Manipulating an ordering*

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Abstract

It is well known that many social decision procedures are manipulable through strategic behavior. Typically, the decision procedures considered in the literature are social choice correspondences. In this paper we investigate the problem of constructing a social welfare function that is non-manipulable. In this context, individuals attempt to manipulate a social ordering as opposed to a social choice.

Using techniques from an ordinal version of fuzzy set theory, we introduce a class of ordinally fuzzy binary relations of which exact binary relations are a special case. Operating within this family enables us to prove an impossibility theorem. This theorem states that all non-manipulable social welfare functions are dictatorial, provided that they are not constant. This theorem generalizes the one in Perote-Peña and Piggins [Perote-Peña, J., Piggins, A., 2007. Strategy-proof fuzzy aggregation rules. J. Math. Econ. vol 43, p564-p580].

1 Introduction

The fact that many social decision procedures are manipulable through strategic behavior is now well understood in the literature. Typically, the social

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decision procedures considered in the literature have been social choice correspondences. A social choice correspondence selects a nonempty subset of the feasible set of alternatives at each profile of individual preferences. If these chosen subsets contain exactly one element, then we obtain the special case of a social choice function.

In this paper, we investigate the problem of constructing a social welfare function that is non-manipulable. In this context, individuals attempt to manipulate a social ordering as opposed to a social choice.[1]

In order to illustrate the importance of strategic behavior in this setting, consider the following example. Imagine that an academic department wishes to appoint a new member, in order to fill a vacant position. Five candidates have been interviewed for the position and each member of the department has been asked to rank them. In accordance with an established procedure, the chair of the department aggregates these individual rankings using the Borda rule and then uses this aggregate ranking to determine who is to receive the offer of appointment. Imagine that the chair makes the initial offer to the candidate who emerges at the top of this aggregate ranking (ties are broken by employing a device that gives each candidate an equal probability of being selected). If the top-ranked candidate rejects the offer then the chair either offers the position to the other tied candidate(s), or moves down the list to the second-placed candidate. This process continues until someone accepts the position.

When placing the candidates in order, each member of the department has no idea as to which of the candidates would accept the position in the event of it being offered to them. For instance, a candidate might decline the position if he or she has already received a better offer elsewhere. This means that it is not just the candidate(s) at the top of the aggregate ordering that matters to the individual department member, but the entire ordering itself becomes relevant.

Naturally, an individual member of the department could behave strategically in such circumstances and submit an insincere ranking as opposed to a sincere one. This is done in the hope that the aggregate ranking which emerges from such strategic behaviour is "closer" to the member's truthful ranking than would otherwise be the case. Therefore, studying the manipulability of social welfare functions is important in its own right. This is the problem we analyze in this paper.

[1]There has been hardly any treatment of manipulation in the context of social welfare functions. Exceptions are Pattanaik (1973) and Bossert and Storcken (1992).

1.1 Preferences

An important feature of the paper is the assumption we make about preferences. Using techniques from an ordinal version of fuzzy set theory, we introduce a class of ordinally fuzzy binary relations (OFBRs) of which exact binary relations are a special case. We use these OFBRs to represent preferences, both individual and social. The assumption that preferences are represented by OFBRs gives us some mathematical generality. However, it can also be given an independent philosophical motivation. In order to illustrate this, let us return to our original example.

Suppose that a member of our hypothetical department is comparing two possible candidates, and what she cares about is how they fare with respect to teaching and research. Imagine that one of the candidates (candidate A) is better at research than the other (candidate B). To complicate matters, imagine that candidate A is worse at teaching than candidate B. How would our department member place these two candidates in order? Often it is hard to say, but not always.

For instance, imagine that candidate A is much better at research than candidate B and only slightly worse at teaching. In such cases, it seems reasonable to suppose that our department member would place A above B in her ranking. The reason for this is that most members of an academic department would probably be willing to trade-off slightly inferior teaching quality in order to acquire a colleague who is significantly better at research.

Unfortunately, things are not always as straightforward as this. For instance, what if candidate A is much worse at teaching than candidate B? In cases like this, it might be extremely difficult for our department member to place the two candidates in a clear order. She might feel that to some extent candidate A is better than candidate B. At the same time however, she might also feel that to some extent candidate B is better than candidate A. These conflicting feelings may be difficult to integrate into a clear expression of preference or indifference.

In such cases we could perhaps describe our department member's preferences as vague. Fuzzy binary relations are a natural mathematical device for representing preferences like these. To see why, recall that in our example the department member feels that to some extent candidate A is better than candidate B, and yet at the same time he or she also feels that to some extent candidate B is better than candidate A. In the version of fuzzy set theory that we use in this paper, which derives from Goguen (1967), these "extents" are elements of a set L of which there are at least two members. Importantly, in Goguen's theory the elements of L are ordered (possibly incompletely) by a binary relation $\succeq$. A special case of this framework is the standard version of fuzzy set theory. In the standard version L is taken to be $[0, 1]$ and the elements of L are ordered by $\geq$.

An OFBR defined on a set of alternatives X is a function $f : X \times X \to L$. If the semantic concept the fuzzy relation f represents is (weak) preference then $f(x, y)$ can be interpreted as the degree of confidence that "x is at least as good as y". This is not the only possible interpretation of $f(x, y)$. It can be interpreted as the degree of truth of the sentence "x is at least as good as y".

1.2 Outline of the paper

This paper is a contribution to the literature on social choice with fuzzy preferences.[2] This literature has been motivated by the idea that fuzziness can have a "smoothing" effect on preference aggregation and so perhaps the famous impossibility results of Arrow (1951) and others can be avoided. Unfortunately, this is not always the case.[3] In fact, in this paper a very strong concept of dictatorship emerges.

We investigate the structure of social welfare functions which, for every permissible profile of fuzzy individual preferences, specify a fuzzy social preference. We show that all social welfare functions that are non-manipulable and not constant must be dictatorial. This means that there is an individual whose fuzzy preferences determine the entire fuzzy social ranking at every profile in the domain of the social welfare function. To prove this theorem, we show that all social welfare functions that are non-manipulable and not constant must satisfy counterparts of independence of irrelevant alternatives and the Pareto condition.

