Spatial Equilibria in a Social Interaction Model

MOSSAY P[*]and PICARD PM[†]

February 2008

Abstract

Social interactions are at the essence of societies and explain the gathering of individuals in villages, agglomerations, or cities. We study the emergence of multiple agglomerations as resulting from the interplay between spatial interaction externalities and competition in the land market. As opposed to Beckmann's original framework (1976), agents get dispersed across several cities distributed along a circle. Spatial equilibrium configurations involve a high degree of spatial symmetry, in terms of city size and location, and can be Pareto-ranked.

Keywords: social interaction, multiple agglomerations, spatial economy.

1 Introduction

A major source of spatial heterogeneity stems from non-market interactions. Social interactions through face-to-face contacts are at the essence of our societies and explain the gathering of individuals in villages, agglomerations, or cities. They translate a psychological need for maintaining relationships with one another, and favor a constant exchange of ideas; see Krugman (1991), Glaeser and Scheinkman (2003), and Fujita and Thisse

[*]University of Reading and Universidad de Alicante. Email: p.mossay@reading.ac.uk

[†]University of Manchester and CORE. Email: pierre.picard@manchester.ac.uk

(2002). In this paper we address the issue of the emergence of multiple agglomerations as the result of the interplay between social interactions and competition in the land market.

The present paper builds on Beckmann's (1976) model. This model provides a simple rationale for the spatial agglomeration of agents as the result of spatial interaction externalities. Agents are distributed along some geographical space and benefit from social interactions with all the other agents. These social interactions provide individual benefits while entailing an individual cost as each one must access to distant agents. Moreover the return of spatial interactions is also balanced by a cost of residence since agents compete for land space. When the benefit of social interactions is larger than the commuting and residence costs, agents prefer to be located close together, which leads to the formation of agglomerations. In his original work, Beckmann considered the case of a one-dimensional spatial economy modelled along a line segment. The resulting equilibrium consists in a uni-modal symmetric - bell-shaped - spatial distribution, where agents agglomerate around the city centre, see Fujita and Thisse (2002).

We revisit Beckmann (1976)'s framework along a line segment and extend it to the case of a spatial economy extending along a circumference. While the modelling along a line segment seems appropriate to describe the internal structure of cities, the formulation along a circumference provides a natural framework to analyze the interaction between multiple agglomerations. Circular spatial frameworks have been studied in 'racetrack economy' models in the context of the new economic geography literature, e.g. Fujita *et al.* (1999), Mossay (2003), or Picard and Tabuchi (2003). Yet, because of the complexity of market interactions, this strand of literature only characterizes the 'flat-earth' spatial equilibrium corresponding to a uniform spatial distribution of agents. In this paper, because our social interaction model has a much simpler structure, we are able to fully characterize the set of spatial equilibria.

Our results are the following. First we determine the equilibrium and first-best spatial distributions of agents along a line segment. In accordance to Fujita and Thisse (2002), the first-best distribution is more concentrated than the equilibrium one; see also Tabuchi (1986). At equilibrium agents choose a too large lot size because they do not internalize other agents' preferences when making their own lot choice. We show that the implementation of the first-best distribution does not require the use of spatial transfers. Furthermore, social interactions generate the emergence of a single city, meaning that multiple cities can't be sustained in equilibrium along a line.

Second we provide a full characterization of spatial equilibria emerging along a circular geographical space. In equilibrium, cities are identical and equally-spaced: cities share the same spatial structure and are separated by equal-size empty hinterlands. We also show that equilibrium configurations can only involve an odd number of cities. Furthermore spatial equilibria can be Pareto-ranked. The total welfare of the spatial economy decreases with the number of cities so that the one-city configuration Pareto dominates all the other configurations. As in the open line framework, the first-best distribution corresponds to a single city that is more concentrated than the equilibrium distribution.

Our paper deals with the endogenous formation of multiple-centre configurations. Like in Fujita and Ogawa (1982), multiplicity of equilibria arise. In order to derive their results, Fujita and Ogawa had to simplify their analysis. An important issue is to know whether their qualitative results (e.g., multiplicity of equilibria) are sensitive to such simplifications. Here, because of the linear structure of the model, we are able to perform a full general equilibrium analysis, and therefore to confirm that mutiple-centre configurations emerge in equilibrium.

We present the model of social interactions in Section 2. We derive the spatial equilibrium and the first-best distributions of the model along an open line in Section 3. Section 4 characterizes spatial equilibria along a circumference. Section 5 ranks these various equilibria and compares them with the socially optimal distribution. Section 6 is devoted to the stability of spatial equilibria.

2 The Model

In this section we present the economic environment. A unit-mass of agents is distributed along a one-dimensional geographical space according to the density $\lambda(x)$ with $\int \lambda(x)dx = 1$. Agents travel along the one-dimensional road and benefit from social contacts with other agents. The social utility that an agent in location x derives from interacting with other agents is given by

$$S(x) = A - \int \lambda(y)T(x-y)dy \tag{1}$$

The first term A denotes the total return from interacting with other agents. The second term reflects the cost of trips when accessing to distant agents. We consider the case of a linear cost function $T(x) = 2\tau\,|x|$, where τ measures the intensity of travel costs. In our model each agent interacts with *all* the other agents so that $S(x) \geq 0$, for any location x. The surplus $S(x)$ can be interpreted in a certain and uncertain context. Indeed, it can be interpreted literally as the utility derived by an individual who plans to interact with all other agents with probability 1. It can also be interpreted as the expected utility of an individual who plans to interact with a subset of agents whom location and identity are not known at the time of the residence choice. Such an interpretation fits the case of shopkeepers, sellers, etc., the case of workers who expect to have several jobs at different locations during their lifetime, and the case of employers who do not have a good idea of the future workers' residences.

In each location x, the residential area is longitudinal to the main road. It is a strip of land space equal to 1, which is connected by its edge to the main road. Agents in location x consume a composite good z and land space $s(x)$. Their utility is given by

$$V(x) = S\,(x) + z - \frac{\beta}{2s(x)}$$

where $S(x)$ is the social utility and β reflects the preference for land. The budget constraint faced by agents is

$$z + R(x)s(x) = Y$$

where Y is the income and $R(x)$ denotes the land rent in x. By using this budget constraint, the utility derived in location x can rewritten as

$$V(x) = S\,(x) - \frac{\beta}{2s(x)} - s(x)R(x) + Y$$

This formulation of the utility function differs from Beckmann's formulation in one respect only: we consider an hyperbolic preference for land rather than the logarithmic preference used by Beckmann. This will allow us to simplify considerably the characterization of equilibria.

Landlords raise the land rent until no worker moves. Let V^* be an equilibrium utility of workers. The bid rent function is given by

$$\Psi(x) = \max_s \frac{S(x) - \beta/(2s) + Y - V^*}{s}$$

which yields the optimal land consumption $s^*(x)$ as satisfying $\beta/(2s^*(x)) = s^*\Psi^*(x) = (S^*(x) + Y - V^*)/2$. At the residential equilibrium, $R^*(x) = \Psi^*(x)$ so that $s^*R^*(x) = \beta/(2s^*(x))$. The indirect utility can then be written as $V(x) = S(x) - \beta/s^*(x) + Y$. Since the total land space is equal to 1 at each location x, the individual land consumption $s^*(x)$ corresponds to $1/\lambda(x)$, so that the indirect utility can be written in term of the population density $\lambda(x)$

$$V(x) = S(x) - \beta\lambda(x) + Y \tag{2}$$

This means that the residents' utility at location x *linearly* increases with the social return $S(x)$ and *linearly* decreases with the residential density $\lambda(x)$. Utility decreases with the residential density because agents compete for land space and thus face higher land prices in more populated areas. The present paper takes advantage of this linear structure to fully characterize spatial equilibria and the optimal spatial distribution. In what follows we assume without much loss of generality that land has no other use than residence so that the opportunity cost of land is zero.

