k+1-m''}(m-1) - w_A^{k+1-m''}(m) \leq w_A^{k+1-m''}(m-2) - w_A^{k+1-m''}(m-1) \tag{14}$$

for $m''=m+1,...,k$.

We find from (4) and (7) that $w_A^{k+1-m}(m-1)=\frac{1}{m}u_B^m$, $w_A^{k+2-m}(m-2)=\frac{1}{m-1}u_B^{m-1}$ and $w_A^{k+1-m}(m-2)=\max\{\frac{1}{m-1}u_B^{m-1}, \frac{1}{m}(u_B^{m-1}+u_B^m)\}$. Thus (13) is equivalent to $\frac{2}{m}u_B^m \leq$

$\frac{1}{m-1}u_B^{m-1} + \max\{\frac{1}{m-1}u_B^{m-1}, \frac{1}{m}(u_B^{m-1} + u_B^m)\}$, and it is easy to see that this inequality holds for either value of $\max\{\frac{1}{m-1}u_B^{m-1}, \frac{1}{m}(u_B^{m-1} + u_B^m)\}$.

About (14), we start by observing that if the inequalities $\mu_m^{k-m} \leq \mu_m^{k-m-1} \leq ... \leq \mu_m^{k-n_B+1}$ hold, then $w_A^{k+1-m''}(m) = \mu_m^{k+1-m''}$ for $m'' = m+1, ..., n_B$. In the opposite case, $\mu_m^{k+1-m''} > \mu_m^{k+1-(m''+1)}$ for some m'' between $m+1$ and $n_B - 1$ and we use $m''(m)$ to denote the smallest m'' for which this inequality holds;[20] notice that by using (4) we find that $\mu_m^{k+1-m''} > \mu_m^{k+1-(m''+1)}$ is equivalent to $\mu_m^{k+1-m''} = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''}) > u_B^{m''+1}$. Then it turns out that $\mu_m^{k+1-m''} > \mu_m^{k+1-(m''+1)}$ for $m'' = m''(m)+1, ..., n_B - 1$,[21] and thus $w_A^{k+1-m''}(m) = \mu_m^{k+1-m''}$ for $m'' = m+1, ..., m''(m)$, and $w_A^{k+1-m''}(m)$ is constantly equal to $\mu_m^{k+1-m''(m)}$ for $m'' = m''(m)+1, ..., n_B$.
Likewise, $\mu_{m-1}^{k+1-m''} > \mu_{m-1}^{k+1-m''-1}$ if and only if $\mu_{m-1}^{k+1-m''} = \frac{1}{m''}(u_B^m + u_B^{m+1} + ... + u_B^{m''}) > u_B^{m''+1}$, and we let $m''(m-1)$ denote the smallest m'' between m and $n_B - 1$ for which this inequality holds. Notice that $m''(m-1) \leq m''(m)$ because $\mu_{m-1}^{k+1-m''} - \mu_m^{k+1-m''} = \frac{1}{m''}u_B^m > 0$. Finally, $\mu_{m-2}^{k+1-m''} > \mu_{m-2}^{k+1-m''-1}$ if and only if $\mu_{m-2}^{k+1-m''} = \frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''}) > u_B^{m''+1}$, and we let $m''(m-2)$ denote the smallest m'' between $m-1$ and n_B for which this inequality is satisfied; we have $m''(m-2) \leq m''(m-1)$ because $\mu_{m-2}^{k+1-m''} - \mu_{m-1}^{k+1-m''} = \frac{1}{m''}u_B^{m-1} > 0$. Thus, as m'' goes from $m+1$ to n_B, $w_A^{k+1-m''}(m-2)$ may become constant at some point, but not later than $w_A^{k+1-m''}(m-1)$, which in turn will not become constant (if it will) later than $w_A^{k+1-m''}(m)$.
Now we prove that (14), or equivalently

