Assessing the Local Stability Properties of Three-Dimensional Discrete Time Dynamical Systems: A Geometrical Argument with Triangles & some Applications to Consumption Complementarities within Suboptimal Competitive Economies

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Abstract†

The difficulties associated with the appraisal of the determinacy properties of a three-dimensional system are circumvented by the introduction of a new geometrical argument. It first brings about a complete and easy-to-use typology of the eigenvalues moduli in discrete time three-dimensional dynamical systems and then provides a new apparatus for assessing from a geometrical standpoint the emergence of local bifurcations. The argument is then applied through an extensive characterisation of the stability properties of two simple benchmark models of intertemporal economic analysis that are augmented by comparison utility arguments.

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I – Introduction

The difficulties associated with the appraisal of the local determinacy properties of a three-dimensional discrete time dynamical system have long deterred a more widespread use of the associated setups in economic theory. This contribution is intended to introduce a range of graphical methods based upon the elementary geometrical properties of a range of triangles over the plane that ought to significantly ease the appraisal of such systems. Having brought about a complete and easy-to-use typology of the eigenvalues moduli, it then illustrates the regards in which, for some third-order dynamical systems, this also corresponds to a new apparatus for assessing from a geometrical standpoint the emergence of local bifurcations.

The current contribution essentially borrows from a range of earlier ones due to Grandmont [17] and Grandmont, Pintus & de Vilder [19]. As they reconsider the role of factors substitutability in competitive economies, these authors have come to introduce a tractable graphical way of assessing local uniqueness or local indeterminacy for dynamical systems of order two. Their approach is based upon a graphical partition of the plane defined from the two coefficients of the second-order characteristic polynomial associated with a two-dimensional dynamical system in the neighbourhood of the steady state. Such a partition is then completed by drawing the critical loci associated to real and complex eigenvalues with unitary modulus, these loci featuring boundaries between stability and unstability zonas. A given economy — a set of fundamental preferences and technological parameterisations — was then to be understood as a point over that plane whilst the appraisal of its local dynamics summarised to the localisation of this point. Letting one of its building parameters vary gives rise to a family of economies, namely a curve over that plane the localisation of which provided insights about the associated qualitative changes undergone by the dynamical properties of the economy. The crux interest of this construction for economic theory stems from its explicit consideration of meaningful and generic concepts without having to resort to specific parametric formulations. Anchoring the argument on original formal developments to assess the stability properties of three-dimensional dynamical systems, the current contribution will argue that most of the key-features that underlay the simplicity and the convenience of the two-dimensional approach can be recovered in the three-dimensional case.

Looking for an appraisal of unrestricted economic setpus through the reference to linear critical loci and basic notions of plane geometry, two key difficulties however quickly emerge as being associated with the conceivability of such an approach for a three-dimensional dynamical system. Firstly, the intricacies of three-dimensional graphs and the intrisic subtelities of the geometry of a three-dimensional space. Secondly, the uprise of a nonlinear critical locus that happens to describe the occurrence of complex eigenvalues with unitary modulus — this was one of the key-ingredients of the two-dimensional construction. The first of these issues shall be circumvented by apprehending the original three-dimensional space — the coordinates of which emerge from the three coefficients of the third-order characteristic polynomial — through a collection of sections defined along a given coordinate and thus of two-dimensional planes. Fortunately enough, such an approach also recovers linear definitions for the critical loci and thus overcomes the second major difficulty of the appraisal of stability issues within a three-dimensional space. A direct byproduct states as the simplicity of the typologies it allows for the moduli of the eigenvalues and thus for the understanding of the boundaries between unstability and stability areas within a three-dimensional dynamical system. Two generic ranges of threefold typologies of the number of moduli inside the unit circle indeed happen to deserve a distinct appraisal according to whether the parameterisation of the plane that is considered proceeds from a value that is, in absolute terms, lower or greater than one. As for the first range of planes with a parameterisation through a coefficient of the characteristic polynomial that is lower than one, the plane will typically be partitioned partioned between areas with one, two or three moduli inside the unit circle. Interestingly, the ultimate «highly stable» occurrence with three moduli inside the unit circle reveals as being associated with economies located within a triangle, a straightforward articulation with the insights of the two-dimensional analysis — such a position therein depicts the occurrence of two moduli inside the unit circle — being then available. The remaining range of planes build from parameterisations through a coefficient of the characteristic polynomial that is greater than one and emerge as requiring at least one modulus to be located outside the unit circle, a plane being then typically partitioned between zonas displaying zero, one or two moduli inside the unit circle. Although a triangle is again available, both its formal definition and its geometric understanding are entirely distinct — they have no direct counterpart in two-dimensional analysis — since it now delimitates a fully unstable occurrence with no modulus inside the unit circle.

The understanding of this typology is further about all that is required for a detailed appraisal of the large range of parameterised economies entailing local properties that fall into the current characterisation: indeed, a thorough understanding of the uniqueness, stability and bifurcations properties of a three-dimensional dynamical system is just a matter of elementary plane geometry techniques. Mainly and for a given family of economies, one is first to locate the plane over which the analysis is to be undertaken and subsequently complete a geometrical characterisation of a curve — it features the role of a fundamental preferences or technology parameter — essentially grounded upon tools already used in a standard two-dimensional analysis. Though such an advanced characterisation of the stability properties and of the bifurcation set is admittedly not available for *any* arbitrary three-dimensional parameterised economy — this preferences or technology parameter cannot appear in the coefficient of the characteristic polynomial that *indexes* the construction of the planes — , the subsequent examples should illustrate that the class of economies for which the current approach provides a useful toolbox is fairly large.

A growing literature has recently been aimed at exploring the consequences of instantaneous utilities parameterised by a direct comparison of the individual consumption to a benchmark stock determined by the consumption of others. The basic insight builds from the postulate according to which individual well-being is not only determined by the intrisic utility of his own consumption but also by one's relative standing — — positional concern, social status. Though the origins of this proposition already appeared in Adam Smith's writing, it was not until the contributions of Duesenberry [14] and Pollak [24] that micro-theoretic foundations were proposed for these ideas, recent empirical assessments being available in Luttmer [22] and Ravina [26]. These ideas have become popular in the recent period with applications in various areas of economics: asset pricing theory and equity premium puzzles with Abel [1], efficiency of the capital accumulation process and optimal taxation with Alonso-Carrera, Caballé & Raurich [3], Liu & Turnovsky [21], Turnovsky & Monteiro [29], the long-run distribution of income and wealth with Garcia-Penalosa & Turnovsky [16] and finally the scope for indeterminacy in dynamic general equilibrium economies with Alonso-Carrera, Caballé & Raurich [7] and Chen

& Hsu [10]. As a matter of fact and at a rough level, the literature distinguishes two forms of consumption benchmarks: the Catching-up with the Joneses which captures the influence of society's past consumption choices — *vide* Abel [1] —; the Keeping-up with the Joneses captures the influence of the society's current choices — — *vide* Gali [15].[1] Finally, whilst a Catching-up with the Joneses for instantaneous utility features a desire to be similar to others, the very possibility that many consumers wish to call attention to themselves cannot a priori be dismissed. Such an observation has led Dupor & Liu [9] to consider utilities displaying Running-away from the Joneses — — as a counterpart of Keeping-up with the Joneses.Though their focus will admittedly more relate to the branch of this literature that deals, as mentioned above, with the scope for indeterminacy in dynamic general equilibrium economies, the purpose of the subsequent applications to this literature of the graphical methods developed for assessing the stability of three-dimensional discrete time dynamical systems is in no way to introduce some new or original idea. It is rather to provide a detailed formal characterisation of the implications on the stability properties of some benchmark environments of the introduction of a class of assumptions that are popular in this literature. Some variations on two benchmark setups for the analysis of intertemporal equilibria based upon capital accumulation, namely the basic Ramsey [14] model of capital accumulation and the Allais [2] - Samuelson [28] model of overlapping generations amended by Diamond [12] in order to account for capital accumulation, will in turn be analysed.