3 Spatial equilibrium along a Line Segment

In this section we formulate the spatial model of social interactions along a line segment as studied in Beckmann (1976) and Fujita Thisse (2003, Chapter 6).

3.1 Spatial equilibrium

A distribution of agents $\lambda(.)$ constitutes a spatial equilibrium if agents have no incentive to relocate. In other words, $\lambda(x)$ is a spatial equilibrium if $V(x) = \overline{V}$ when $\lambda(x) > 0$ and $V(x) < \overline{V}$ when $\lambda(x) = 0$. In equilibrium we should have that $V'(x) = V''(x) = 0$ for all x where $\lambda(x) > 0$. In this paper, any area with a positive population is referred to as a city. As in Beckmann (1976) and Fujita and Thisse (2002) we characterize the spatial distribution along a line segment. In addition to this, we show that spatial equilibrium implies the emergence of a single city.

First let us consider a single city located along the interval $[-b, b]$, $b > 0$. By the differentiating the social utility (1) with respect to x, we have that

$$S'(x) = 2\tau \int_x^{-b} \lambda(y)dy - 2\tau \int_b^x \lambda(y)dy$$

$$S''(x) = -2\tau\ \lambda(x) - 2\tau\ \lambda(x) = -4\tau\ \lambda(x)$$

Because of linear travel costs, $S''(x)$ reduces to a linear function of λ. Hence, from relation (2) a necessary condition for equilibrium is $V''(x) = S''(x) - \beta\lambda''(x) = 0$, which leads to

$$\lambda''(x) + \delta^2\lambda(x) = 0 \quad \text{where} \quad \delta^2 \equiv 4\tau/\beta \tag{3}$$

The solution to this differential equation is given by

$$\lambda(x) = c \cos\delta\,(x - x_o) \tag{4}$$

where c and x_o are constants to be determined. Because $R^*(\pm b) = (\lambda(\pm b))^2\,\beta/2 = 0$ and because $\int \lambda(x)dx = 1$, the equilibrium spatial structure of the city and the boundary are given by

$$\lambda^*(x) = \frac{\delta}{2}\cos\delta x \quad \text{and} \quad b^* = \frac{\pi}{2\delta}$$

This describes the spatial the structure of a single city. We must also ensure that each agent is willing to interact with all the other agents so that $S(x) > 0$ for all x in $(-b^*, b^*)$. We need that $A > \int_{-b}^{b} \lambda^*(y)T(y-b)dy$ or equivalently $A > \tau\pi/\delta$.

An important issue is whether multiple cities can co-exist in equilibrium. The answer turns out to be negative. To show this, consider a set of cities possibly separated by empty hinterlands. Let the supports of the $M \geq 2$ cities be the closed intervals $[a_m, b_m]$, $m = 1, 2, 3, ..., M$, with $b_m < a_{m+1}$. Indeed, within each city, Equation (3) holds and $R^*(a_m) = R^*(b_m) = 0$, so that $\lambda(x)$ can be written as $\lambda_m(x) = c_m \cos[\delta(x - x_m)]$, $\forall x \in [a_m, b_m]$, with $x_m = (b_m - a_m)/2$. From Equations (1) and (2), we get

$$V(x) = A - 2\tau \sum_{m=1}^{M} \int_{a_m}^{b_m} \lambda(y)\,|x - y|\,dy - \beta\lambda(x)$$

$$V'(x) = -2\tau\left[1-\int_{a_m}^{b_m}\lambda_m(y)dy\right]-2\tau\int_{a_m}^{x}\lambda_m(y)dy+2\tau\int_{x}^{b_m}\lambda_m(y)dy-\beta\lambda_m'(x),\ \forall x\in(a_m,b_m)$$

When a resident relocates to her right, she looses access to the residents to her left either in other cities (first term) or in her own city (second term) while she gains a better access to the residents to her right within her city (third term) and faces an increase in land rent (resp. a decrease in land rent) if $x \in (a_m, x_m)$, (resp. $x \in (x_m, b_m)$), (last term). In particular, at the centre of city M, $\lambda_M'(x_M) = 0$ and

$$V'(x_M) = -2\tau\left[1-\int_{a_M}^{b_M}\lambda_M(y)dy\right] = -2\tau(1-L_M) < 0$$

where L_M denotes the population in city M. Therefore, residents in city M have always an incentive to move leftward, and no spatial configuration with $M \geq 2$ cities can't be sustained in equilibrium.

We have the following Proposition.

Proposition 1 If $A > \tau\pi/\delta$, the spatial equilibrium along a line segment is unique and involves a single city.

The *distribution is symmetric* with respect to location $x = 0$ and *concave*. This result is similar to Beckmann's result except that here the city structure is nowhere convex because of our hyperbolic preference for residential space. Note that the equilibrium utility level in the city is given by

$$V^* = V^*(x=0) = A - \int_{-b}^{b}\lambda^*(y)T(y)dy - \beta\lambda^*(0) = A - \frac{\pi-2}{\delta} - \frac{1}{2}\beta\delta$$

This means that the *equilibrium utility level decreases with the travel cost* τ and *increases with the preference for residential space* β *provided that* $\tau < \pi/2 - 1$.

3.2 First-best Spatial Distribution

In this subsection we determine the first-best distribution of agents as opposed to the equilibrium distribution analyzed so far. The utilitarian planner compensates agents with lump sump transfers $r(x)$ and the utility function is assumed to be transferable so that $U(x) = V(x) + r(x)$. The planner maximizes the total welfare

$$\max_{\lambda(.)} W = \textstyle\int_{-b}^{b} U(x)\lambda(x)dx$$

subject to the budget balance $\int_{-b}^{b} r(x)\lambda(x)dx = 0$, and $[-b, b]$ denotes the support of the optimal city. Because of the budget balance, the total welfare is independent of the transfer distribution and is equal to $W = \int_{-b}^{b} V(x)\lambda(x)dx$. In Appendix $A.1$, we show that the first-best distribution satisfies the following relation

$$S(x) - \beta\lambda(x) = A/2 \tag{5}$$

This relationship implies that *at the optimum,* $V(x)$ *is constant* and equal to $A/2$. Since $U(x) = A/2 + r(x)$, the planner does not need any redistributive instrument to guarantee the equal treatment of residents at the socially optimal spatial distribution. The transfers $r(x)$ are useless instruments for the planner and can be set to zero.