$$2w_A^{k+1-m''}(m-1) \leq w_A^{k+1-m''}(m-2) + w_A^{k+1-m''}(m), \tag{15}$$

is satisfied for $m'' = m+1, ..., k$.
Step 1 The case of $m+1 \leq m'' < m''(m-2)$. Then $w_A^{k+1-m''}(m-2) = \frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''})$, $w_A^{k+1-m''}(m-1) = \frac{1}{m''}(u_B^m + u_B^{m+1} + ... + u_B^{m''})$ and $w_A^{k+1-m''}(m) = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$. As a consequence, (15) reduces to $2u_B^m \leq u_B^{m-1} + u_B^m$, which is satisfied.
Step 2 The case of $m''(m-2) \leq m'' < m''(m-1)$. Then $w_A^{k+1-m''}(m-2) = \frac{1}{m''(m-2)}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''(m-2)}) > \frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''})$, $w_A^{k+1-m''}(m-1) = \frac{1}{m''}(u_B^m + u_B^{m+1} + ... + u_B^{m''})$ and $w_A^{k+1-m''}(m) = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$. We know that (15) would hold if $w_A^{k+1-m''}(m-2)$ were equal to $\frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''})$, thus (15) *a fortiori* holds since $w_A^{k+1-m''}(m-2) > \frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''})$.

[20] If $\mu_m^{k-m} \leq \mu_m^{k-m-1} \leq ... \leq \mu_m^{k-n_B+1}$, then we set $m''(m) = n_B$. A similar remark applies to $m''(m-1)$ and $m''(m-2)$ defined below.

[21] We know that $\mu_m^{k+1-m''} > \mu_m^{k+1-(m''+1)}$ is equivalent to $u_B^{m+1} + ... + u_B^{m''} > m''u_B^{m''+1}$, and when this inequality is satisfied at $m'' = m''(m)$ we find that it is satisfied also at $m'' = m''(m)+1$ since $u_B^{m''+1} \geq u_B^{m''+2}$.

Step 3 The case of $m''(m-1) \leq m'' < m''(m)$. Then $w_A^{k+1-m''}(m-2) = \frac{1}{m''(m-2)}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''(m-2)})$, $w_A^{k+1-m''}(m-1) = \frac{1}{m''(m-1)}(u_B^m + u_B^{m+1} + ... + u_B^{m''(m-1)}) > \frac{1}{m''}(u_B^m + u_B^{m+1} + ... + u_B^{m''})$ and $w_A^{k+1-m''}(m) = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$. We know from step 2 that (15) holds at $m'' = m''(m-1) - 1$. As m'' increases to $m''(m-1)$, and beyond, $w_A^{k+1-m''}(m-1)$ and $w_A^{k+1-m''}(m-2)$ remain constant while $w_A^{k+1-m''}(m)$ increases. Thus (15) is still satisfied.

Step 4 The case of $m''(m) \leq m'' \leq n_B$. Then $w_A^{k+1-m''}(m-2) = \frac{1}{m''(m-2)}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''(m-2)})$, $w_A^{k+1-m''}(m-1) = \frac{1}{m''(m-1)}(u_B^m + u_B^{m+1} + ... + u_B^{m''(m-1)})$ and $w_A^{k+1-m''}(m) = \frac{1}{m''(m)}(u_B^{m+1} + ... + u_B^{m''(m)})$. We know from step 3 that (15) holds at $m'' = m''(m) - 1$. As m'' increases to $m''(m)$, and beyond, we have that $w_A^{k+1-m''}(m-1)$, $w_A^{k+1-m''}(m-2)$ and $w_A^{k+1-m''}(m)$ all remain constant; thus (15) still holds.[22]

Step 5 The case of $m'' = n_B + 1, ..., k$. From (8) we see that in this case (15) is reduced to $2w_A^{k+1-n_B}(m-1) \leq w_A^{k+1-n_B}(m-2) + w_A^{k+1-n_B}(m)$, and we have proved in step 4 that this inequality is satisfied.