Of course, this result is a fuzzy version of the Gibbard-Satterthwaite theorem but in the context of social welfare functions.[4] A proof of this theorem first appeared in Perote-Peña and Piggins (2007). However, in that paper L was assumed to be $[0, 1]$. That proof involved an unnecessarily

[2] A comprehensive survey of the growing literature is Salles (1998). The papers that are closest to this one are Barrett, Pattanaik and Salles (1992) and Basu, Deb and Pattanaik (1992). Both of these papers consider "ordinal" approaches to fuzziness. The former paper deals with social choice theory explicitly.

[3] Barrett, Pattanaik and Salles (1986), Dutta (1987), Banerjee (1994), Richardson (1998) and Dasgupta and Deb (1999). Results using lattice theory have been obtained by Leclerc (1984, 1991) and Leclerc and Monjardet (1995).

[4] Gibbard (1973) and Satterthwaite (1975). An important precursor to the present study in the case of exact preferences is Pattanaik (1973). To the best of our knowledge, the only other papers that consider the manipulability problem in a fuzzy framework are Tang (1994), Abdelaziz, Figueira and Meddeb (2008), Côrte-Real (2007) and Perote-Peña and Piggins (2007a, 2007b).

complicated argument involving vectors. This paper contains a new proof of this theorem which is considerably simpler than the original. Some features of the original proof are retained, but the complicated vector argument has been replaced. Finally, at the end of the paper, we consider a way of circumventing our impossibility theorem.

What is responsible for the impossibility theorem? The requirement that the social welfare function should not be constant is mild. It requires that, for each pair of social alternatives, two profiles exist in the domain of the social welfare function which produce different social values in $\{0, 1\}$ for this pair. This is reminiscent of the standard non-imposition axiom in social choice theory. This condition rules out social welfare functions which assign constant values to pairs of alternatives, irrespective of individual preferences.

Given this, one can suspect that the theorem derives from our non-manipulation condition and the assumption we make about the transitivity of fuzzy preferences. Let us deal with non-manipulation first. On first impressions, it does seem desirable for social welfare functions to be immune to preference misrepresentation. Indeed, this normative position seems to be implicitly accepted in the large literature on strategy-proof social choice.[5] If we accept this normative position, then obviously we need some way of prohibiting profitable misrepresentation within the framework of social welfare functions. How do we accomplish this?

The condition we employ can be described as follows. Take any pair of alternatives (a, b) and any fuzzy preference profile in which you truthfully report your preferences. At this profile you are confident to some degree that "a is at least as good as b". However, imagine that at this profile the social welfare function assigns a *larger* social degree of confidence to (a, b) than the one you happen to hold. Then, if the social welfare function is non-manipulable, whenever you misrepresent your preferences the social degree assigned to (a, b) will *either rise, or remain constant*. Conversely, if the social degree assigned to (a, b) is *smaller* than your individual (a, b) value, whenever you misrepresent your preferences the social degree assigned to (a, b) will *either fall, or remain constant*.

Loosely speaking, what this means is as follows. Whenever someone unilaterally switches from telling the truth to lying, the fuzzy social ranking moves "at least as far away" from their truthful ranking as was initially the case. In other words, whenever someone misrepresents their preferences, the "distance" between their truthful ranking and the social ranking (weakly) increases. In such circumstances, individuals do not gain by misrepresenting

[5]A contrary view is expressed by Dowding and van Hees (2006). An introduction to the literature is Barberà (2001).

their preferences. We say that a social welfare function is non-manipulable if and only if it satisfies this property.

Obviously there may be other ways of formulating a non-manipulation condition within the framework of social welfare functions, but this one strikes us as a natural place to start. Weaker conditions are possible, but inevitably they would be more controversial as conditions of non-manipulation

We now deal with transitivity. As with all papers that deal with fuzzy preferences, the way the transitivity condition is formulated is crucial. In this paper, our fuzzy preferences (both individual and social) are assumed to satisfy "max-min" transitivity. This is probably the most widely used transitivity assumption in the literature on fuzzy relations, although it is somewhat controversial.[6] It is certainly the case that weaker transitivity conditions exist. However, we are aware of no experimental evidence which shows that people with fuzzy preferences *do not satisfy* max-min transitivity. In the absence of such evidence, we feel that this condition cannot simply be dismissed out of hand. This fact, combined with its status as the most widely used condition in the literature, means that max-min transitivity is a natural place to start.

With these arguments in mind, it is worth emphasising that our objective in this paper is to establish a baseline impossibility theorem which can serve as a useful benchmark for future work.

2 The model

Environment

Let A be a set of social alternatives with $\#A \geq 3$. Let $N = \{1, ..., n\}$, $n \geq 2$, be a finite set of individuals.

Preferences

A fuzzy binary relation (FBR) over $A \times A$ is a function $f : A \times A \to [0, 1]$. An exact binary relation over $\overline{A}$ is an FBR g such that $g(A \times A) \subseteq \{0, 1\}$.

Let T denote the set of all FBRs over $\overline{A}$.

Let H be the set of all $r \in T$ which satisfy the following two conditions.

(i) For all $(a, b) \in \overline{A}$, $r(a, b) + r(b, a) \geq 1$.

(ii) For all distinct $a, b, c \in A$, $r(a, c) \geq \min \{r(a, b), r(b, c)\}$.

The FBRs in H will be interpreted as fuzzy weak preference relations. Thus if $r_i \in H$ is interpreted as the fuzzy weak preference relation of indi-

[6] Barrett and Pattanaik (1989) and Dasgupta and Deb (1996, 2001).

vidual i, then $r_i(a,b)$ is to be interpreted as the degree to which individual i is confident that "a is at least as good as b".[7]

Property (i) is the fuzzy counterpart of the traditional completeness condition, whereas property (ii) is the familiar max-min transitivity condition. Notice that within H are all (exact) complete and transitive weak preference relations. This is one reason why the fuzzy approach to preferences is popular. It generalizes the traditional theory.