Furthermore, the first-best distribution is also characterized by $V'(x) = V''(x) = 0$. This leads to the same general solution as in relation (4), $\lambda^{FB}(x) = c\cos\delta x$ where $\delta^2 = 4\tau/\beta$, where c is a constant to be determined. Since the city has to host the total population, $\int \lambda(x)dx = 1$, we have that

$$c^{FB} = \frac{1}{2}\frac{\delta}{\sin\delta b}$$

Since the utility level $S(x) - \beta\lambda(x)$ is equal to $A/2$ for any x, the boundary $b^{FB}$of the first-best city has to solve

$$\delta b + \cot\delta b = A/(\delta\beta) \tag{6}$$

This equation has a unique solution when $b \in (0, \pi/\delta)$. Yet we must ensure that each agent interacts with *all* the others, that is, $A > T(2b) = 4\tau b \iff A/(\delta\beta) > \delta b$. Under this restriction, the unique solution b belongs to the interval $(0, \pi/(2\delta))$. Therefore, we get $b^{FB} < b^* = \pi/(2\delta)$. The first-best city has a narrower support than the equilibrium city. Because the first-best and the equilibrium cities host the same number of residents, the density of residents must be larger at the first-best ($c^{FB} > c^*$). Finally, because the first-best and the equilibrium distributions are cosine functions with same spatial frequency δ, the borders of the first-best city must be densely populated ($\lambda(b^{FB}) > 0 = \lambda(b^*)$).

Proposition 2 The first-best city can be implemented without the use of spatial transfers and has a narrower support than that of the equilibrium distribution ($\lambda^{FB}(x) = c^{FB}\cos\delta x$, with $c^{FB} > c^*$ and $b^{FB} > b^*$).

4 Spatial Equilibrium along a Circle

In this section we consider the spatial interactions taking place along a circular geographical space. Because agents may access to other agents by travelling to the right or to the left, they will be sensitive to the fact that other agents may be located in the opposite location along the circumference. Our main result is that spatial equilibria may involve multiple cities. Yet, the equilibrium characterization is more difficult to obtain along the circumference than along the line segment. A major contribution of this paper is to provide a full characterization of multiple agglomerations in equilibrium.

To obtain this result, we proceed in several steps. As in previous section, we first derive a necessary equilibrium condition (Lemma 1). This condition expresses the trade-off between the residence cost and the accessing cost to other agents. We then derive another necessary equilibrium condition (Lemma 2) which simply states that an equilibrium distribution is of made of pieces, each of which corresponds to the cosine function as determined in previous section. Then we show that in equilibrium cities can't face each other along the circumference (Lemma 3), which subsequently implies that no equilibrium with an even number of cities, can exist (Lemma 4). Finally we show that in equilibrium cities are equally populated and equally-spaced along the circumference (Lemma 5). All those results are summarized in Proposition 3. Whereas it may be intuitive that these spatial distributions constitute equilibria, it is far from obvious a priori to exclude other asymmetric patterns in terms of size or location. Proofs are relegated to Appendix B.

We discuss spatial configurations involving cities separated by empty hinterlands. M denotes the total number of cities and $[a_m, b_m]$ the support of city m so that the support of λ can be written as supp $\lambda = \bigcup_{m=1}^{M}[a_m, b_m]$. Let H be the set of empty hinterlands, i.e. 'empty' locations where $\lambda(x) = 0$. Thus, $H = [0, 1]/\text{supp } \lambda$. At equilibrium we must have that $V(x) = V^*$, $\forall x \in \text{supp}\lambda$ and $V(x) < V^*$, $\forall x \in H$.

Consider some agent located in city k so that $x \in [a_k, b_k]$. We define $P^+(x)$ (resp. $P^-(x)$) as the population that is located at a clockwise (resp. counterclockwise) distance from x smaller than $1/2$. This means that $P^+(x)$ and $P^-(x)$ divide the total population into that at the right and that at the left of x.

Lemma 1 In equilibrium $P^+(x) - P^-(x) = \lambda'(x)[\beta/(2\tau)]$, $\forall x \in supp\ \lambda$.

This condition expresses the equilibrium trade-off between the residence cost and the accessing cost: an increase in residence cost must be compensated by a better access to distant agents. So as to illustrate Lemma 1, suppose that $\lambda'(x) < 0$, so that by moving to his right, an agent enjoys a lower residence cost. Lemma 1 says that this gain in terms of residence cost is balanced by a larger accessing cost. This means that the population that the agent gets closer to, that is the population at his right (P^+), is less numerous that the population he gets further away from, that is the population at his left (P^-). *The marginal cost of residence by moving to the right or to the left corresponds to the marginal gain of accessing to agents.*

Differentiating once more the indirect utility yields another necessary condition $V''(x) = 0$. Each piece of the equilibrium distribution is determined by a spatial structure similar to the one given by expression (4).

This city structure is summarized in the following Lemma.

Lemma 2 Suppose that $\lambda(x)\lambda(x+1/2) = 0$ for every location $x \in [a_k, b_k]$, $k = 1, ..., M$. Then, an equilibrium distribution is given by $\lambda(x) = c_k \cos \delta(x - x_k)$ where $\delta^2 = 4\tau/\beta$, $\delta(b_k - a_k) = \pi$, $x_k = (b_k - a_k)/2$, and c_k is a positive constant.

Therefore, *the shape of cities is given by the same cosine function* as in the case of a line segment. Lemma 2 applies when $\lambda(x)\lambda(x + 1/2) = 0$, that is when *cities do not face each other along the circumference*, meaning that if location x is inhabited then location $x + 1/2$ should not be inhabited. As an illustration, let us show that a spatial configuration consisting of 2 symmetric cities as determined by Lemma 2, located at the North and the South of the circumference ($x = 0$ and $x = 1/2$) can't be sustained in equilibrium. The supports of these cities are $[-b, b]$ and $[1/2 - b, 1/2 + b]$. By applying Lemma 1 in locations x and $x + 1/2$, we get

$$\lambda'(x) + \lambda'(x + 1/2) = \frac{2\tau}{\beta}\left[P^+(x) - P^-(x) + P^+(x + 1/2) - P^-(x + 1/2)\right]$$

This expression is equal to 0 given that $P^+(x) = P^-(x+1/2)$ and $P^-(x) = P^+(x+1/2)$, which leads to an inconsistency given that in our example, $\lambda'(x) = \lambda'(x + 1/2) \neq 0$ if

$x \neq 0$. The above condition says that if $\lambda'(x) > 0$ then $\lambda'(x + 1/2) < 0$. The following lemma generalizes this result to the case of any admissible spatial distribution.

Lemma 3 Generically there exists no spatial equilibrium - except the uniform distribution - that involves cities facing each other along the circumference, that is $\nexists\ x$ s.t. $\lambda(x)\lambda(x + 1/2) \neq 0$.

An implication of this Lemma is that the spatial equilibrium distribution involves either empty hinterlands or a uniform distribution of agents (the flat earth distribution). Note that spatial distributions corresponding to $c \cos \delta x$, $\forall x \in [0, 1]$, are equilibria to the extent that δ^{-1} is a multiple of the perimeter of the circumference, that is $\delta^{-1} \in \mathbb{N}^{+}$. These equilibria are clearly not generic. If we omit these non-generic equilibria, the uniform distribution is the only spatial distribution with no hinterland.