11.5 Proof of Corollary 1

We know from Proposition 4 that $w_A^{k-m} = \frac{1}{m+1}u_B^{m+1}$ and that $w_A^{k+1-m''} = \max\{\frac{1}{m+1}u_B^{m+1}, \frac{1}{m+2}(u_B^{m+1} + u_B^{m+2}), ..., \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})\}$ (recall that $\mu_m^{k+1-m''} = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$) for $m'' = m+2, ..., n_B$. Given that $\frac{1}{m+1}u_B^{m+1} \geq u_B^{m+2}$, we infer that $\frac{1}{m+1}u_B^{m+1} \geq u_B^{m+3} \geq ... \geq u_B^{m''}$. This implies that $\frac{1}{m''}(u_B^{m+1}+...+u_B^{m''}) \leq \frac{1}{m''}[u_B^{m+1}+\frac{1}{m+1}u_B^{m+1}(m''-m-1)] = \frac{1}{m+1}u_B^{m+1}$. Thus, $\max\{\frac{1}{m+1}u_B^{m+1}, \frac{1}{m+2}(u_B^{m+1} + u_B^{m+2}), ..., \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})\} = \frac{1}{m+1}u_B^{m+1}$ and $w_A^{k+1-m''} = \frac{1}{m+1}u_B^{m+1}$ for $m'' = m+2, ..., n_B$.

11.6 Proof of Lemma 3

For B, it is possible to block B_A out if and only if the following inequalities are satisfied:

$$U_B(q_B) - P_B > U_{AB}(q_A, q_B) - P_A - P_B \quad (16)$$

$$U_B(q_B) - P_B \geq U_A(q_A) - P_A \quad (17)$$

Given (q_A, P_A), in order to relax (16) it is the best for B to choose $q_B = n_B$, as the left hand side of (16) increases (weakly) more quickly with respect to q_B than the right hand side; at $q_B = n_B$, (16) reduces to $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$. Furthermore, the highest P_B

[22] By invoking very similar argument to the ones used in steps 1-4 we can deal with the case in which $m''(m-2) = m''(m-1)$, or $m''(m-1) = m''(m)$, or $m''(m-2) = m''(m-1) = m''(m)$. We skip the details for the sake of brevity.

consistent with (17) is $U_B(q_B) - U_A(q_A) + P_A$, and this is maximized at $q_B = n_B$. Therefore, blocking out B_A is feasible if and only if $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$, and in such a case it yields B a payoff of $U_B(n_B) - U_A(q_A) + P_A$.

Conversely, for B to accommodate B_A, it is necessary and sufficient that (q_B, P_B) satisfies $U_{AB}(q_A, q_B) - P_A - P_B \geq \max\{U_A(q_A) - P_A, U_B(q_B) - P_B\}$, or equivalently

$$P_B \leq U_{AB}(q_A, q_B) - U_A(q_A) \quad (18)$$

$$P_A \leq U_{AB}(q_A, q_B) - U_B(q_B) \quad (19)$$

Hence, by accommodating B_A, B can realize a profit of $P_B = U_{AB}(q_A, q_B) - U_A(q_A)$

Suppose now that $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$, so that it is feasible for B to block B_A out. Then it is easy to see that $U_B(n_B) - U_A(q_A) + P_A > U_{AB}(q_A, q_B) - U_A(q_A)$, which means that B gets a strictly larger profit by pushing B_A out than by accommodating. Therefore, B will block M_A whenever this it feasible, i.e. when $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$; B will instead accommodate when $P_A \leq U_{AB}(q_A, n_B) - U_B(n_B)$. In the latter case, it is profitable for B to choose $q_B = \bar{q}(q_A)$ because this maximizes $U_{AB}(q_A, q_B) - U_A(q_A)$. We have thus identified B's strategy in any SPNE.

11.7 Proof of Proposition 8

As we know from the proof of Lemma 3, for B it is possible to push B_A out if and only if (16) and (17) are satisfied. Again, the highest P_B which satisfies (17) is $P_B = U_B(q_B) - U_A(q_A) + P_A$, and it is maximized with respect to q_B at $q_B = n_B$. About (16), it is still true that it is most relaxed when $q_B = n_B$, but it is important to notice that D cannot buy both B_A and B_B when $q_A + q_B > k$, since he cannot distribute all objects in both bundles. Thus, (16) reduces to $P_A > U_{AB}(q_A, n_B) - U_B(n_B) = U_A(q_A)$ when $q_A \leq k - n_B$ and becomes irrelevant when $q_A > k - n_B$.[23] In the latter case, B can block B_A out if and only if $P_A > U_A(q_A) - U_B(n_B)$, otherwise $P_B = U_B(n_B) - U_A(q_A) + P_A \leq 0$.