The infinite-horizon environment of Ramsey is first augmented to account for past Catching-up with the Joneses and contemporaneous Keeping-up with the Joneses arguments in the instantaneous utility of the consumers. On a formal basis, it is to be stressed that the implied three-dimensional discrete time dynamical system — it is associated with an intertemporal competitive equilibrium with lagged and contemporaneous consumption spillover effects — is somewhat particular in being based upon a pair of predetermined variables. The graphical methods developed in the first part of the article allow to derive a range of conclusions: as long as the contemporaneous spillover effects stemming from aggregate consumption do not question the concavity of utility, the implications of spillover effects stemming from earlier aggregate consumption will not question the local uniqueness of the steady state. Second and more interestingly, it is the very conjunction of positive spillover effects stemming from current consumption and past consumption that has the more dramatic implications on the stability properties of the economy. Highly complicated scenarios with strong sensibilities with respect to factors substituability but also with respect to the size of these spillovers then indicate a natural area where the dimension of the stable manifold exceeds the number of predetermined variables, breaks the uniqueness result and opens road for the so-called *expectations-driven fluctuations*. A modified Diamond [12] model of overlapping generations is then considered, the current benchmark model also accounting for labour supply and an extra useless asset — it is due to Benhabib & Laroque [8]. Whilst its original formulation was already associated with a three-dimensional discrete dynamical system, it is currently augmented by contemporaneous Keeping-up with the Joneses effects in the preferences of the individuals. On a formal basis, it differs from the preceding modified version of the Ramsey

[1] The Catching-up with the Joneses hypothesis is actually an external habit formation setup where the consumption benchmark is an externality. This contrasts with the internal habit formation setup where the reference is the consumer's own past consumption.

setup by being based upon a unique predetermined variable but also because, in being locally associated with a characteristic polynomial for which two coefficients — out of three, against one out of free for the Ramsey setup — explicitly depend on the bifurcation parameter, it provides a more canonical illustration of the usefulness of the graphical methods introduced by this contribution. The insights of the analysis essentially confirm the ones of the Ramsey environment, pointing the Keeping-up from the Joneses assumption as a candidate to violate the gross substitutability assumption retained on the preferences and then unequivocally bending the economic system towards widened areas characterised by excessive dimensions for the stable manifold.

The geometrical techniques are introduced in Section II. Section III deals with the comparison utility model, Section IV builds upon an elaboration of the latter within the model of overlapping generations. Some computations are provided in a final appendix.

II – A Geometric Argument for the Appraisal of Three-Dimensional Dynamical Systems

This section will first unveil a collection of simple geometric pictures underlying the typology of eigenvalues and the emergence of local bifurcations in discrete three-dimensional dynamical systems.

II.1 – A Geometric Picture for the Critical Loci

Letting the equilibrium dynamics of an economy be described by a system: $y_{t+1} = G(y_t)$, $y_t \in \mathbb{R}^3_+$, steady states equilibria are the roots of $\bar{y} - G(\bar{y}) = 0$. The characterisation of the local dynamics nearby a given steady equilibrium proceed from the appraisal of an associated linear map $z_{t+1} = \mathscr{J} z_t$ for $\mathscr{J} \triangleq DG(\bar{y})$ the Jacobian matrix of $G(\cdot)$ evaluated at $\bar{y}$ and $z_t \triangleq y_t - \bar{y}$ the deviation from the steady state. The eigenvalues of the matrix $\mathscr{J}$ are the zeroes of the following third order polynomial:

$$(1) \qquad \begin{aligned} \mathscr{P}(z) &= (z_1 - z)(z_2 - z)(z_3 - z) \\ &= -z^3 + \big(z_1 + z_2 + z_3\big)z^2 - \big(z_1 z_2 + z_1 z_3 + z_2 z_3\big)z + z_1 z_2 z_3 \\ &= -z^3 + \mathscr{T} z^2 - \mathscr{M} z + \mathscr{D} \end{aligned}$$

for $\mathscr{T}, \mathscr{M}$ and $\mathscr{D}$ that respectively denote the trace, the sum of the principal minors of order two and the determinant of the Jacobian matrix $\mathscr{J} \triangleq DG(\bar{y})$.
The locus such that the coefficients $\mathscr{T}, \mathscr{M}, \mathscr{D}$ satisfy $\mathscr{P}(+1) = 0$ is a plane — henceforward refered to as the saddle-node critical plane —whose characteristic equation is given by:

$$(2) \qquad -1 + \mathscr{T} - \mathscr{M} + \mathscr{D} = 0.$$

Generically, a saddle-node bifurcation[2] will occur in its neighbourhood when the triple $(\mathscr{T}, \mathscr{M}, \mathscr{D})$ crosses this plane and the uniqueness properties of the steady state will be lost.

[2] Vide Devaney [11] or Grandmont [17].

Similarly, the locus such that the coefficients $\mathscr{T}, \mathscr{M}, \mathscr{D}$ satisfy $\mathscr{P}(-1) = 0$ is a plane — henceforward refered to as the flip critical plane — whose characteristic equation is given by:

(3) $\quad 1 + \mathscr{T} + \mathscr{M} + \mathscr{D} = 0.$

Generically, a flip bifurcation will occur in its neighbourhood when the triple $(\mathscr{T}, \mathscr{M}, \mathscr{D})$ crosses this plane and two-period cycles will emerge. Lastly, when a pair of nonreal characteristic roots exhibiting an unitary norm emerges, the remaining eigenvalue, *e.g.*, z_3, sumarises to the product of the eigenvalues $\mathscr{D}$. The latter thus becomes a characteristic root, *i.e.*, $\mathscr{P}(\mathscr{D}) = 0$. Solving, the characteristic polynomial restates as $\mathscr{P}(z) = (\mathscr{D} - z)\mathscr{Q}(z)$, for $\mathscr{Q}(z) = z^2 - (\mathscr{T} - \mathscr{D})z + \mathscr{M} - (\mathscr{T} - \mathscr{D})\mathscr{D}$. A standard analysis of $\mathscr{Q}(\cdot)$ then indicates that the locus of coefficients $\mathscr{T}, \mathscr{M}$ and $\mathscr{D}$ such that two roots are complex conjugate with unitary modulus is a *ruled surface* — *i.e.*, a surface generated by straight-lines — that shall henceforward refered to as the Poincaré-Hopf critical surface, is delimited[3] by $|\mathscr{T} - \mathscr{D}| < 2$ and defined from letting the determinant of $\mathscr{Q}(\cdot)$ undergo a value of 1, namely:

(4) $\quad \mathscr{M} - 1 - \big(\mathscr{T} - \mathscr{D}\big)\mathscr{D} = 0.$

Generically, a Poincaré-Hopf bifurcation will occur when the triple $(\mathscr{T}, \mathscr{M}, \mathscr{D})$ crosses the complex interior component of this surface and quasi-periodic equilibria will emerge in its neighbourhood. The central difficulty in the appraisal of this ultimate locus through a three-dimensional graph stems from its nonlinear shape, namely the appearance of the quadratic expression $(\mathscr{T} - \mathscr{D})\mathscr{D}$ in its definition. Interestingly, it is however circumvented upon the consideration of a collection of projections indexed by $\mathscr{D}$ of this surface over the plane defined by $\mathscr{T}$ and $\mathscr{M}$. An analysis with a strong two-dimensional flavour — any of the aforementioned critical loci can anew be represented through a straight-line or a segment — being then conceivable in the space of the two other coefficients $\mathscr{T}$ and $\mathscr{M}$.