It turns out that a consequence of Lemma 3 is to exclude the possibility of having an even number of cities. As an illustration, let us explain the argument for a configuration involving an even number of identical cities. By Lemma 3, we know that these cities can't be located symmetrically along the circumference. By applying Lemma 1 at the centre x_m of a city, we get that $P^{+}(x_m) = P^{-}(x_m)$ because the land rent gradient is nil at the centre ($\lambda'(x_m) = 0$). This means that the populations at the right and the left of the city centre x_m are equal, which is inconsistent with our example since one side of the city will involve an even number of cities while the other side will involve an odd number of cities, given that the total number of cities is even. In this illustration, the argument applies because cities are of equal size. The following Lemma extends this argument to spatial distributions involving cities of different size.

Lemma 4 Amy non-uniform spatial equilibrium displays an odd number of cities.

What is left to be determined is the size of cities and their location along the circumference. In Appendix B, we apply the argument used in Lemma 4 to pairs of cities located on opposite sides of the circumference. It can then be shown that such pairs of cities have an identical population size at equilibrium. By inference, all cities must have the same size. Furthermore, once cities have an identical size, it is easy to understand why they should be equally-spaced along the circle. It is because any asymmetry in the location of these cities would necessarily confer an advantage to residents of some city and a disavantage to residents of some other city, thus precluding equilibrium. We summarize our results in the following Proposition.

Proposition 3 Any non-uniform spatial equilibrium configuration displays an odd number of identical equally-spaced cities.

In contrast to Beckmann's result, *multiple-city configurations do emerge along a circular geographical space. They imply the existence of empty hinterlands and a high degree of spatial symmetry in terms of size and location.* According to Proposition 3, *configurations with an even number of cities are excluded.*

5 Pareto-Ranking of Equilibria and Optimum

In this section we rank spatial equilibria involving many cities and the uniform distribution of agents in the sense of Pareto. We then compare the Pareto dominating equilibrium with the first-best distribution. Proofs are relegated to Appendix C.

Consider a spatial equilibrium with an odd number M of identical equally-spaced cities. With no loss of generality, we assume that the first city is located at $x = 0$. In equilibrium, the support of cities should be fit the unit perimeter of the circumference ($2M\pi/\delta < 1$), so that the maximum number of cities is given by $M^{\max} = int(\delta/(2\pi))$. On the other hand, since the total population is 1, we have that $M \int_{-\frac{\pi}{2\delta}}^{\frac{\pi}{2\delta}} c\cos(\delta x)\, dx = 1$, meaning that $c = \delta/(2M)$. These two last conditions put a bound $M^{\max}$ on the admissible number of cities and relate the size c of a city to the total number of cities M.

In equilibrium the utility is the same for all residents and can be identified to the utility of the resident located at $x = 0$, which is given by

$$V^*(M) = A - \beta\lambda(a_1 + \frac{\pi}{2\delta}) - \sum_{i=1}^{M} \int_{a_i}^{a_i+\frac{\pi}{\delta}} T(a_1 + \frac{\pi}{2\delta}, y)\lambda(y)dy$$

where a_i corresponds to the left-border of city i. Developments given in Appendix $C.1$

lead to

$$V^*(M) = A - \underbrace{\tau\frac{\pi - 2}{\delta M}}_{\text{Cost of accessing to his own city}} - \underbrace{\tau\frac{M^2 - 1}{2M^2}}_{\text{Cost of accessing to other cities}} - \underbrace{\frac{2\tau}{\delta M}}_{\text{Residence cost}}$$

The first term represents the benefit of social interaction, the second one the agent's travel cost to other agents in their own city, the third one the travel cost to agents in other cities, and the last one the land rent. It can be shown that $V^*(M)$ is a decreasing function in the admissible interval $[1, \delta/(2\pi)]$.

Proposition 4 If $\delta > 2\pi$ (resp. $\delta < 2\pi$), then the Pareto dominating spatial configuration involves a single city. Otherwise it corresponds to the uniform distribution (flat earth).

Spatial equilibria can be ranked in the sense of Pareto: *the smaller the number of cities, the larger the total welfare of the equilibrium distribution*. Of course, when no city can fit the unit perimeter, then the only possible equilibrium is the flat earth distribution, $\lambda(x) = 1$.

We now determine the first-best distribution of residents along the circumference.

Proposition 5 When $\delta > \pi$ (resp. $\delta < \pi$), the optimal spatial configuration corresponds to a single city (resp. the uniform spatial distribution of agents).

As in Beckmann's framework, *the social optimum involves a single city which is more concentrated than the equilibrium distribution*. Of course, this occurs provided that the optimal city can fit the unit perimeter. Otherwise, the first-best corresponds to the uniform distribution of agents. While an increase of the travelling cost τ favours the optinal agglomeration, an increase of the preference for residential space β favors the optimal uniform distribution of residents.

6 Conclusion

We have studied a spatial model of social interactions. We have shown that in Beckmann's framework -that is, along a line segment- only a single city emerges. On the other hand, along a circle, our model leads to the endogenous formation of multiple-centre configurations. Because of the linear structure of the model, we are able to perform a full general equilibrium analysis and to characterize equilibrium configurations. Cities are identical and equally-spaced along the circle. The smaller the number of cities, the larger the total welfare of the spatial economy. The first-best distribution corresponds to a single city which is more concentrated than the equilibrium city.

Appendix

Appendix A: Proof of $S(x) - \beta\lambda(x) = A/2$

Proof. Total welfare can successively be expressed as

$$\begin{aligned} W &= \int_{-a}^{a}[A - \int_{-a}^{a}\lambda(y)T(x-y)dy - \beta\lambda(x)]\lambda(x)dx \\ &= \int_{-a}^{a}\left[A\lambda(x) - \beta\lambda(x)^2\right]dx - \int_{-a}^{a}\int_{-a}^{a}T(x-y)\lambda(y)\lambda(x)dydx \end{aligned}$$

Consider any infinitesimally small variation $\widetilde{\lambda}(x)$ around the optimal solution $\lambda(x)$. Then the variation of the functional W is given by variation of the first term in the above integral

$$\int_{-a}^{a}A\widetilde{\lambda}(x) - 2\beta\widetilde{\lambda}(x)\lambda(x)dx$$

and by variation of the second term in that integral

$$\begin{aligned} &\int_{-a}^{a}\int_{-a}^{a}\left[T(x-y)\widetilde{\lambda}(y)\lambda(x) + T(x-y)\lambda(y)\widetilde{\lambda}(x)\right]\ dydx \\ &= \int_{-a}^{a}\int_{-a}^{a}T(x-y)\widetilde{\lambda}(y)\lambda(x)\ dydx\ + \int_{-a}^{a}\int_{-a}^{a}T(x-y)\lambda(y)\widetilde{\lambda}(x)\ dydx \\ &= \int_{-a}^{a}\int_{-a}^{a}T(y-x)\widetilde{\lambda}(x)\lambda(y)\ dxdy\ + \int_{-a}^{a}\int_{-a}^{a}T(x-y)\lambda(y)\widetilde{\lambda}(x)\ dxdy \\ &= \int_{-a}^{a}\int_{-a}^{a}\left[T(y-x) + T(x-y)\right]\widetilde{\lambda}(x)\lambda(y)\ dydx \\ &= \int_{-a}^{a}\int_{-a}^{a}2T(x-y)\lambda(y)dy\ \widetilde{\lambda}(x)dx \end{aligned}$$

where we substitute x for y in the first term in the second equality and where we use symmetry of $T(x)$ in the last equality. Hence the variation of the objective W is equal to

$$\widetilde{W} = \int_{-a}^{a}\left\{\left[A\widetilde{\lambda}(x) - 2\beta\widetilde{\lambda}(x)\lambda(x)\right] - \int_{-a}^{a}2T(x-y)\lambda(y))dy\widetilde{\lambda}(x)\right\}dx$$