If instead B wants to accommodate, then (q_B, P_B) needs to satisfy (18)-(19) and $q_B \leq k - q_A$, otherwise D cannot buy $B_A \& B_B$; thus the right hand side of (19) is equal to $U_A(q_A)$. In order to maximize the right hand side of (18), B chooses $\hat{q}_B(q_A) = \min\{k - q_A, n_B\}$ and earns profit $U_B(\hat{q}_B)$, as long as $P_A \leq U_A(q_A)$. Therefore, in the case of $q_A > k - n_B$ and $U_A(q_A) - U_B(n_B) < P_A \leq U_A(q_A)$, B can choose between accommodating and pushing out B_A. By comparing the respective profits $U_B(k - q_A)$ and $U_B(n_B) - U_A(q_A) + P_A$, we see that B pushes B_A out when $P_A > U_A(q_A) + U_B(k - q_A) - U_B(n_B)$.

[23] In other words, when $q_A > k - n_B$, by choosing $q_B = n_B$ firm B reduces the possible choices of D to buying only M_A or buying only M_B.

By using these results, at stage one A infers that he can earn $U_A(q_A)$ by choosing $q_A \le k - n_B$, while he can make $U_A(q_A) + U_B(k - q_A) - U_B(n_B)$ if $q_A > k - n_B$. Since $k \ge n_B$, we have that $m_A^* \ge k - n_B$ and thus 1's profit is maximized at $q_A = m_A^*$.

11.8 Proof of Proposition 9

(i) Suppose that B does not bundle, and let S_B the set of products D buys from B, with profit $\sum_{j \in S_B} p_B^j$. Then, let B offer the bundle composed of the products in S_B, at the price $\sum_{j \in S_B} p_B^j$. With respect to the previous setting, D has now less flexibility in his purchases since he cannot buy only a few products in S_B. However, he can still buy the same products he was buying previously, and at the same aggregate price. Thus D will buy the same products of A as before, and the bundle of B. This means that, by bundling suitably a set of products, B can make at least the same profit as with individual sales.
(ii) Suppose that A does not practice bundling. Then, at stage two, B chooses q_B and P_B after observing $w_A^1, ..., w_A^k$. We start by finding B's best reply. First notice that q_B needs to satisfy $u_B^{q_B} \ge w_A^{k-q_B+1}$, otherwise D will not distribute all products included in B_B. Then, in order to determine the optimal P_B, we have that D's payoff is $U_B(q_B) - (w_A^{k-q_B+1} + w_A^{k-q_B+2} + ... + w_A^k)$ if he buys B_B, while it is $w_A^1 + ... + w_A^k$ otherwise. Thus, the optimal P_B is $U_B(q_B) - (w_A^{k-q_B+1} + w_A^{k-q_B+2} + ... + w_A^k)$, and the optimal q_B is the largest value which satisfies such that $u^{q_B} \ge w_A^{k-q_B+1}$.

In the case that A wants to sell $k-m$ units, he needs to choose $w_A^1 = ... = w_A^{k-m} = u_B^{m+1}$ (that requires $u_A^{k-m} > u_B^{m+1}$, or equivalently $k-m \le m_A^*$) and, for instance, $w_A^{k-m+1} = ... = w_A^k = 0$. The profit A can make by selling $k-m$ products with no bundling is therefore $u_A^1 + ... + u_A^{k-m} - (k-m)u_B^{m+1}$. In the case that A chooses to bundle, and still wants to sell $k - m \le m_A^*$ units, we know from Lemma 3(ii) that the highest profit he can make is $U_{AB}(k-m, n_B) - U_B(n_B)$, which is equal to $u_A^1 + ... + u_A^{k-m} - (u_B^{m+1} + ... + u_B^{n_B})$, the value of the $k-m$ best units of A minus the value of the $n_B - m$ worst units of B. In order to conclude the proof, we show that $u_A^1 + ... + u_A^{k-m} - (k-m)u_B^{m+1} \le u_A^1 + ... + u_A^{k-m} - (u_B^{m+1} + ... + u_B^{n_B})$. This inequality is equivalent to $u_B^{m+1} + ... + u_B^{n_B} \le (k-m)u_B^{m+1}$, which is satisfied as (i) the number of terms on the left hand side is $n_B - m \le k - m$; (ii) each of the terms on the left hand side is not larger than u_B^{m+1}. In particular, equality holds if and only if $k = n_B$ and $u_B^{m+1} = u_B^{n_B}$.

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