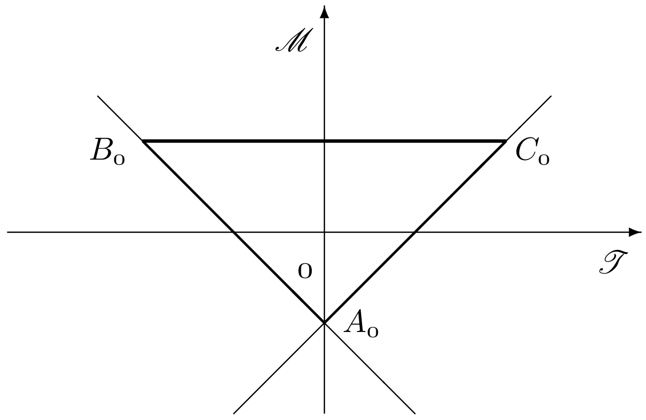
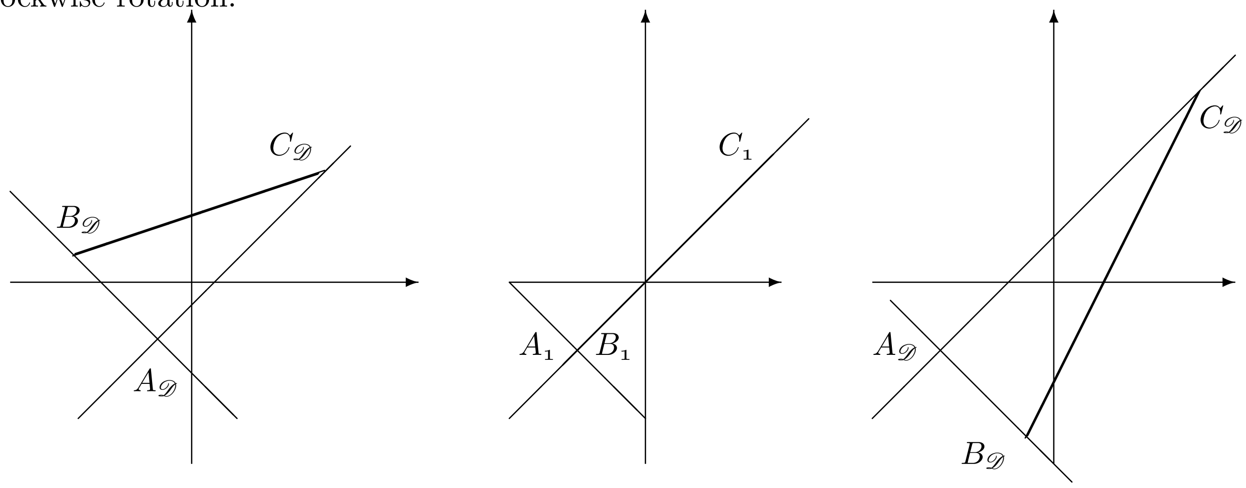


Figure 1: Benchmark case $\mathscr{D} = 0$.

More explicitly and first introducing the benchmark case $\mathscr{D} = 0$ on Figure 1, the set of coefficients $(\mathscr{T}, \mathscr{M})$ such that $\mathscr{P}(+1) = 0$ and $\mathscr{P}(-1) = 0$ respectively boil down to the saddle-node and flip critical lines (A_0C_0) and (A_0B_0) — the index 0 refers to the value of the parameter $\mathscr{D}$ under which the whole picture is drawn — whilst the corresponding set for two nonreal eigenvalues with unitary norm is depicted by the horizonal Poincaré-Hopf critical

[3]This follows from the restriction for a negative sign for the discriminant associated to $\mathscr{Q}(\cdot)$.

segment $[B_0C_0]$. This gives rise to a construction familiar from the two-dimensional analysis, namely the triangle $(A_0B_0C_0)$ defined by $|\mathscr{T}| < |1+\mathscr{M}|$ and $|\mathscr{M}| < 1$.
As $\mathscr{D}$ is increased over $\mathbb{R}_+$ and as illustrated on Figure 2, the slopes of $(A_\mathscr{D}C_\mathscr{D})$ and $(A_\mathscr{D}B_\mathscr{D})$ are unmodified whilst the segment $[B_\mathscr{D}C_\mathscr{D}]$, of slope $\mathscr{D}$, essentially follows a translated counter-clockwise rotation.

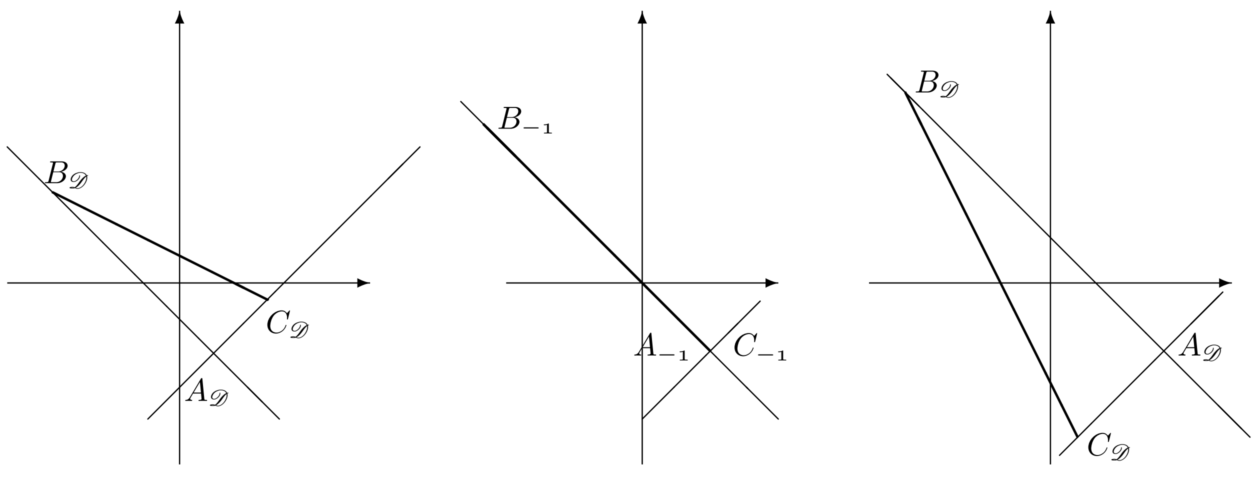


$\left[\mathscr{D} \in]0,1[\right]$ $\left[\mathscr{D} = 1\right]$ $\left[\mathscr{D} \in]1,+\infty)\right]$

Figure 2: Translated counterclockwise rotation as $\mathscr{D}$ is increased over $\mathbb{R}_+$.

The parameterised coordinates of $A_\mathscr{D}$, $B_\mathscr{D}$ and $C_\mathscr{D}$ respectively derive from the solving of (2) and (3), (3) and (4), (2) and (4). They list as:

$$(5)\qquad \begin{aligned} (\mathscr{T}_{A_\mathscr{D}}, \mathscr{M}_{A_\mathscr{D}}) &= (-\mathscr{D}, -1), \\ (\mathscr{T}_{B_\mathscr{D}}, \mathscr{M}_{B_\mathscr{D}}) &= (-2+\mathscr{D}, 1-2\mathscr{D}), \\ (\mathscr{T}_{C_\mathscr{D}}, \mathscr{M}_{C_\mathscr{D}}) &= (2+\mathscr{D}, 1+2\mathscr{D}). \end{aligned}$$



$\left[\mathscr{D} \in]-1,0[\right]$ $\left[\mathscr{D} = -1\right]$ $\left[\mathscr{D} \in (-\infty,-1[\right]$

Figure 3: Translated clockwise rotation as $\mathscr{D}$ is decreased over $\mathbb{R}_-$.