Social welfare function

A social welfare function (SWF) is a function $\Psi : H^n \to H$.

Intuitively, an SWF specifies a fuzzy social weak preference relation given an n-tuple of fuzzy individual weak preference relations (one for each individual). The elements of H^n are indicated by $(r_1, ..., r_n)$, $(r_1^{'}, ..., r_n^{'})$, etc. We write $r = \Psi(r_1, ..., r_n)$, $r^{'} = \Psi(r_1^{'}, ..., r_n^{'})$ and so on (where Ψ is the SWF). We write $r(a,b)$ to denote the restriction of r to (a,b), and $r^{'}(a,b)$ to denote the restriction of $r^{'}$ to (a,b) and so on.

Manipulation

We denote by $(r_1, .., r_i^{'}, .., r_n) \in H^n$ the profile obtained from $(r_1, .., r_i, .., r_n)$ when individual i replaces $r_i \in H$ with $r_i^{'} \in H$.

We write $r_{-i} \otimes r_i^{'} = \Psi(r_1, .., r_i^{'}, .., r_n)$ and $r_{-i} \otimes r_i^{'}\{a,b\}$ denotes the restriction of $r_{-i} \otimes r_i^{'}$ to (a,b). Similarly, $r_{-i-j} \otimes r_i^{'} \otimes r_j^{'} = \Psi(r_1, .., r_i^{'}, .., r_j^{'}, ..r_n)$ and $r_{-i-j} \otimes r_i^{'} \otimes r_j^{'}\{a,b\}$ denotes the restriction of $r_{-i-j} \otimes r_i^{'} \otimes r_j^{'}$ to (a,b).

An SWF Ψ is *non-manipulable* if and only if it satisfies the following property.

(NM) For all $(a,b) \in \overline{A}$, all $(r_1, ..., r_n) \in H^n$, all $i \in N$ and all $r_i' \in H$, both (i) and (ii) hold.

(i) $r(a,b) < r_i(a,b) \to r_{-i} \otimes r_i^{'}\{a,b\} \leq r(a,b)$.

(ii) $r(a,b) > r_i(a,b) \to r_{-i} \otimes r_i^{'}\{a,b\} \geq r(a,b)$.

To clarify the nature of this condition, consider Figure 1.

[7] It is possible to factor out of a fuzzy weak preference relation a fuzzy strict preference relation, and a fuzzy indifference relation. There are several ways of doing this (Dasgupta and Deb, 2001). However, this issue does not arise in this paper. All of our results require the fuzzy weak preference relation only.

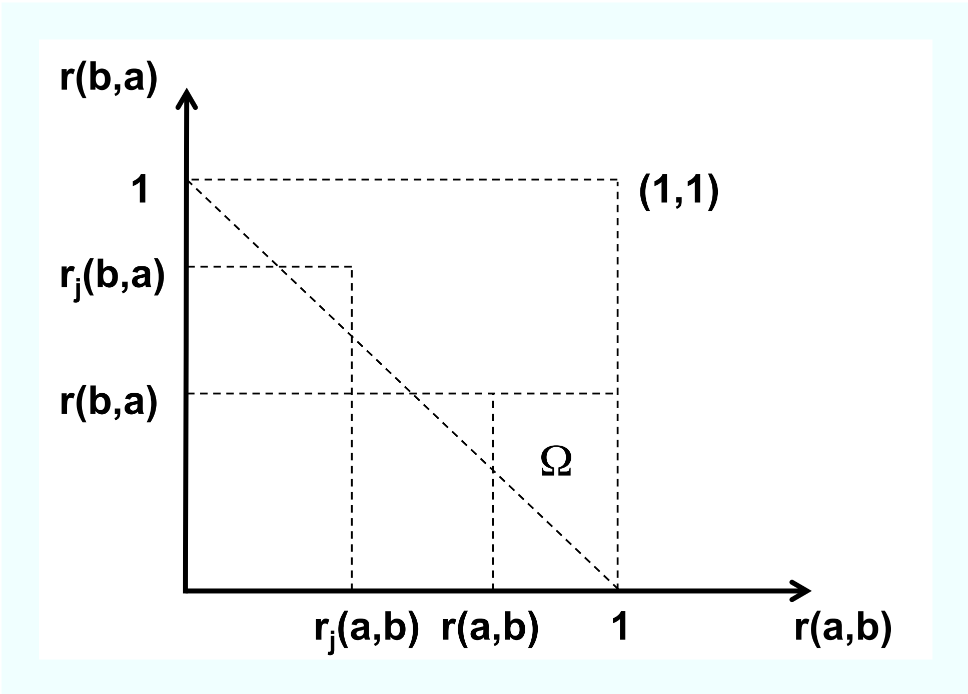

Figure 1: An illustration of the NM condition.

In Figure 1, we restrict attention to the ordered pair (a, b) and the ordered pair (b, a). Individual j's fuzzy preferences are denoted by the vector $(r_j(a, b)$, $r_j(b, a))$. Social preferences are denoted by the vector $(r(a, b),\ r(b, a))$. If individual j misrepresents his preferences, then the new vector of social values $(r^*(a, b),\ r^*(b, a))$ is constrained to lie in Ω.

Finally, we need a condition that eliminates trivially non-manipulable SWFs.

An SWF Ψ is not constant if and only if it satisfies the following property.
(NC) For all $(a, b) \in \overline{A}$, there exists $(r_1, ..., r_n), (r_1^{'}, ..., r_n^{'}) \in H^n$ such that $r(a, b) = 0$ and $r^{'}(a, b) = 1$.

As stated earlier, this condition is mild. It stipulates that, for each pair of social alternatives, two profiles exist in the domain of the social welfare function which produce different social values in $\{0, 1\}$ for this pair. This condition rules out social welfare functions which assign constant values to pairs of alternatives, irrespective of individual preferences.

Let P denote the set of all subsets of N. A non-empty subset of N is called a coalition. Given a coalition $C = \{i_1, ..., i_m\}$, and given $(r_1, ..., r_n) \in H^n$, $r_C(a, b)$ denotes $(r_{i_1}(a, b), ..., r_{i_m}(a, b)) \in [0, 1]^m$.