At the optimum, $\widetilde{W}$ must be equal to zero for any $\widetilde{\lambda}(x)$ around the optimal $\lambda(x)$. This implies that $A - 2\beta\lambda(x) - \int_{-a}^{a} 2T(x-y)\lambda(y)dy = 0$. By using the definition of $S(x)$, we get finally get $S(x) - \beta\lambda(x) = A/2$. ■

Appendix B.1: Proof of Lemma 1

Proof. Let us define I_k^+ (resp. I_k^-) to be set of indices of cities that are located at a clockwise (resp. counterclockwise) distance from interval k inferior to 1/2. We consider an agent located at $x \in [a_k, b_k]$. When $x + 1/2 \notin H$, we denote by j_k the interval index to which $x + 1/2$ belongs to. The utility of an agent located in city k can be written as

$$\begin{aligned} V(x) = A &- 2\tau[\textstyle\sum_{i\in I_k^+} \int_{a_i}^{b_i} (y-x)\lambda(y)dy + \sum_{i\in I_k^-} \int_{a_i}^{b_i} (1-(y-x))\lambda(y)dy] \\ &- 2\tau[\textstyle\int_{a_k}^{x} (x-y)\lambda(y)dy + \int_{x}^{b_k} (y-x)\lambda(y)dy] - \beta\lambda(x) \\ &- 2\tau\chi_{\mathrm{supp}\lambda}(x+1/2)\left[\textstyle\int_{a_{j_k}}^{x+1/2} (y-x)\lambda(y)dy + \int_{x+1/2}^{b_{j_k}} (1-(y-x))\lambda(y)dy\right] \end{aligned}$$

where $\chi_{\mathrm{supp}\lambda}$ denotes a characteristic function so that $\chi_{\mathrm{supp}\lambda}(x)$ is equal 1, if $x \in$suppλ, and 0 otherwise. By differentiation with respect to x, we get

$$-2\tau\textstyle\sum_{i\in I_k^+} \int_{a_i}^{b_i} (-1)\lambda(y)dy - 2\tau\sum_{i\in I_k^-} \int_{a_i}^{b_i} (1)\lambda(y)dy - 2\tau\int_{a_k}^{x} (1)\lambda(y)dy - 2\tau\int_{x}^{b_k} (-1)\lambda(y)dy$$

$$+2\tau\chi_{\mathrm{supp}\lambda}(x+1/2)\left[\textstyle\int_{a_{j_k}}^{x+1/2} \lambda(y)dy - \int_{x+1/2}^{b_{j_k}} \lambda(y)dy\right] - \beta\lambda'(x) = 0$$

We get the stated result by writing $P^+(x) = \left(\sum_{i\in I_k^+} \int_{a_i}^{b_i} + \int_{x}^{b_k} + \chi_{\mathrm{supp}\lambda}(x+1/2)\int_{a_{j_k}}^{x+1/2}\right)\lambda(y)dy$ and $P^-(x) = \left(\sum_{i\in I_k^-} \int_{a_i}^{b_i} + \int_{a_k}^{x} + \chi_{\mathrm{supp}\lambda}(x+1/2)\int_{x+1/2}^{b_{j_k}}\right)\lambda(y)dy$. ■

Appendix B.2. : Proof of Lemma 2

Proof. Let us define I_k^+ (resp. I_k^-) to be set of indices of cities that are located at a clockwise (resp. counterclockwise) distance from interval k inferior to 1/2. We consider an agent located at $x \in [a_k, b_k]$ such that $x + 1/2 \in H$. By the proof of Lemma 1 in Appendix B.1 the FOC $V'(x) = 0$ can be written as

$$\begin{aligned} &-2\tau\textstyle\sum_{i\in I_k^+} \int_{a_i}^{b_i} (-1)\lambda(y)dy - 2\tau\sum_{i\in I_k^-} \int_{a_i}^{b_i} (1)\lambda(y)dy \\ &-2\tau\textstyle\int_{a_k}^{x} (1)\lambda(y)dy - 2\tau\int_{x}^{b_k} (-1)\lambda(y)dy - \beta\lambda'(x) = 0 \end{aligned}$$

By further differentiation we get

$$-2\tau\lambda(x)-2\tau\lambda(x)-\beta\lambda^{''}(x)=0$$

$$\lambda''(x)+\delta^2\lambda(x)=0$$

where $\delta^2=4\tau/\beta$. The general solution to this differential equation is given by

$$\lambda(x)=c_k\cos[\delta\,(x-x_k)]$$

where c_k and x_k are constants. Note that $\lambda(a_k)$ and $\lambda(b_k)$ can't be strictly positive. For instance, if $\lambda(b_k)$ were strictly positive, then agents in location b_k would have an incentive to move to the hinterland in location $b_k+\varepsilon$ with $\varepsilon>0$. By doing so they would save a finite marginal residence cost while facing only an infinitesimal marginal accessing cost. Therefore $\delta(b_k-a_k)=\pi$ and $x_k=(b_k-a_k)/2$. ■

Appendix B.3: Proof of Lemma 3

In order to prove Lemma 3 we need two other Lemmas (B.1), (B.2).

Lemma B.1 If an agent were located in $x\in[a_k,b_k]$ with $x+1/2\in$suppλ, then the equilibrium distribution would be given by $\lambda(x)=C+c\cos[\delta'(x-x_k)]$ where $\delta'^2=2\delta^2=8\tau/\beta$ and c, C and x_k are some constants.

Proof. Consider an agent located at $x\in[a_k,b_k]$ such that $x+1/2\in$suppλ -i.e. $x+1/2$ does not belong to the hinterland-. From the proof of Lemma 1 in Appendix B.1, the FOC $V'(x)=0$ can be written as

$$\begin{aligned}-2\tau{\textstyle\sum_{i\in I_k^+}}\int_{a_i}^{b_i}(-1)\lambda(y)dy-2\tau{\textstyle\sum_{i\in I_k^-}}\int_{a_i}^{b_i}(1)\lambda(y)dy-2\tau\int_{a_k}^{x}(1)\lambda(y)dy\\ -2\tau\int_{x}^{b_k}(-1)\lambda(y)dy+2\tau\left[\int_{a_{j_k}}^{x+1/2}\lambda(y)dy-\int_{x+1/2}^{b_{j_k}}\lambda(y)dy\right]-\beta\lambda'(x)=0\end{aligned}$$