It is worth emphasising that on Figure 2, the Poincaré-Hopf and the saddle-node critical loci coincide and merge for $\mathscr{D} = 1$ in the sense that $A_1 = B_1$. A counterpart scenario is available on Figure 3 where negative values are considered for $\mathscr{D}$. Similarly, the Poincaré-Hopf and the flip critical loci coincide and merge for $\mathscr{D} = -1$ in the sense that $A_{-1} = C_{-1}$. These mergers imply that the definition of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ is modified as $|\mathscr{D}|$ goes through one. Namely, starting from

$$\text{(6a)}\quad |\mathscr{T}+\mathscr{D}| < 1+\mathscr{M} \quad \text{for} \quad |\mathscr{D}|<1$$
$$\mathscr{M} < 1+(\mathscr{T}-\mathscr{D})\mathscr{D}.$$

it becomes

$$\text{(6b)}\quad |1+\mathscr{M}| < \mathscr{T}+\mathscr{D} \quad \text{for} \quad \mathscr{D}>1$$
$$\mathscr{M} > 1+(\mathscr{T}-\mathscr{D})\mathscr{D}.$$

and

$$\text{(6c)}\quad |1+\mathscr{M}| > \mathscr{T}+\mathscr{D} \quad \text{for} \quad \mathscr{D}<-1$$
$$\mathscr{M} > 1+(\mathscr{T}-\mathscr{D})\mathscr{D}.$$

Figures 1, 2 and 3 illustrate how such a three-dimensional parameterised construction, still organised around a triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$, keeps on proceeding from the same lines as the traditional two-dimensional one, but to the qualification that the slope of the Poincaré-Hopf segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$ being given by $\mathscr{D}$, it will vary accordingly. whilst the change from (6a) to (6b) has made clear that the geometric meaning of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ is distinct between the configurations for which $|\mathscr{D}|<1$ and the ones for which $|\mathscr{D}|>1$, an important extra property is worth emphasising at that stage: $(0,0) \in (A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ for any $\mathscr{D} \in \mathbb{R}\setminus\{-1,+1\}$. As this will be illustrated by the forthcoming subsection, this assumes a direct and straightforward articulation with the number of eigenvalues moduli inside the unit circle.

II.2 – A Typology of the Eigenvalues

In order to reach the essence of the argument about the cardinality of stable eigenvalues, consider Figure 1 and the basic configuration for which $\mathscr{D}=0$. An economy within the triangle $(A_0B_0C_0)$ — this means for values of $\mathscr{T}$ and $\mathscr{M}$ that remain close to zero — displays three eigenvalues with modulus inside the unit circle. As this was further clarified by (6), the definition of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ is unaltered as $|\mathscr{T}+\mathscr{D}|<1+\mathscr{M}$ and $\mathscr{M}<1+(\mathscr{T}-\mathscr{D})\mathscr{D}$ as long as $\mathscr{D}$ spans $]-1,1[$, that indicates the uniform occurence of three moduli inside the unit circle under this joint configuration. Incidently, this can similarly be understood by noticing that the origin $(0,0)$ of the above representations over the planes $(\mathscr{T},\mathscr{D})$ is associated to the satisfaction of $z^3=\mathscr{D}$: for $\mathscr{D}<1$, this corresponds to the occurrence of a *triple* real eigenvalue with an absolute value lower than one. As long as the system is maintained in the interior of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ for $|\mathscr{D}|<1$, its stability properties are then ruled by three eigenvalues with moduli inside the unit circle. Consider then an upward perturbation on Figure 1. The system will cross the segment $[B_0C_0]$: this implies that the modulus of the complex eigenvalues gets out of the unit circle and there only remains a unique eigenvalue with norm less than one. When one, after a rightward or a leftward perturbation, leaves the origin stability area by crossing (A_0C_0) or (A_0B_0), the position of a unique eigenvalue with respect to the unit circle will be modified and the system falls in an area with two stable eigenvalues. Finally, a downward perturbation from any of these areas will lead the system within an area that exhibits one modulus within the unit circle. As for the above characterisation of the interior of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$, the formal definitions — in terms of the coefficients $\mathscr{T}$ and $\mathscr{D}$ — of these areas remains unaltered as long as $\mathscr{D}$ spans $]-1,1[$, such a line of reasoning straightforwardly generalises to the whole range of values of $\mathscr{D}$ such that $|\mathscr{D}|<1$. This eventually establishes the typologies — the number of stable eigenvalues indicated between parenthesis on these figures — portrayed by the left components of Figures 4 and 5.

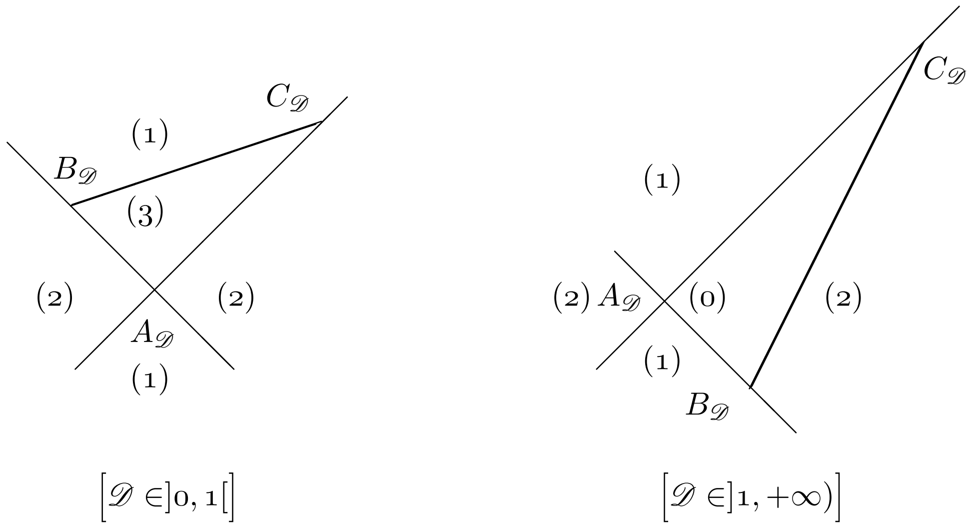
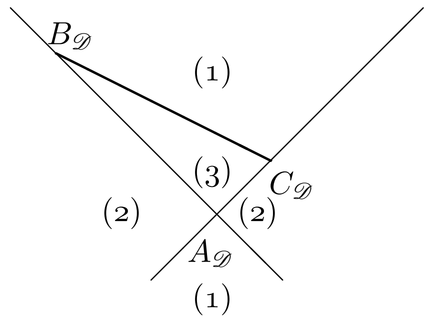


Figure 4: Typologies for $\mathscr{D}\in]0,1[$ and $\mathscr{D}>1$.

Completing the same line of reasoning on the right components of Figure 4 and 5, for $|\mathscr{D}|>1$, though the origin $(0,0)$ of the above representations over the planes $(\mathscr{T},\mathscr{D})$ remains located within the interior of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$, from (6), the definition of the latter is now modified to $|\mathscr{T}+\mathscr{D}|>1+\mathscr{M}$ and $\mathscr{M}>1+\big(\mathscr{T}-\mathscr{D}\big)\mathscr{D}$. The associated satisfaction of $z^3=\mathscr{D}$ indeed now coresponds to the occurrence of a *triple* real eigenvalue with an absolute value that is greater than one. From the same lines of reasoning as above, the interior of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ then uniformly indicates the holding of three eigenvalues with a modulus that is greater than one. Starting from this position, a perturbation that results in the crossing of the segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$ should then be interpreted as follows: the modulus of the complex eigenvalues gets in the unit circle and there only remains a unique eigenvalue with norm greater than one. Similarly, perturbations entailing the crossing of $(A_{\mathscr{D}}C_{\mathscr{D}})$ or $(A_{\mathscr{D}}B_{\mathscr{D}})$ from a initial position inside the $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ when $\mathscr{D}>1$ indicate that the position of a unique real eigenvalue will be modified with respect to the unit circle and the system falls in an area with two unstable eigenvalues. Finally, a subsequent perturbation resulting in a crossing of $(A_{\mathscr{D}}B_{\mathscr{D}})$ after an initial crossing of $(A_{\mathscr{D}}C_{\mathscr{D}})$ or in a crossing of $(A_{\mathscr{D}}C_{\mathscr{D}})$ after an initial crossing of $(A_{\mathscr{D}}B_{\mathscr{D}})$ would lead the system within an area that exhibits two modulus within the unit circle.