Note that $r_N(a,b)$ denotes $(r_{i_1}(a,b),...,r_{i_n}(a,b)) \in [0,1]^n$. So given $N = \{1,...,n\}$, $r_{i_1}(a,b)$ in the vector $r_N(a,b)$ denotes individual 1's fuzzy preference over (a,b), $r_{i_n}(a,b)$ in the vector $r_N(a,b)$ denotes individual n's fuzzy preference over (a,b) and so on.

We write $r_C(a,b) \geq r'_C(a,b)$ if $r_i(a,b) \geq r'_i(a,b)$ for all $i \in C$.

We now introduce some other properties that SWFs might satisfy.

An SWF Ψ is Arrow-like if and only if it satisfies the following two properties.

(IIA) For all $(r_1,...,r_n)$, $(r'_1,...,r'_n) \in H^n$, and all $(a,b) \in \overline{A}$,

$$r_N(a,b) = r'_N(a,b) \text{ implies } r(a,b) = r'(a,b).$$

(PC) For all $(r_1,...,r_n) \in H^n$, and all $(a,b) \in \overline{A}$,

$$\max_{i \in N} \{r_N(a,b)\} \geq r(a,b) \geq \min_{i \in N} \{r_N(a,b)\}.$$

Of course, IIA is our version of Arrow's (1951) independence of irrelevant alternatives condition but in the framework of fuzzy aggregation. Similarly, PC is our version of the Pareto condition.[8]

An SWF Ψ is neutral if and only if it satisfies the following property. For all $(r_1,...,r_n)$, $(r'_1,...,r'_n) \in H^n$, and all $(a,b),(c,d) \in \overline{A}$,

$$r_N(a,b) = r'_N(c,d) \text{ implies } r(a,b) = r'(c,d).$$

In order to explain this condition, let Ψ be a neutral SWF. Take any $(r_1,...,r_n) \in H^n$ and any $(a,b) \in \overline{A}$. At this profile we have $r_N(a,b) \in [0,1]^n$. In addition, assume that at this profile we also have $r_N(c,d) = r_N(a,b)$ for some $(c,d) \in \overline{A}$. In other words, each individual's (c,d) value is identical to their (a,b) value. Then, since Ψ is neutral, we have $r(a,b) = r(c,d)$. This means that the social value assigned by Ψ to (c,d) is identical to the value assigned to (a,b). Furthermore, if $r_N(a,b) = \overline{r}_N(c,d)$ at some *other* profile $(\overline{r}_1,...,\overline{r}_n) \in H^n$ then we have $r(a,b) = \overline{r}(c,d)$ as well. In other words, neutrality is both an intra-profile condition and an inter-profile condition. Loosely speaking, neutrality means that the names of the alternatives do not matter.

An SWF Ψ is dictatorial if and only if there exists an individual $i \in N$ such that for all $(a,b) \in \overline{A}$, and for every $(r_1,...,r_n) \in H^n$, $r_i(a,b) = r(a,b)$.

In order to explain this condition, let Ψ be a dictatorial SWF. Then there is an individual (the dictator) who can impose on society his fuzzy preferences at every profile in the domain of the social welfare function.

[8] Some authors like García-Lapresta and Llamazares (2001) call this condition "compensative".

3 The theorem

We now state and prove our theorem.

Any non-manipulable SWF that is not constant is dictatorial.

The proof of this theorem involves a number of steps.

Let Ψ be a non-manipulable SWF that is not constant. Then Ψ is Arrow-like.

Let Ψ be a non-manipulable SWF that is NC. We start by proving that Ψ must satisfy IIA. Assume, by way of contradiction, that Ψ does not satisfy IIA. Therefore, $\exists(a,b) \in \overline{A}$ and $\exists(r_1,...,r_n),(r_1^{'},...,r_n^{'}) \in H^n$ with $r_j(a,b) = r_j^{'}(a,b)$ $\forall j \in N$ such that $r(a,b) \neq r^{'}(a,b)$. Consider the following sequence of fuzzy preference profiles:

$$\begin{aligned}
\mathbf{R}^{(0)} &= (r_1,...,r_n),\\
\mathbf{R}^{(1)} &= (r_1',r_2,..,r_n),\\
\mathbf{R}^{(2)} &= (r_1',r_2',r_3,..,r_n),\\
&...\\
\mathbf{R}^{(n)} &= (r_1',...,r_n').
\end{aligned}$$

Assume, without loss of generality, that $r(a,b) = k$ and $r^{'}(a,b) = k'$ with $k' > k$. First of all, compare $r_{-1} \otimes r_1^{'}\{a,b\}$ with $r(a,b)$. There are two possibilities.

Case 1. $r_{-1} \otimes r_1^{'}\{a,b\} \neq k$. If $k < r_{-1} \otimes r_1^{'}\{a,b\} \leq r_1(a,b)$ or $k > r_{-1} \otimes r_1^{'}\{a,b\} \geq r_1(a,b)$ then NM is violated in the move from $\mathbf{R}^{(0)}$ to $\mathbf{R}^{(1)}$. Similarly, if $k \leq r_1(a,b) < r_{-1} \otimes r_1^{'}\{a,b\}$ then NM is violated either in the move from $\mathbf{R}^{(0)}$ to $\mathbf{R}^{(1)}$ or in the move from $\mathbf{R}^{(1)}$ to $\mathbf{R}^{(0)}$. If $k \geq r_1(a,b) > r_{-1} \otimes r_1^{'}\{a,b\}$ then NM is violated either in the move from $\mathbf{R}^{(0)}$ to $\mathbf{R}^{(1)}$ or in the move from $\mathbf{R}^{(1)}$ to $\mathbf{R}^{(0)}$. Finally, if $r_1(a,b) > k > r_{-1} \otimes r_1^{'}\{a,b\}$ or $r_1(a,b) < k < r_{-1} \otimes r_1^{'}\{a,b\}$ then NM is violated in the move from $\mathbf{R}^{(1)}$ to $\mathbf{R}^{(0)}$. The only case remaining is Case 2.