By further differentiation we get

$$-4\tau\lambda(x)+4\tau\lambda(x+1/2)-\beta\lambda''(x)=0$$

By applying Lemma 1 in locations x and $x+1/2$, we have that $\lambda'(x)+\lambda'(x+1/2)=0$ so that $\lambda(x+1/2)=C-\lambda(x)$, and the previous equation leads to

$$\lambda''(x)+\delta'^2\lambda(x)=C'$$

where ${\delta'}^2 = 2\delta^2 = 8\tau/\beta$, $C' = 4\tau C/\beta$. The solution general to this equation

$$\lambda(x) = C'' + c'_k \cos[\delta'(x - x'_k)]$$

where C'', c'_k and x'_k are constants. ■

We now construct a spatial configuration with two cities that face each other along the circumference. W.l.o.g. the two cities are centered in locations $x = 0$ and $x = 1/2$. The large city lies in the interval $[-a, a]$ while the small one lies on the opposite side of the circumference in the interval $[1/2 - a', 1/2 + a']$, with $a > a' > 0$. Hence we have a range $(-a', a')$ in which $\lambda(x)\lambda(x + 1/2) > 0$ and two ranges $[-a, -a']$ and $[a', a]$ such that $\lambda(x) > 0$ and $\lambda(x + 1/2) = 0$. Land rent are zero at the city boundaries so that $\lambda(a) = \lambda(-a) = \lambda(1/2 - a') = \lambda(1/2 + a') = 0$. Note that in the intervals $(a, 1/2 - a')$ to the right of the large city, and $(1/2 + a', 1 - a)$ to the left of the large city, there may be other cities. We characterize the shape of any candidate equilibrium distribution. In this candidate distribution, the small city displays k bumps and the large city $k + 1$ bumps.

Lemma B.2 If two cities were to face each other in locations 0 and 1/2, then they would have necessarily the following shape

$$\begin{array}{ll} \lambda(x) = C\left[1 + (-1)^k \cos\delta' x\right] & \text{for } x \in [-a', a'] \\ \lambda(x) = 2C\cos\delta\,(x - a') & \text{for } x \in [-a, -a'] \cup [a', a'] \\ \lambda(x) = C\left[1 - (-1)^k \cos\delta'\,(x - 1/2)\right] & \text{for } x \in [1/2 - a', 1/2 + a'] \end{array}$$

where $a' = k\pi/\delta'$ with $k \in \mathbb{N}_{++}$, $a = a' + \pi/(2\delta)$, and C is a positive constant. Also there would be no other city in $[1/2 - a, 1/2 - a\prime] \cup [1/2 + a', 1/2 + a]$.

Proof. Since the two cities face each other in the intervals $(-a', a')$ and $(1/2-a', 1/2+a')$, Lemmas $B.1$ gives

$$\begin{array}{ll} \lambda(x) = C - c\cos\delta' x & , x \in (-a', a) \\ \lambda(x) = C + c\cos\delta'(x - 1/2) & , x \in (1/2 - a', 1/2 + a') \end{array}$$

Since λ is positive, $C \geq |c|$. Since $\lambda(1/2 \mp a') = 0$, this gives $C = |c|$, and if $c > 0$ (resp. $c < 0$), then $\delta' a' = (2k - 1)\pi$ (resp. $\delta' a' = 2k\pi$), $k \in \mathbb{N}_{++}$. Thus the distribution can be written as

$$\lambda(x) = C - (-1)^k C \cos \delta' x \qquad , x \in (-a', a)$$

$$\lambda(x) = C + (-1)^k C \cos \delta'(x - 1/2) \quad , x \in (1/2 - a', 1/2 + a')$$

with some constant $C > 0$ and $\delta' a' = k\pi$, $k \in \mathbb{N}_{++}$.

Since in the intervals $(-a, -a')$ and (a', a) the large city doesn't face any other city, Lemma 2 gives $\lambda(x) = c' \cos \delta(x - x')$. Because V and λ are continuously differentiable, $\lambda(a'^{+}) = \lambda(a'^{-})$ and $\lambda'(a'^{+}) = \lambda'(a'^{-})$. This implies $c' = 2C$ and $x' = a'$. Finally, since $\lambda(a) = 0$, we have that $\delta(a - a') = \pi/2$. ■

We now prove Lemma 3.

Proof. As mentioned before already, in the intervals $(a, 1/2 - a')$ to the right of the larger city, and $(1/2 + a', 1 - a)$ to the left of the larger city, there may be other cities. Their total population $Q^+ = \int_a^{1/2-a'} \lambda(y)dy$ is located at the mean distance $x_{Q^+} = \int_a^{1/2-a'} y\lambda(y)dy/Q^+$. Similarly, in the interval $(1/2 + a', 1 - a)$ to the left of the larger city there may be other cities that we summarize by a total population Q^- located at a mean distance x_{Q^-}. Thus $(Q^+, Q^-, x_{Q^+}, x_{Q^-})$ are exogenous parameters whereas (a, a') are to be determined by equilibrium conditions.

The total population is given by the relation

$$\int_{-a}^{a} \lambda^*(y)dy + \int_{1/2-a'}^{1/2+a'} \lambda^*(y)dy + Q^+ + Q^- = 1 \tag{7}$$

We now determine the indirect utility at locations $x = 0$ and at $x = 1/2$

$$V(0) = A - 4\tau \int_0^a y\lambda^*(y)dx - 4\tau \int_{1/2-a'}^{1/2} y\lambda^*(y)dx - 2\tau x_{Q^+} Q^+ - 2\tau(1 - x_{Q^-})Q^- - \beta \underbrace{\lambda(0)}_{0 \text{ or } 2C}$$

$$V(1/2) = A - 4\tau \int_0^a (1/2 - y)\, \lambda^*(y)dx - 4\tau \int_{1/2-a'}^{1/2} (1/2 - y)\, \lambda^*(y)dx$$
$$- 2\tau\, (1/2 - x_{Q^+})\, Q^+ - 2\tau\, (x_{Q^-} - 1/2)\, Q^- - \beta \underbrace{\lambda(1/2)}_{2C \text{ or } 0}$$

Thus, $V(1/2) = V(0)$ implies

$$0 = 2 \int_0^a (2y - 1/2)\, \lambda^*(y)dx + 2 \int_{1/2-a'}^{1/2} (2y - 1/2)\, \lambda^*(y)dx$$
$$+ (2x_{Q^+} - 1/2)\, Q^+ + (-2x_{Q^-} + 3/2)\, Q^- \mp \beta c$$

By using expression (7) we get

$$2\int_0^a y\lambda^*(y)dx + 2\int_{1/2'}^{1/2+a'} y\lambda^*(y)dx + x_{Q^+}Q^+ + x_{Q^-}Q^- = 1/4 \tag{8}$$

This means that for a given configuration of cities $(Q^+, x_{Q^+}, Q^-, x_{Q^-})$, expressions (7) and (8) define a system two equations with one unknown C. Because expression (7) is a sum of $\int \lambda^*(y)dx$ and expression (8) is a sum of $\int y\lambda^*(y)dx$ they cannot be linear combinations for any measurable set of parameters (β, τ). This system is therefore over-determined and there exists no solution C that solves the equilibrium. ■

Appendix B.4: Proof of Lemma 4

Proof. By applying Lemma 1 at the centre x_k of each city k, we get $P^+(x_k) - P^-(x_k) = 0$, $k = 1, 2, ..., n$. These conditions can be written in the following matrix form

$$\underbrace{\begin{bmatrix} 0 & a_{12} & \cdots & a_{1M} \\ -a_{12} & 0 & \cdots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1M} & -a_{2M} & \cdots & 0 \end{bmatrix}}_{A} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_M \end{bmatrix} = 0$$

where $a_{ij} \in \{-1, +1\}$ indicates whether $j \in I_i^-$ (city j is a right-neighbor of city i) or $j \in I_i^+$ (city j is a left-neighbor of city i). We refer to matrix A as the 'neighboring' matrix.