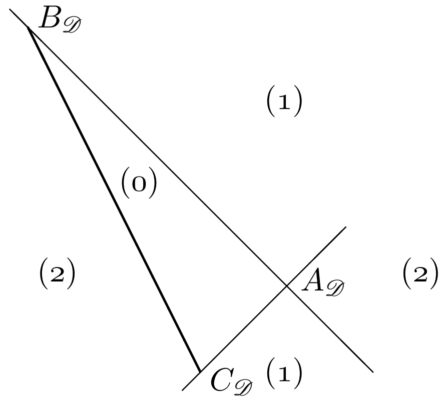


Figure 5: Typologies for $\mathscr{D} \in]-1,0[$ and $\mathscr{D} < -1$.

This eventually establishes the typologies — the number of stable eigenvalues indicated between parenthesis on these figures — portrayed by the right components of Figures 4 and 5. As this soon will become clear, this collection of figures introduces an alternative benchmark for analysing the stability properties of parameterised economies that entails the treatment of a three-dimensional dynamical system.

III – Catching-up with the Joneses / Keeping-up with the Joneses: A Basic Representative Agent Example

III.1 – The Setup

This section will consider a variation of the neo-classical growth model that is augmented by an outward-looking specification of intertemporal preferences for the representative individuals; the level of utility derived from a given amount of instantaneous consumption is thus assumed to exhibit extra dependencies with respect to previous and contemporaneous society consumption standards. More explicitly, these preferences state as

$$(7) \qquad \sum_{t=0}^{\infty} \beta^t u(c_t; C_t, C_{t-1}),$$

for $\beta \in]0,1[$, c_t, C_t and C_{t-1} respectively their positive rate of marginal impatience, their consumption at date $t \geq 0$ and the average consumption across all consumers at the current and at the previous date, $u(\cdot; C, C_{-1})$ being a continuous concave instantaneous utility function which maps $\mathbb{R}_+$ into $\mathbb{R}$, is of class C^k, $k \geq 3$, over[4] $\mathbb{R}_+^*$ and satisfies the Inada conditions at the origin, namely and for a given pair (C_t, C_{t-1}), the holding of $\lim_{c\to 0} \partial u(c, C, C_{-1})/\partial c = \infty$, $\lim_{c\to\infty} \partial u(c, C, C_{-1})/\partial c = 0$. Further, $\partial u(c, C, C_{-1})/\partial c > 0$, $\partial^2 u(c, C, C_{-1})/\partial c^2 < 0$ for any $c \in \mathbb{R}_+^*$. Besides, the Keeping-up and the Catching-up with the Joneses dimension of this formulation are respectively ensured by the holding of $\partial^2 u(c, C, C_{-1})/\partial c \partial C > 0$ $\partial^2 u(c, C, C_{-1})/\partial c \partial C_{-1} > 0$ prevail, *i.e.*, mimetism effects with respect to contemporaneous or earlier consumption standards. In opposition to this, respectively contemporaneous and

[4] $\mathbb{R}_+^* =]0, +\infty)$.

lagged Running-away from the Joneses dimensions are ensured by the converse occurrences of $\partial^2 u(c, C, C_{-1})/\partial c \partial C < 0$ and $\partial^2 u(c, C, C_{-1})/\partial c \partial C_{-1} < 0$ that translates the potential from a repulsive dimension from previous consumption standards.
The capital stock accumulates according to

$$K_{t+1} = F\big(K_t, L\big) - c_t,$$

for K_t and L respectively the t-value, $t \geq 0$ of the capital stock and the fixed amount of the labour input, $F(\cdot, \cdot)$ an aggregate production function that is continuous, maps $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$, is of class C^3 over $\mathbb{R}_+^* \times \mathbb{R}_+^*$, homogeneous of degree one, satisfies the Inada conditions and $\partial^2 F(K, L)/\partial K^2 < 0$, $\partial^2 F(K, L)/\partial L^2 < 0$ for any $K, L > 0$, the capital stock having been assumed to fully depreciate at each period of time. Instead of attempting to complete an extensive characterisation of the symmetric competitive equilibrium with externalities associated with the occurrence of $C_{t-1} = c_{t-1}$ and $C_t = c_t$ at any $t \geq 0$, the current exposition shall proficiently take advantage the simpler centralised approach introduced by Kehoe, Levine & Romer [20]. This builds from considering a Pareto problem parameterised by a sequence of externalities $\{C_{t-1}, C_t\}_{t=1}^{+\infty}$:

$$\begin{aligned} \underset{\{c_t, K_t\}}{\text{Maximise}} & \sum_{t=0}^{+\infty} \delta^t u\big(c_t, C_t, C_{t-1}\big) \\ \text{s.t.} \quad & K_{t+1} \leq F\big(K_t, L\big) - c_t, \\ & K_0 \text{ given}, K_t, c_t \geq 0. \end{aligned}$$

A solution to this optimisation problem satisfies all of the conditions that characterise a symmetric competitive equilibrium with externalities but, possibly, the extra symmetry requesites $C_{t-1} = c_{t-1}$ and $C_t = c_t$ at each $t \geq 0$. For any $\{C_{t-1}\}_{t=1}^{+\infty}$, the earlier strict concavity assumptions on $u(\cdot; C, C_{-1})$ and $F(\cdot, L)$ imply that such an optimisation problem has a unique solution. Establishing the existence of an interior intertemporal competitive equilibrium with externalities for which $C_{t-1} = c_{t-1}\big(\{C_{t-1}\}_{t=1}^{+\infty}\big)$ and $C_t = c_t\big(\{C_{t-1}\}_{t=1}^{+\infty}\big)$ at any $t \geq 0$ is a significantly more involving task: being outside of the current line of concerns, it shall not be discussed further. Assuming then its existence, a symmetric competitive equilibrium with externalities is then described by the holding of:

$$\begin{aligned} (8) \qquad & \frac{\partial u}{\partial c}\big(c_t; c_t, x_t\big) - \beta \frac{\partial F}{\partial K}\big(K_t, L\big) \frac{\partial u}{\partial c}\big(c_{t+1}; c_{t+1}, x_{t+1}\big) = 0, \\ & K_{t+1} - F\big(K_t, L\big) + c_t = 0, \\ & x_{t+1} - c_t = 0, \\ & \lim_{t \to \infty} \delta^t \cdot \frac{\partial u}{\partial c}\big(c_t; c_t, x_t\big) \cdot K_t = 0. \end{aligned}$$