Case 2. $k = r_{-1} \otimes r_1^{'}\{a,b\}$.

We now proceed to move from $\mathbf{R}^{(1)}$ to $\mathbf{R}^{(2)}$ by changing the fuzzy preferences of individual 2. However, we can treat this case in exactly the same manner as the move from $\mathbf{R}^{(0)}$ to $\mathbf{R}^{(1)}$ and so $r_{-1-2} \otimes r_1^{'} \otimes r_2^{'}\{a,b\} = k$. Repeating this argument for each individual ensures that when we reach $\mathbf{R}^{(n)}$ we have $r^{'}(a,b) = k$ which contradicts the assumption that $r^{'}(a,b) = k' > k$.

Therefore, Ψ satisfies IIA.

We now prove that Ψ satisfies PC.

First of all, we prove that Ψ satisfies the following property.

(*) For all $(r_1,...,r_n),(r_1^{'},...,r_n^{'}) \in H^n$ and every $(a,b) \in \overline{A}$, (i) if $r_N(a,b) = \mathbf{0}^n$ then $r(a,b) = 0$ and (ii) if $r_N^{'}(a,b) = \mathbf{1}^n$ then $r^{'}(a,b) = 1$.

To see that (*) holds note that NC implies that there exists $(r_1, ..., r_n) \in H^n$ such that $r(a,b) = 0$. Let $(r_1^*, ..., r_n^*) \in H^n$ denote a profile such that $r_N^*(a,b) = \mathbf{0}^n$. If $r_N(a,b) = \mathbf{0}^n$ then (i) of (*) holds immediately by IIA. Assume that $r_N^*(a,b) \neq r_N(a,b)$. Therefore, $\exists Q \subseteq N$ such that $r_j(a,b) \neq 0$ for all $j \in Q$. Let $q \in Q$ and note that $r_{-q} \otimes r_q^*\{a,b\} = 0$. If not, then NM is violated in the move from $(r_1, .., r_q, .., r_n) \in H^n$ to $(r_1, .., r_q^*, .., r_n) \in H^n$. If $Q \backslash \{q\}$ is non-empty then let $z \in Q \backslash \{q\}$ and note that $r_{-q-z} \otimes r_q^* \otimes r_z^*\{a,b\} = 0$. If not, then NM is violated in the move from $(r_1, .., r_q^*, r_z, .., r_n) \in H^n$ to $(r_1, .., r_q^*, r_z^*, .., r_n) \in H^n$. Simply repeating this argument for the remaining members of Q ensures that $r^*(a,b) = 0$. Since $(r_1^*, ..., r_n^*) \in H^n$ is arbitrary, part (i) of (*) is proved.

The proof of part (ii) of (*) is similar and therefore is omitted. We now prove that Ψ satisfies PC.

We prove by contradiction. Assume that $\exists (a,b) \in \overline{A}$, $\exists (\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ such that $\widehat{r}(a,b) < \min_{i \in N} \{\widehat{r}_N(a,b)\}$. Note that if $\widehat{r}_N(a,b) = \mathbf{1}^n$ then (*) implies that $\widehat{r}(a,b) = 1$ which contradicts the assumption that $\widehat{r}(a,b) < \min_{i \in N} \{\widehat{r}_N(a,b)\}$. So assume that $\widehat{r}_N(a,b) \neq \mathbf{1}^n$.

Consider any preference profile $(r_1^*, ..., r_n^*) \in H^n$ such that $r_i^*(a,b) = 1$ for all $i \in N$.

Consider the following sequence of fuzzy preference profiles:

$$\begin{aligned}
\mathbf{G}^{(0)} &= (\widehat{r}_1, ..., \widehat{r}_n), \\
\mathbf{G}^{(1)} &= (r_1^*, \widehat{r}_2, .., \widehat{r}_n), \\
\mathbf{G}^{(2)} &= (r_1^*, r_2^*, \widehat{r}_3, .., \widehat{r}_n), \\
&\dots \\
\mathbf{G}^{(n)} &= (r_1^*, ..., r_n^*).
\end{aligned}$$

Consider $\mathbf{G}^{(1)}$. If $\widehat{r}_{-1} \otimes r_1^*\{a,b\} > \widehat{r}(a,b)$ then NM is violated in the move from $\mathbf{G}^{(0)}$ to $\mathbf{G}^{(1)}$. If $\widehat{r}_{-1} \otimes r_1^*\{a,b\} < \widehat{r}(a,b)$ then NM is violated in the move from $\mathbf{G}^{(1)}$ to $\mathbf{G}^{(0)}$. Therefore, $\widehat{r}_{-1} \otimes r_1^*\{a,b\} = \widehat{r}(a,b)$.

We can repeat this argument as we move from $\mathbf{G}^{(1)}$ to $\mathbf{G}^{(2)}$ and so $\widehat{r}_{-1-2} \otimes r_1^* \otimes r_2^*\{a,b\} = \widehat{r}(a,b)$. Again, repeating this argument for each individual ensures that when we reach $\mathbf{G}^{(n)}$ we have $r^*(a,b) = \widehat{r}(a,b)$. However, this contradicts (*) and so $\widehat{r}(a,b) \geq \min_{i \in N} \{\widehat{r}_N(a,b)\}$.

In order to complete the proof that Ψ satisfies PC, assume that $\exists (a,b) \in \overline{A}$, $\exists (\overline{r}_1, ..., \overline{r}_n) \in H^n$ such that $\overline{r}(a,b) > \max_{i \in N} \{\overline{r}_N(a,b)\}$. Note that if $\overline{r}_N(a,b) = \mathbf{0}^n$ then (*) implies that $\overline{r}(a,b) = 0$ which contradicts the assumption that $\overline{r}(a,b) > \max_{i \in N} \{\overline{r}_N(a,b)\}$. So assume that $\overline{r}_N(a,b) \neq \mathbf{0}^n$.