It turns out that the determinant of a matrix can be expressed as $\det A = \sum_{\gamma\in\Gamma} \varepsilon(\gamma) \prod_{\gamma_i} a_{i\gamma_i}$, where γ is a permutation of $\{1, 2, ..., M\}$, Γ the set of derangements of $\{1, 2, ..., M\}$, and $\varepsilon : \Gamma \to \{-1, 1\}$. Given that the number of such derangements is odd when M is even and $a_{ij} \in \{-1, 1\}$ for $j \neq i$, $\det A$ corresponds to a sum of an odd number of terms equal to -1 or $+1$. Therefore, whenever M is even, $\det A$ is non-zero and the only solution to $AP = 0$, is $P = 0$. Note that when M is odd, $\det A = 0$ because $A = -A^T$. ■

Appendix B.5: Proof of Proposition 3

We firstly prove that cities are equally populated and then that they are equidistant.

Let M be an odd number of cities that are clockwisely indexed as $i = 1, 2, ..., M$, let P_i be the population of city i (or $i - kM$ if $Mk < i \leq M(k+1)$, $k \in \mathcal{N}$) and let $\#I_i^+$ (resp. $\#I_i^-$) to number of cities that are located the right (resp. left) of the centre of city i.

We firstly need to define a concept of symmetry in the location of cities.

Definition (Neighborhood Symmetry) Spatial distribution displays the neighborhood symmetry if each city has the same number of cities on its left and on its right: $\#I_i^+ = \#I_i^- = (M-1)/2$, $\forall i$.

We also define the pairs of cities located on opposite sides of the circumference as it follows.

Definition (Paired Cities) Consider the centre x_i of some city i. Cross the disk and reach the symmetric location $x + 1/2$ just in front. Move clockwise (resp. counterclockwise) to the next first city, say city j. Then cross back the disk and reach the symmetric location of city $x_{j+1/2}$. Move counterclockwise (resp. clockwise) back to city i. If no other city is met before reaching city i, we say that cities i and j are clockwise (resp. counterclockwise) paired.

Given this definition, we readily get the three following lemma.

Lemma B.5.1 If cities i and j are paired, then $P_i = P_j$.

Proof. By applying Lemma 1 at the centres of cities i and j. ■

Lemma B.5.2 Under neighborhood symmetry, $P_i = \overline{P}$, $\forall i$.

Proof. Under neighborhood symmetry, each city can be clockwise and counterclockwise paired. By contradiction. Assume that some city i can't be paired. We know that it has $(M-1)/2$ right- and left- neighbors. If it can't be paired then one meets at least one other city when coming back to city i in the last stage of the pairing construction. Then apply Lemma 1 to the centre of that city. This necessarily violates the neighborhood symmetry.

Then each city i can be paired to cities $i+\frac{M+1}{2}$ and $i+\frac{M-1}{2}$, and $P_i = P_{i+\frac{M+1}{2}} = P_{i+\frac{M-1}{2}}$. This implies that $P_i = P_{i+1}$, $\forall i$. (Note: we need a notation for $i+\frac{M+1}{2}$ and $i+\frac{M-1}{2}$) ■

Lemma B.5.3 Neighborhood symmetry holds.

Proof. We show that if neighborhood symmetry didn't hold, then there would exist a city with negative population.

Step 1. If neighborhood symmetry does not hold, then $\exists i$: city i can't be paired clockwise. This is because if neighborhood symmetry does not hold, then $\exists i$: the number of right neighbors $\neq$ number of left neighbors. Consider the clockwise pairing of cities, but city i. At maximum, $2\min$(number of right neighbors, number of left neighbors) can be paired clockwise. This means at least max(number of right neighbors, number of left neighbors)-min(number of right neighbors, number of left neighbors) cities remain unpaired among the $M-1$ cities. This number is necessarily even. Even by taking care of the clockwise pairing of city i, there will always remain at least a city that can't be paired.

Step 2. Partition cities in cities that can be paired clockwise and cities that cannot be paired clockwise. Take two neighbor cities i (that can't be paired clockwise) and $i+1$ (that can be paired clockwise). Apply Lemma 1 at the centre of cities i and $i+1$. Then apply FOC1 at $i+1$. This implies that $P_i + P_{\text{city paired to } i+1} = 0$ so that population of some city should be negative. ■

It naturally follows that cities are equally populated.

Lemma B.5.4 Neighborhood symmetry holds and all cities are equally populated, $P_i = \overline{P} = 1/M$, $\forall i$.

Given this lemma we can show the following result. Let city centres are denoted by x_i, $i = 1, ..., M$.

Lemma B.5.5 Cities are equidistant: $x_i - x_{i-1} = 1/M$.

Proof. By Lemma B.5.4, we know that $P_i = P/M$. The interaction costs for agents located in city centres are given by

$$i < (M-1)/2 : C_i = 2\tau\{\sum_{k=1}^{k=i+(M-1)/2} |x_k - x_i| + \sum_{k=i-(M-1)/2+M}^{M} [1 - (x_k - x_i)\}$$
$$i = (M-1)/2 : C_{(M-1)/2} = 2\tau\{\sum_{k \neq (M-1)/2} \left|x_k - x_{(M-1)/2}\right|$$
$$i > (M-1)/2 : C_i = 2\tau\{\sum_{k=i-(M-1)/2+1}^{k=M} |x_k - x_i| + \sum_{k=1}^{i+(M-1)/2-M} [1 - (x_i - x_k)]\}$$

Because of the neighborhood symmetry and because these costs C_i should be equal -say to C-, we have that

$$Ax = b$$

where $b^T = [C-(M-1)/2, ..., C-1, C, C+1, ..., C+(M-1)/2]/(2\tau)$, and A is the neighborhood matrix introduced in Appendix $B.4$.

It turns out that matrix A has rank $M-1$. This is because the minor (i,i) of A is a neighborhood matrix corresponding a configuration where city i has been removed, and thus is of $M-1$ since the determinant of a neighborhood matrix is non zero when the number of cities is even, see Proof of Lemma 4 in Appendix $B.4$. Then the only solution to $Ax = b$ is necessarily $x_i - x_{i-1} = M^{-1}$, $\forall i$. ■