Under the earlier assumptions on the technology and an extra restriction on preferences, namely the satisfaction of $\partial^2 u(c, C, C_{-1})/\partial c^2 + \partial^2 u(c, C, C_{-1})/\partial c \partial C \neq 0$ for any $(c, C, C_{-1}) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*$, system (8) defines a three-dimensional first-order dynamical system $\big[K_{t+1}, c_{t+1}, x_{t+1}\big]' = \Psi\big(K_t, c_{t+}, x_{t+1}\big)$. The emergence of a third extra dimension with respect to the canonical two-dimensional Ramsey setting univocally springs from the consideration of lagged consumption externalities in the preferences.[5] A remarkable formal property of the equilibrium dynamical system (8) is that it builds from two predetermined variables, namely the capital stock and past consumption, such considerations being essential for the understanding of its uniqueness properties by the next section. An interior steady state position is then defined as a triple $\{K^\star, c^\star, x^\star\} \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*$ which satisfies:

$$\text{(9a)} \quad 1/\beta - \frac{\partial F}{\partial K}\big(K^\star, L\big) = 0,$$

$$\text{(9b)} \quad K^\star - F\big(K^\star, L\big) + c^\star = 0,$$

$$\text{(9c)} \quad x^\star - c^\star = 0.$$

Its existence being ensured by the Inada conditions on the technology and the preferences, the concavity assumption of the technology further implies that it is unique. From Appendix 1, a linearisation of the dynamical system around this steady state gives the following expressions for the coefficients of the characteristic polynomial $\mathscr{P}(z) = -z^3 + \mathscr{T} z^2 - \mathscr{M} z + \mathscr{D}$:

$$\text{(10)} \quad \begin{aligned} \mathscr{T} &= 1 - \frac{\eta^{\mathscr{E}_{-1}}}{\eta + \eta^{\mathscr{E}}} - \frac{1}{\eta + \eta^{\mathscr{E}}} \frac{1-s}{\sigma} \left(\frac{1}{\beta s} - 1 \right) + \frac{1}{\beta}, \\ \mathscr{M} &= \left(1 - \frac{\eta^{\mathscr{E}_{-1}}}{\eta + \eta^{\mathscr{E}}} \right) \frac{1}{\beta} - \frac{\eta^{\mathscr{E}_{-1}}}{\eta + \eta^{\mathscr{E}}}, \\ \mathscr{D} &= -\frac{1}{\beta} \frac{\eta^{\mathscr{E}_{-1}}}{\eta + \eta^{\mathscr{E}}}, \end{aligned}$$

for

$$\text{(11)} \quad \begin{aligned} &\sigma \triangleq \frac{\partial F}{\partial K}(K/L,1)\frac{\partial F}{\partial L}(K/L,1) \Big/ F(K/L,1)\frac{\partial^2 F}{\partial K \partial L}(K/L,1), \\ &1-s \triangleq \frac{\partial F}{\partial L}(K/L,1) \Big/ F(K/L,1), \; s \triangleq \frac{\partial F}{\partial K}(K/L,1) \cdot \frac{K}{L} \Big/ F(K/L,1), \\ &\eta \triangleq \frac{\partial^2 u}{\partial c^2}(c,C,C_{-1})c \Big/ \frac{\partial u}{\partial c}(c,C,C_{-1}), \; \eta^{\mathscr{E}} \triangleq \frac{\partial^2 u}{\partial c \partial C}(c,C,C_{-1})C \Big/ \frac{\partial u}{\partial c}(c,C,C_{-1}), \\ &\eta^{\mathscr{E}_{-1}} \triangleq \frac{\partial^2 u}{\partial c \partial C_{-1}}(c,C,C_{-1})C_{-1} \Big/ \frac{\partial u}{\partial c}(c,C,C_{-1}), \end{aligned}$$

that respectively depict the elasticity of substitution between capital and labour, the share of labour and the share of capital, the intertemporal elasticity of substitution in consumption and a pair of outward-looking comparison utility coefficients defined from the marginal utility on consumption, all these being considered at their steady state values.

[5] A careful reader will have already noticed that, had such a lagged consumption argument been internalised by the agent, the ensued dynamical system would have displayed a dimension of four. As this was however argued by Ryder & Heal [27] in the first characterisation of such an environment, the analysis is then complexified further by the potential for utility satiation, such an occurrence having no counterpart in the present setting.

III.2 – The Analysis

It is noticed that $\mathscr{D}$ does not depend in (10) on the elasticity of substitution between capital and labour, *i.e.*, σ. In accordance with the earlier approach and selecting $1/\sigma$ as the tuning parameter, such a feature will bring about the possibility of a purely geometric argument for appraising the dynamical properties of the model.
The current setting is moreover slightly particular on a formal basis. As a matter of fact, since

$$\text{(12a)}\quad \frac{\partial \mathscr{T}}{\partial(1/\sigma)}(1/\sigma) = -\frac{1-s}{\eta+\eta^{\mathscr{E}}}\left(\frac{1}{\beta s}-1\right),$$

$$\text{(12b)}\quad \frac{\partial \mathscr{M}}{\partial(1/\sigma)}(1/\sigma) = 0,$$

the parameterised curve[6] $\Delta(1/\sigma)$ simplifies to a straight-line with a slope given by

$$\text{(13)}\quad \frac{\partial[\Delta(1/\sigma)]}{\partial(1/\sigma)} = \frac{\partial[\mathscr{M}(1/\sigma)]/\partial(1/\sigma)}{\partial[\mathscr{T}(1/\sigma)]/\partial(1/\sigma)} = 0$$

and an origin provided by $\Delta(0) = (\mathscr{T}(0), \mathscr{M}(0))$. It is further remarked that

$$\text{(14)}\quad -1 + \mathscr{T}(0) - \mathscr{M}(0) + \mathscr{D} = 0,$$

this latter equation meaning that $\Delta(0)$ will locate on the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$. Finally, notice that, from (5), the comparison with the ordinate of the origin of $\Delta(1/\sigma)$ with the corresponding values of $\mathscr{M}_{A_{\mathscr{D}}}$, $\mathscr{M}_{C_{\mathscr{D}}}$ and $\mathscr{M}_{B_{\mathscr{D}}}$ namely the respective ordinates of the intercrossings of the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$ with the critical line $(A_{\mathscr{D}}B_{\mathscr{D}})$ and the critical segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$ that were argued throughout Section II as being central to the stability analysis, will respectively detail as

$$\text{(15a)}\quad \mathscr{M}(0) + 1 = (1/\beta + 1)\left[1 - \frac{\eta^{\mathscr{E}-1}}{\eta+\eta^{\mathscr{E}}}\right],$$

$$\text{(15b)}\quad \mathscr{M}(0) - 2\mathscr{D} - 1 = (1/\beta - 1)\left[1 + \frac{\eta^{\mathscr{E}-1}}{\eta+\eta^{\mathscr{E}}}\right],$$

$$\text{(15c)}\quad \mathscr{M}(0) + 2\mathscr{D} - 1 = -\frac{\eta^{\mathscr{E}-1}}{\eta+\eta^{\mathscr{E}}}(3/\beta + 1) + (1/\beta - 1),$$

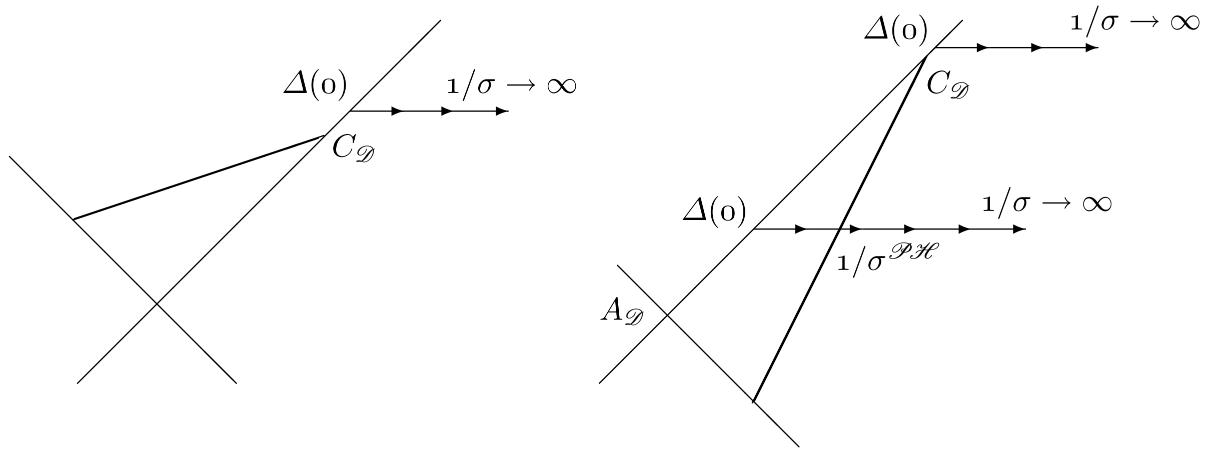
where it it remarked that $\mathscr{M}_{A_{\mathscr{D}}} \lesseqgtr \mathscr{M}_{C_{\mathscr{D}}} \iff -1 \lesseqgtr -\eta^{\mathscr{E}-1}/\beta(\eta+\eta^{\mathscr{E}})$. Otherwise stated, $\mathscr{M}_{A_{\mathscr{D}}} > \mathscr{M}_{C_{\mathscr{D}}}$ for $\mathscr{D} \in]-1, 1[\cup]1, +\infty)$ whereas $\mathscr{M}_{A_{\mathscr{D}}} < \mathscr{M}_{C_{\mathscr{D}}}$ for $\mathscr{D} \in (-\infty, -1[$.
To sum up and on a methodological basis, the subsequent argument shall hence first be organised around the sign of $\eta + \eta^{\mathscr{E}}$ that emerges as the key parameter of the local stability properties of this economy. First and from (14a), it allows, whatever the value of $\mathscr{D}$, for locating the position of the whole $\Delta(1/\sigma)$ with respect to the line $(A_{\mathscr{D}}C_{\mathscr{D}})$ associated with the occurrence of an eigenvalue of $+1$. Second and together with the value of $\eta^{\mathscr{E}-1}$, it is the most significant determinant of $\mathscr{D}$ and thus of the plane $(\mathscr{T}, \mathscr{M})$ over which the analysis is