Consider any preference profile $(r_1^{**}, ..., r_n^{**}) \in H^n$ such that $r_i^{**}(a,b) = 0$ for all $i \in N$.

Consider the following sequence of fuzzy preference profiles:

$$\begin{aligned}\mathbf{H}^{(0)} &= (\overline{r}_1, ..., \overline{r}_n),\\ \mathbf{H}^{(1)} &= (r_1^{**}, \overline{r}_2, .., \overline{r}_n),\\ \mathbf{H}^{(2)} &= (r_1^{**}, r_2^{**}, \overline{r}_3, .., \overline{r}_n),\\ &\cdots\\ \mathbf{H}^{(n)} &= (r_1^{**}, ..., r_n^{**}).\end{aligned}$$

Consider $\mathbf{H}^{(1)}$. If $\overline{r}_{-1} \otimes r_1^{**}\{a,b\} < \overline{r}(a,b)$ then NM is violated in the move from $\mathbf{H}^{(0)}$ to $\mathbf{H}^{(1)}$. If $\overline{r}_{-1} \otimes r_1^{**}\{a,b\} > \overline{r}(a,b)$ then NM is violated in the move from $\mathbf{H}^{(1)}$ to $\mathbf{H}^{(0)}$. Therefore, $\overline{r}_{-1} \otimes r_1^{**}\{a,b\} = \overline{r}(a,b)$.

We can repeat this argument as we move from $\mathbf{H}^{(1)}$ to $\mathbf{H}^{(2)}$ and so $\overline{r}_{-1-2} \otimes r_1^{**} \otimes r_2^{**}\{a,b\} = \overline{r}(a,b)$. Again, repeating this argument for each individual ensures that when we reach $\mathbf{H}^{(n)}$ we have $r^{**}(a,b) = \overline{r}(a,b)$. However, this contradicts (*) and so $\overline{r}(a,b) \leq \max_{i \in N} \{\overline{r}_N(a,b)\}$.

Therefore, Ψ satisfies PC.

We have proved that Ψ is Arrow-like.

In what follows $\mathbf{w}^n$ denotes the vector with $w \in [0,1]$ in all n places.

An Arrow-like SWF Ψ is neutral.

Case 1: If $(a,b) = (c,d)$ then the result follows immediately from the fact that Ψ is Arrovian. Case 2: $(a,b),(a,d) \in \overline{A}$. Take $(r_1, ..., r_n) \in H^n$ such that $r_N(b,d) = \mathbf{1}^n$. Then PC implies that $r(b,d) = 1$. Since r is max-min transitive, we have $r(a,d) \geq \min\,\{r(a,b), r(b,d)\}$. Therefore, $r(a,d) \geq \min\,\{r(a,b), 1\}$ and so $r(a,d) \geq r(a,b)$.

In addition, since $r_N(b,d) = \mathbf{1}^n$ and individual preferences are max-min transitive, it follows that $r_N(a,d) \geq r_N(a,b)$.

Select a profile $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ such that $\overline{r}_N(b,d) = \mathbf{1}^n$ and $\overline{r}_N(d,b) = \mathbf{1}^n$. From the argument above we know that $\overline{r}(a,d) \geq \overline{r}(a,b)$ and $\overline{r}_N(a,d) \geq \overline{r}_N(a,b)$. However, an identical argument shows that $\overline{r}(a,b) \geq \overline{r}(a,d)$ and $\overline{r}_N(a,b) \geq \overline{r}_N(a,d)$. Therefore, it must be the case that $\overline{r}(a,b) = \overline{r}(a,d)$ and $\overline{r}_N(a,b) = \overline{r}_N(a,d)$.

Since $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1, ..., r_n) \in H^n$ such that $r_N(b,d) = \mathbf{1}^n$ and $r_N(d,b) = \mathbf{1}^n$. Let F^n denote the set of such profiles. Take any profile $(\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ such that $\widehat{r}_N(a,b) = \widehat{r}_N(a,d)$. Then there exists a profile $(r_1^{'}, ..., r_n^{'}) \in F^n$ such that $\widehat{r}_N(a,b) = \widehat{r}_N(a,d) = r_N^{'}(a,b) = r_N^{'}(a,d)$. IIA implies that $\widehat{r}(a,b) = \widehat{r}(a,d) = r^{'}(a,b) = r^{'}(a,d)$.

Take any pair of distinct profiles $(r_1^{''}, ..., r_n^{''}), (r_1^{*}, ..., r_n^{*}) \in H^n$ such that $r_N^{''}(a,b) = r_N^{*}(a,d)$. Then there exists a profile $(r_1^{**}, ..., r_n^{**}) \in F^n$ such that $r^{''}_N(a,b) = r^*_N(a,d) = r^{**}_N(a,b) = r^{**}_N(a,d)$. IIA implies that $r^{''}(a,b) = r^*(a,d) = r^{**}(a,b) = r^{**}(a,d)$.

Case 3: $(a,b),(c,b) \in \overline{A}$. Take $(r_1,...,r_n) \in H^n$ such that $r_N(a,c) = \mathbf{1}^n$. Then PC implies that $r(a,c) = 1$. Since r is max-min transitive, we have $r(a,b) \geq \min\ \{r(a,c), r(c,b)\}$. Therefore, $r(a,b) \geq \min\ \{1, r(c,b)\}$ and so $r(a,b) \geq r(c,b)$.

In addition, since $r_N(a,c) = \mathbf{1}^n$ and individual preferences are max-min transitive, it follows that $r_N(a,b) \geq r_N(c,b)$.

Select a profile $(\overline{r}_1,...,\overline{r}_n) \in H^n$ such that $\overline{r}_N(a,c) = \mathbf{1}^n$ and $\overline{r}_N(c,a) = \mathbf{1}^n$. From the argument above we know that $\overline{r}(a,b) \geq \overline{r}(c,b)$ and $\overline{r}_N(a,b) \geq \overline{r}_N(c,b)$. However, an identical argument shows that $\overline{r}(c,b) \geq \overline{r}(a,b)$ and $\overline{r}_N(c,b) \geq \overline{r}_N(a,b)$. Therefore, it must be the case that $\overline{r}(a,b) = \overline{r}(c,b)$ and $\overline{r}_N(a,b) = \overline{r}_N(c,b)$.