Appendix C.1: Ranking of Equilibria

Consider an equilibrium with an odd number M odd of identical equidistant cities. The equilibrium utility is given by

$$
\begin{aligned}
V^*(M) &= A-\beta\lambda(a_1+\frac{\pi}{2\delta})-\sum\nolimits_{i=1}^{M}\int_{a_i}^{a_i+\frac{\pi}{\delta}}T(a_1+\frac{\pi}{2\delta},y)\lambda(y)dy \\
&= A-\beta\lambda(a_1+\frac{\pi}{2\delta})-\int_{a_1}^{a_1+\frac{\pi}{\delta}}T(a_1+\frac{\pi}{2\delta},y)\lambda(y)dy \\
&-\sum\nolimits_{i=2}^{\frac{M+1}{2}}\int_{a_i}^{a_i+\frac{\pi}{\delta}}T(a_1+\frac{\pi}{2\delta},y)\lambda(y)dy-\sum\nolimits_{i=\frac{M+1}{2}+1}^{M}\int_{a_i}^{a_i+\frac{\pi}{\delta}}T(a_1+\frac{\pi}{2\delta},y)\lambda(y)dy \\
&= A-\beta\lambda(a_1+\frac{\pi}{2\delta})-2\tau\int_{a_1}^{a_1+\frac{\pi}{2\delta}}(a_1+\frac{\pi}{2\delta}-y)\lambda(y)dy-2\tau\int_{a_1+\frac{\pi}{2\delta}}^{a_1+\frac{\pi}{\delta}}(y-(a_1+\frac{\pi}{2\delta}))\lambda(y)dy \\
&-2\tau\sum\nolimits_{i=2}^{\frac{M+1}{2}}\int_{a_i}^{a_i+\frac{\pi}{\delta}}(y-(a_1+\frac{\pi}{2\delta}))\lambda(y)dy-2\tau\sum\nolimits_{i=\frac{M+1}{2}+1}^{M}\int_{a_i}^{a_i+\frac{\pi}{\delta}}(1-(y-(a_1+\frac{\pi}{2\delta})))\lambda(y)dy \\
&= A-\frac{4\tau}{\delta^2}\frac{\delta}{2M}-2\tau\frac{\pi-2}{\delta^2}\frac{\delta}{2M}-\tau\frac{M^2-1}{2M^2} \\
&= A-\underbrace{\frac{2\tau}{\delta M}}_{\text{Residence Cost}}-\underbrace{\tau\frac{\pi-2}{\delta M}}_{\text{Accessing Cost within the city}}-\underbrace{\tau\frac{M^2-1}{2M^2}}_{\text{Accessing Cost to other cities}}
\end{aligned}
$$

6.1 Appendix C.2: Proof of Proposition 4

Proof. As $\partial_M V^* = \frac{M\pi-\delta}{M^3\delta}\tau$, $\partial_M V^* = 0$ for $M = \delta/\pi > 1$ since $\delta > 2\pi$. This means that $\partial_M V^* < 0$ in the interval $[1, \delta/(2\pi)]$. Thus $V^*(M)$ decreases with M, and the maximum of $V^*(M)$ is reached when $M = 1$. The flat-earth welfare is given by $V(\text{flat earth}) = \int_0^1 \left[A-\beta-\int_0^1 T(x,y)dy\right]dx = A-\beta-\frac{\tau}{2}$. It is always inferior to $V^*(M=1)$ when the single city is an equilibrium. ■

6.2 Appendix C.3: Proof of Proposition 5

We now derive the first best spatial distribution on the perimeter of the unit circumference (Proposition 5). We assume that the opportunity cost of land is 0, so that the first best spatial configuration solves

$$
\max_{\lambda(.)}\int_{\mathcal{C}}\left[A-\int_{\mathcal{C}}T(x,y)\lambda(y)dy-\frac{\beta}{2}\lambda(x)-r(x)\right]\lambda(x)dx
$$

$$
\text{st. } \int_{\mathcal{C}}\lambda(x)dx = 1
$$

$$
\int_{\mathcal{C}}r(x)\lambda(x)dx = 0
$$

where $r(.)$ represents spatial transfers. The optimum can be implemented without the use of transfers and we have that

$$\max_{\lambda(.)} \int_{\mathcal{C}} \left[A - \int_{\mathcal{C}} T(x,y)\lambda(y)dy - \frac{\beta}{2}\lambda(x) \right] \lambda(x)dx$$

$$\text{st. } \int_{\mathcal{C}} \lambda(x)dx = 1$$

The Lagrange function is

$$L = \int_{\mathcal{C}} \left[A - \int_{\mathcal{C}} T(x,y)\lambda(y)dy - \frac{\beta}{2}\lambda(x) \right] \lambda(x)dx - \mu(\int_{\mathcal{C}} \lambda(x)dx - 1)$$

The first variation gives

$$\delta L = \int_{\mathcal{C}} \left[A - \mu - \beta\lambda(x) - 2\int_{\mathcal{C}} T(x,y)\lambda(y)dy \right] \delta\lambda(x)dx = 0$$

Since at the optimum $\delta L = 0$, we have that

$$\int_{\mathcal{C}} T(x,y)\lambda(y)dy + \frac{\beta}{2}\lambda(x) = \frac{A-\mu}{2}$$

Since $S(x) - \frac{\beta}{2}\lambda(x) = A - \int_{\mathcal{C}} T(x,y)\lambda(y)dy - \frac{\beta}{2}\lambda(x)$, this leads to

$$S(x) - \frac{\beta}{2}\lambda(x) = \frac{A+\mu}{2}$$

It means that at the optimum $V(x) = S(x) - \frac{\beta}{2}\lambda(x)$ is constant. Compared to the decentralized equilibrium, $\beta/2$ appears instead of β. As a consequence the optimum corresponds then to having a single city which is more concentrated than the spatial equilibrium involving a single city. The optimal welfare is then given by $V^*(M = 1, \beta/2)$

$$V^*(M=1,\beta/2) = A - \frac{\tau}{2} + \frac{\tau}{2} - \pi\frac{\tau}{\sqrt{\frac{4\tau}{\beta/2}}} = A - \frac{\pi}{2\sqrt{2}}\sqrt{\tau\beta}$$

References

[1] Beckmann, M.J. (1976), Spatial Equilibrium and the Dispersed City, in: Y.Y. Papageorgiou (ed.), *Mathematical Land Use Theory*, Lexington, MA:Lexington Books, 117-25.

[2] Fujita, M., P. Krugman and A. Venables (1999), *The Spatial Economy. Cities, Regions and International Trade*, MIT Press.

[3] Fujita, M. and H. Ogawa (1980), Equilibrium Land Use Patterns in a NonMonocentric City, *Journal of Regional Science*, 20:4.

[4] Fujita, M. and H. Ogawa (1982), Multiple Equilibria and Structural Transition of Non-Monocentric Urban Configurations, *Regional Science and Urban Economics*, 12.

[5] Fujita, M. and J.-F. Thisse (2002), Economics of Agglomeration: Cities, Industrial Location, and Regional Growth, Cambridge University Press.

[6] Glaeser, E. and J. Scheinkman (2003), Non-Market Interactions, in: M. Dewatripont, L.P. Hansen, and S. Turnovsky (Eds.), Advances in Economics and Econometrics: Theory and Applications, Eight World Congress, Cambridge University Press.

[7] Hotelling, H. (1929), Stability in Competition, *Economic Journal*.

[8] Krugman P. (1991) Increasing returns and economic geography, *Journal of Political Economy* 99, 483-499.

[9] Mossay P. (2003) Increasing returns and heterogeneity in a spatial economy, *Regional Science and Urban Economics* 33, 419-444.

[10] Picard P. and Tabuchi T. (2003), Natural agglomeration, CORE DP 2003/101, Catholic University of Louvain.

[11] Tabuchi T. (1986), Urban Agglomeration Economies in a Linear City, *Regional Science and Urban Economics* 16:421-436.

[12] Salop S.C. (1979) Monopolistic competition with an outside good, *Bell Journal of Economics* 10, 141-56.