[6]Although $\mathscr{M}$ does not explicitly depend upon $1/\sigma$, in order to clarify further the approach that is to be systematically followed to assess the geometrical properties of the curve $\Delta(1/\sigma)$, such a fictitious indexation is maintained in the subsequent exposition.

then to be completed. An eventual step builds from locating $\Delta(1/\sigma)$ upon that plane: recalling from (13) that the slope of $\Delta(1/\sigma)$ is invariantly nil, the localisation simplifies to firt place $\Delta(0)$, namely $\mathscr{T}(0)$ and then analyse from (12a) the implications of positive values for $1/\sigma$.

a/ Running-away from the Joneses: $\eta + \eta^{\mathscr{E}} < 0$

From (12a), this configuration is associated with $\partial[\mathscr{T}(1/\sigma)]/\partial(1/\sigma) > 0$: otherwise stated, the origin of the curve $\Delta(1/\sigma)$ is located on the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$ whilst its entire shape belongs to the area associated with occurrence of $\mathscr{P}(+1) > 0$ on its right-hand side. As mentioned above, the second step of the characterisation then builds from the consideration of the sign of $\eta^{\mathscr{E}_{-1}}$.



$-\eta^{\mathscr{E}_{-1}}/(\eta + \eta^{\mathscr{E}}) \in]0, \beta[$ $\qquad$ $-\eta^{\mathscr{E}_{-1}}/(\eta + \eta^{\mathscr{E}}) > \beta$

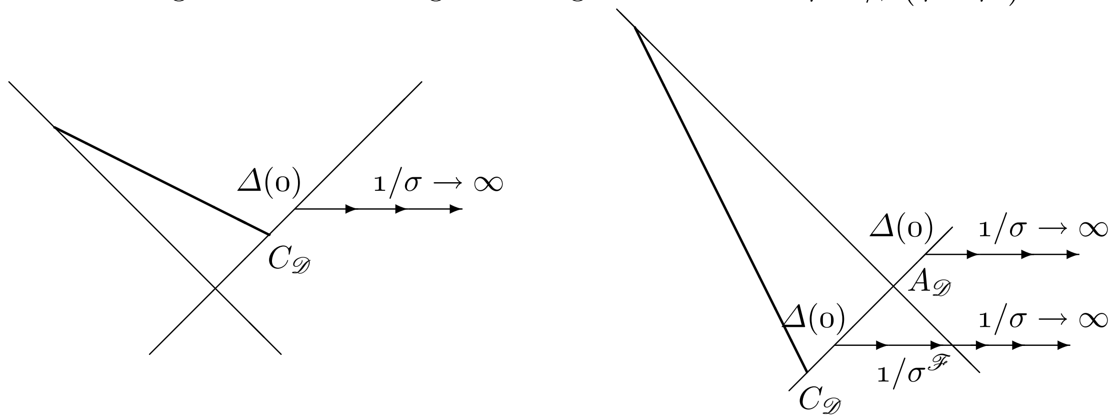
Figure 6: Bifurcation schemes for $\eta + \eta^{\mathscr{E}} < 0$ and $\eta^{\mathscr{E}_{-1}} > 0$.

Firstly focusing on the occurrence of a catching up with the Joneses specification $\eta^{\mathscr{E}_{-1}} > 0$, Figure 6 first takes advantage of the implied occurrence of $\mathscr{D} > 0$ and displays the two admissible configurations according to the sign of $\mathscr{D} - 1 = -\eta^{\mathscr{E}_{-1}}/\beta(\eta + \eta^{\mathscr{E}}) - 1$. Firstly and on the L.H.S. of Figure 6, $-\eta^{\mathscr{E}_{-1}}/\beta(\eta + \eta^{\mathscr{E}}) \in]0, 1[$: from (15), the negativeness of $-\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) - \beta$ immediatly implies that ordinate of $\Delta(0)$ is located above the one of $A_{\mathscr{D}}$. As for the respective position with respect to the one of $C_{\mathscr{D}}$ and from (15b), this configuration being associated with the occurrence of $-\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) < \beta < 1$, it similarly derives that the ordinate of the origin will be located above the one of $C_{\mathscr{D}}$. All this implies that the economy will be characterised by two moduli in the unit circle and hence be locally determinate — the dynamical system being characterised by two predetermined variables — for any $1/\sigma > 0$. Secondly and on the R.H.S. of Figure 6, $-\eta^{\mathscr{E}_{-1}}/\beta(\eta + \eta^{\mathscr{E}}) > 1$: whilst the position of the ordinate of the origin straightforwardly remains located above the one of $A_{\mathscr{D}}$, its position with respect to the one of $C_{\mathscr{D}}$ is reversed. Indeed, this configuration being associated with the satisfaction of $\beta + \eta^{\mathscr{E}_{-1}}/\beta(\eta + \eta^{\mathscr{E}}) < 0$, two configurations are admissible. First and for $-\eta^{\mathscr{E}_{-1}}/(\eta + \eta^{\mathscr{E}}) > 1$, the ordinate of the origin of $\Delta(1/\sigma)$ will be located above the one of $C_{\mathscr{D}}$: in accordance with the configuration depicted on Figure 6a, the economy will similarly be characterised by two moduli inside the unit circle and hence display local uniqueness for any $1/\sigma > 0$. Second and oppositely, for $-\eta^{\mathscr{E}_{-1}}/(\eta + \eta^{\mathscr{E}}) \in]\beta, 1[$, the ordinate of the origin of $\Delta(1/\sigma)$ will be located above the one of $A_{\mathscr{D}}$ but below the one of $C_{\mathscr{D}}$. This would imply the existence of a critical $1/\sigma^{\mathscr{PH}}$, namely:

$$1/\sigma^{\mathscr{PH}} = \frac{\left[1+\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}})\right](1/\beta-1)\left[1-\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}})\right]}{(1-s)(1/\beta s-1)(1/\beta)\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}})^2},$$

corresponding to the crossing of the critical segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$ and such that for $1/\sigma \in]0, 1/\sigma^{\mathscr{PH}}[$, the dynamical system assumes no modulus inside the unit circle and is thus locally unstable. It then typically undergoes a Poincaré-Hopf bifurcation for $1/\sigma = 1/\sigma^{\mathscr{PH}}$ and eventually assumes two moduli inside the unit circle and becomes locally unique for any $1/\sigma > 1/\sigma^{\mathscr{PH}}$.