Since $(\overline{r}_1,...,\overline{r}_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1,...,r_n) \in H^n$ such that $r_N(a,c) = \mathbf{1}^n$ and $r_N(c,a) = \mathbf{1}^n$. Let G^n denote the set of such profiles. Take any profile $(\widehat{r}_1,...,\widehat{r}_n) \in H^n$ such that $\widehat{r}_N(a,b) = \widehat{r}_N(c,b)$. Then there exists a profile $(r^{'}_1,...,r^{'}_n) \in G^n$ such that $\widehat{r}_N(a,b) = \widehat{r}_N(c,b) = r^{'}_N(a,b) = r^{'}_N(c,b)$. IIA implies that $\widehat{r}(a,b) = \widehat{r}(c,b) = r^{'}(a,b) = r^{'}(c,b)$.

Take any pair of distinct profiles $(r^{''}_1,...,r^{''}_n),(r^*_1,...,r^*_n) \in H^n$ such that $r^{''}_N(a,b) = r^*_N(c,b)$. Then there exists a profile $(r^{**}_1,...,r^{**}_n) \in G^n$ such that $r^{''}_N(a,b) = r^*_N(c,b) = r^{**}_N(a,b) = r^{**}_N(c,b)$. IIA implies that $r^{''}(a,b) = r^*(c,b) = r^{**}(a,b) = r^{**}(c,b)$.

Case 4: $(a,b),(c,d) \in \overline{A}$ with a,b,c,d distinct. Take $(r_1,...,r_n) \in H^n$ such that $r_N(b,d) = r_N(d,b) = r_N(a,c) = r_N(c,a) = \mathbf{1}^n$. Then PC implies that $r(d,b) = 1$. Since r is max-min transitive, we have $r(a,b) \geq \min\ \{r(a,d), r(d,b)\}$. Therefore, $r(a,b) \geq \min\ \{r(a,d), 1\}$ and so $r(a,b) \geq r(a,d)$. However, an identical argument shows that $r(a,d) \geq r(a,b)$ and so $r(a,b) = r(a,d)$.

In addition, since $r_N(d,b) = r_N(b,d) = \mathbf{1}^n$ and individual preferences are max-min transitive, it follows that $r_N(a,b) = r_N(a,d)$.

We can repeat this argument to show that $r(a,d) = r(c,d)$ and $r_N(a,d) = r_N(c,d)$. Since $(r_1,...,r_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1,...,r_n) \in H^n$ such that $r_N(b,d) = r_N(d,b) = r_N(a,c) = r_N(c,a) = \mathbf{1}^n$. Let J^n denote the set of such profiles.

Take any profile $(\widehat{r}_1,...,\widehat{r}_n) \in H^n$ such that $\widehat{r}_N(a,b) = \widehat{r}_N(c,d)$. Then there exists a profile $(r^{'}_1,...,r^{'}_n) \in J^n$ such that $\widehat{r}_N(a,b) = \widehat{r}_N(c,d) = r^{'}_N(a,b) = r^{'}_N(c,d)$. IIA implies that $\widehat{r}(a,b) = \widehat{r}(c,d) = r^{'}(a,b) = r^{'}(c,d)$.

Take any pair of distinct profiles $(r^{''}_1,...,r^{''}_n),(r^*_1,...,r^*_n) \in H^n$ such that $r^{''}_N(a,b) = r^*_N(c,d)$. Then there exists a profile $(r^{**}_1,...,r^{**}_n) \in J^n$ such that $r^{''}_N(a,b) = r^*_N(c,d) = r^{**}_N(a,b) = r^{**}_N(c,d)$. IIA implies that $r^{''}(a,b) =$

$r^*(c,d) = r^{**}(a,b) = r^{**}(c,d)$.

Case 5: $(a,b),(b,a) \in \overline{A}$. Take any profile $(r_1,...,r_n) \in H^n$ such that $r_N(a,b) = r_N(a,c) = r_N(b,c) = r_N(b,a)$. Cases (2) and (3) imply that $r(a,b) = r(a,c) = r(b,c) = r(b,a)$. Let W^n denote the set of such profiles. Take any profile $(\overline{r}_1,...,\overline{r}_n) \in H^n$ such that $\overline{r}_N(a,b) = \overline{r}_N(b,a)$. Then there exists a profile $(r_1^{'},...,r_n^{'}) \in W^n$ such that $\overline{r}_N(a,b) = \overline{r}_N(b,a) = r_N^{'}(a,b) = r_N^{'}(b,a)$. IIA implies that $\overline{r}(a,b) = \overline{r}(b,a) = r^{'}(a,b) = r^{'}(b,a)$.

Take any pair of distinct profiles $(r_1^{''},...,r_n^{''}),(r_1^*,...,r_n^*) \in H^n$ such that $r_N^{''}(a,b) = r_N^*(b,a)$. Then there exists a profile $(r_1^{**},...,r_n^{**}) \in W^n$ such that $r_N^{''}(a,b) = r_N^*(b,a) = r_N^{**}(a,b) = r_N^{**}(b,a)$. IIA implies that $r^{''}(a,b) = r^*(b,a) = r^{**}(a,b) = r^{**}(b,a)$.

Given that Ψ is neutral we can complete the proof in the following way. Take any $(a,b) \in \overline{A}$. Take a profile $(r_1,...,r_n) \in H^n$ such that $r_N(a,b) = \mathbf{0}^n$. By PC it must be the case that $r(a,b) = 0$. Take some other profile $(r_1^*,...,r_n^*) \in H^n$ such that $r_N^*(a,b) = 1$. By PC it must be the case that $r^*(a,b) = 1$.

Consider the following sequence of fuzzy preference profiles:

$$\begin{aligned}
\mathbf{W}^{(0)} &= (r_1,...,r_n),\\
\mathbf{W}^{(1)} &= (r_1^*,r_2,..,r_n),\\
\mathbf{W}^{(2)} &= (r_1^*,r_2^*,r_3,..,r_n),\\
&\;\;...\\
\mathbf{W}^{(n)} &= (r_1^*,...,r_n^*).
\end{aligned}$$

At some profile in this sequence, say $\mathbf{W}^{(2)}$, the social value assigned to (a,b) must rise from zero to a value strictly greater than zero. By PC, the latest this can happen is when we reach $\mathbf{W}^{(n)}$. We shall assume, without loss of generality, that this happens at $\mathbf{W}^{(2)}$ when individual 2 raises his or her (a,b) value from 0 to 1.

Now consider the profile $\mathbf{W}^{(\alpha)} = (r_1^*,r_2,r_3^*,..,r_n^*)$. We claim that the value of (a,b) at this profile is 0. To see this note that, by assumption, the value of (a,b) at $\mathbf{W}^{(1)}$ is zero. We can construct a profile $(\widehat{r}_1,...,\widehat{r}_n) \in H^n$ in which individuals have the following preferences over three alternatives a,b and c. Individual preferences over (a,b) at this profile are the same as they are over (a,b) at $\mathbf{W}^{(\alpha)}$. Individual preferences over (a,c) at this profile are the same as they are over (a,b) at $\mathbf{W}^{(1)}$. Finally, individual preferences over (b,c) at this profile are the same as they are over (a,b) at $\mathbf{W}^{(2)}$. We write $aRb \longleftrightarrow \widehat{r}(a,b) = 1$ and $aPb \longleftrightarrow aRb \wedge \widehat{r}(b,a) = 0$. Therefore at $(\widehat{r}_1,...,\widehat{r}_n) \in H^n$ individuals hold the following preferences:

Individual 1: $aRbRc$

Individual 2: $bRcPa$

Everyone else: $cPaRb$.

Neutrality implies that $\widehat{r}(a,b)$ is identical to the value (a,b) takes at $\mathbf{W}^{(\alpha)}$. Similarly, it implies that $\widehat{r}(b,c) = s > 0$ and that $\widehat{r}(a,c) = 0$. Note that by max-min transitivity $\widehat{r}(a,c) \geq \min\{\widehat{r}(a,b), s\}$ and so $\widehat{r}(a,b) = 0$. Therefore, the value (a,b) takes at $\mathbf{W}^{(\alpha)}$ is zero. At $\mathbf{W}^{(\alpha)}$ individual 2 assigns the value 0 to (a,b) but everyone else assigns the value 1. Despite this, the social value of (a,b) is 0. Neutrality implies that this will remain the case whenever these preferences are replicated over any other pair of distinct social alternatives at any profile.

Now consider any profile $\mathbf{W}^{(\alpha\alpha)} = (\overline{r}_1, ..., \overline{r}_n) \in H^n$ in which individual 2 assigns the value 0 to some pair of distinct social alternatives. Furthermore, at this profile, individual 1 assigns the value t_1 to this pair, individual 3 assigns the value t_3 to this pair, and so on with $t_1, t_3, ..., t_n \in [0,1]$. The NM condition implies that the social value assigned to this pair must remain at 0 for all $t_1, t_3, ..., t_n \in [0,1]$.

Let us now return to $\mathbf{W}^{(\alpha)}$. To recall, individual 2 assigns the value 0 to (a,b) at this profile but everyone else assigns the value 1. Despite this, the social value of (a,b) is 0. Completeness implies that, at this profile, individual 2 must assign the value 1 to (b,a) and so must society. This is true irrespective of everyone else's (b,a) value. Neutrality implies that any profile $\mathbf{W}^{(\beta\beta)} = (\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ in which individual 2 assigns the value 1 to some pair of distinct social alternatives, and in which individual 1 assigns the value t_1^* to this pair, individual 3 assigns the value t_3^* to this pair, and so on with $t_1^*, t_3^*, ..., t_n^* \in [0,1]$, must be consistent with the social welfare function assigning a value of 1 to this pair.

To see that individual 2 is a dictator, fix some ordered pair (a,b). By the above argument, whenever individual 2 assigns a value of 1 to this pair then so must society, irrespective of everyone else's (a,b) value. Imagine now that individual 2 lowers his or her (a,b) value to some value in $[0,1)$. If this value is 0 then the social value of (a,b) must be 0 due to the argument above about $\mathbf{W}^{(\alpha\alpha)}$. Imagine that individual 2 selects a value v where $1 > v > 0$. If the social value of (a,b) exceeds v then individual 2 can profitably misrepresent by lowering his or her value to 0. If the social value of (a,b) is below v then individual 2 can profitably misrepresent by raising his or her value to 1. Neither of these things can happen and so the social value must be equal to v.

We have proved that individual 2 can impose his (a,b) preferences on society at every profile in the domain of the social welfare function Ψ. Since Ψ is neutral individual 2 is a dictator.

4 A possibility theorem

One way of circumventing the impossibility theorem is to relax the assumption that social preferences are transitive. This is actually Fishburn's position expressed in Fishburn (1970). Fishburn argues that the concept of a social welfare function is untenable since it assumes social transitivity. Fishburn claims in this article that he has yet to see a convincing argument made for social transitivity. For Fishburn, transitivity is a much less appealing assumption than Arrow's "independence of irrelevant alternatives".

Following Fishburn's lead, we can state a possibility result under the assumption that social preferences are not transitive.

There exists a function $\Phi : H^n \to T$ which is non-manipulable, not constant and not dictatorial.

Imagine that the number of individuals is odd. Define the function $\Phi : H^n \to T$ as follows. For all $(a, b) \in \overline{A}$, all $(r_1, ..., r_n) \in H^n$, $r(a, b) = \text{median}_{i \in N} \{r_i(a, b)\}$. If the number of individuals is even then define the function $\Phi : H^n \to T$ as follows. For all $(a, b) \in \overline{A}$, all $(r_1, ..., r_n) \in H^n$, $r(a, b) = \max_{i \in N} \{r_i(a, b)\}$. Both of these functions are non-manipulable and they are not constant. Moreover, neither is dictatorial.

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