Secondly analysing the occurrence of a lagged running away from the Joneses specification $\eta^{\mathscr{E}_{-1}} < 0$, Figure 7 first takes advantage of the implied occurrence of $\mathscr{D} < 0$ and displays the two admissible configurations according to the sign of $\mathscr{D} + 1 = -\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}}) + 1$.



$-\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) \in]-\beta, 0[$ $\qquad$ $-\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) < -\beta$

Figure 7: Bifurcation schemes for $\eta+\eta^{\mathscr{E}} < 0$ and $\eta^{\mathscr{E}_{-1}} < 0$.

From (15b), it is first immediate that, for both occurrences, the ordinate of $\Delta(0)$ will be located above the one of $C_{\mathscr{D}}$. From the earlier exposition, the latter is itself located above the one of $A_{\mathscr{D}}$, but for the case $\mathscr{D} \in (-\infty, -1[$. This is confirmed by the L.H.S. of Figure 7 that focuses on the configuration $-\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}}) \in]-1, 0[$: as $1-\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}}) > \beta-\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}}) > 0$, the ordinate of the origin of $\Delta(0)$ locates above the ones of $A_{\mathscr{D}}$ and $C_{\mathscr{D}}$, that implies that the economy will be characterised by two moduli in the unit circle and hence be locally indeterminate for any $1/\sigma > 0$. Secondly and on the R.H.S. of Figure 7, $-\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}}) < -1$, a configuration for which the ordinate of $C_{\mathscr{D}}$ is located below the one of $A_{\mathscr{D}}$. Two configurations are then to be distinguished according to whether the ordinate of $\Delta(0)$ being is below or above the one of $A_{\mathscr{D}}$. For $-\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) \in]-1, -\beta[$, the ordinate of $\Delta(0)$ is located above the one of $A_{\mathscr{D}}$, the economy will be characterised by two moduli inside the unit circle and hence display a local determinacy for any $1/\sigma > 0$. In opposition to this and for $-\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) < -1$, the ordinate of $\Delta(0)$ is located between the one of $C_{\mathscr{D}}$ and the one of $A_{\mathscr{D}}$ and there exists a critical value $1/\sigma^{\mathscr{F}}$ available as:

$$1/\sigma^{\mathscr{F}} = 2\frac{(1+1/\beta)\left[1-\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}})\right] / (\eta+\eta^{\mathscr{E}})}{(1-s)(1/\beta s-1)}$$

such that for $1/\sigma \in]0, 1/\sigma^{\mathscr{F}}[$, the dynamical system assumes only one modulus inside the unit circle and is locally unstable. It then typically undergoes a flip bifurcation for $1/\sigma = 1/\sigma^{\mathscr{F}}$

and eventually assumes two moduli inside the unit circle and becomes locally unique for any $1/\sigma > 1/\sigma^{\mathscr{F}}$.

b/ Keeping-up with the Joneses: $\eta + \eta^{\mathscr{E}} > 0$

In contradiction with Figures 6 and 7 and from (12a), this configuration will be associated with $\partial[\mathscr{T}(1/\sigma)]/\partial(1/\sigma) < 0$: otherwise stated, whilst the origin of line $\Delta(1/\sigma)$ is still located on the line $(A_{\mathscr{D}}C_{\mathscr{D}})$, its entire shape for $1/\sigma > 0$ now belongs to the area associated with occurrence of $\mathscr{P}(+1) < 0$ on its left-hand side. As for the earlier running away from the Joneses configuration — $\eta + \eta^{\mathscr{E}} < 0$ —, the second step of the characterisation is going to build from the consideration of the sign of $\eta^{\mathscr{E}_{-1}}$.

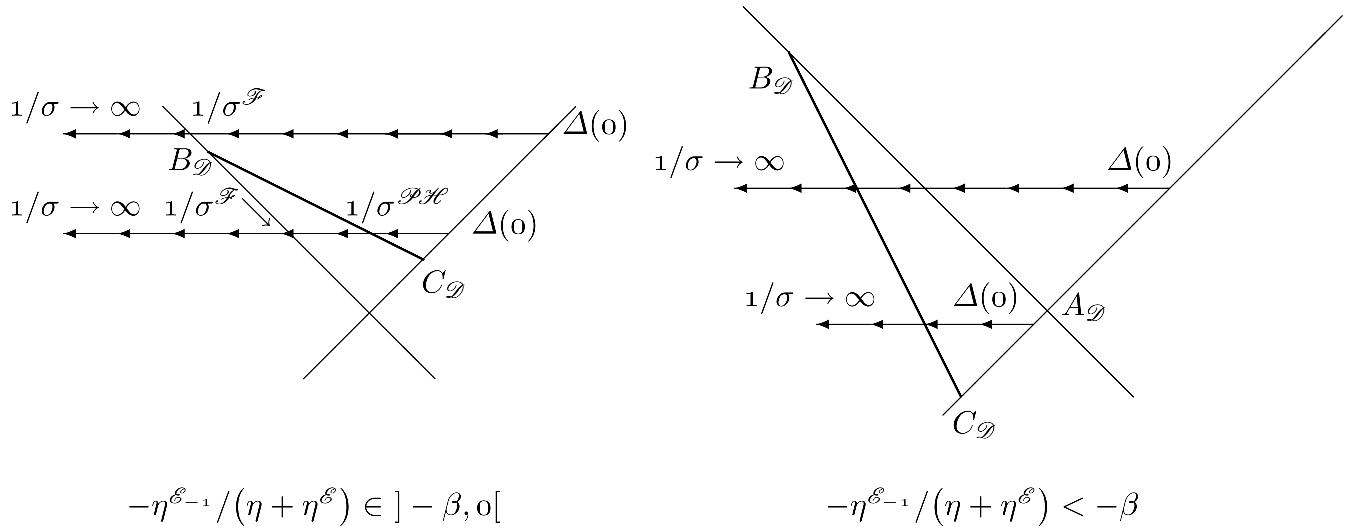


Figure 8: Typologies for $\eta + \eta^{\mathscr{E}} > 0$ and $\eta^{\mathscr{E}_{-1}} > 0$.

Firstly focusing on the occurrence of a catching-up with the Joneses specification $\eta^{\mathscr{E}_{-1}} > 0$, Figure 8 first hinges upon the implied occurrence of $\mathscr{D} < 0$ and displays the two admissible configurations according to the sign of $\mathscr{D} + 1 = -\eta^{\mathscr{E}_{-1}}/\beta(\eta + \eta^{\mathscr{E}}) + 1$. Firstly and on the L.H.S. of Figure 8, $-\eta^{\mathscr{E}_{-1}}/\beta(\eta + \eta^{\mathscr{E}}) \in]-1, 0[$. From (15b), the ordinate of $\Delta(0)$ is located above the one of $C_{\mathscr{D}}$ that is itself located above the one of $A_{\mathscr{D}}$. New to this configuration is however the role played by the ordinate of $B_{\mathscr{D}}$, where it shall henceforward be assumed that $(1-\beta)/(3+\beta) < \beta$. When the ordinate of $\Delta(0)$ is also located above the one of $B_{\mathscr{D}}$, namely and from (15c), for $\eta^{\mathscr{E}_{-1}}/(\eta + \eta^{\mathscr{E}}) \in]0, (1/\beta - 1)\beta/(3/\beta + 1)[$, there exists a $1/\sigma^{\mathscr{F}}$ such that for $1/\sigma \in]0, 1/\sigma^{\mathscr{F}}[$, the dynamical system assumes one modulus inside the unit circle and is locally unstable. It then typically undergoes a flip bifurcation for $1/\sigma = 1/\sigma^{\mathscr{F}}$, eventually assumes two moduli inside the unit circle and becomes locally unique for $1/\sigma > 1/\sigma^{\mathscr{F}}$. In opposition to this and when the ordinate of $\Delta(0)$ is located between the one of $C_{\mathscr{D}}$ and the one of $B_{\mathscr{D}}$, *i.e.*, for $\eta^{\mathscr{E}_{-1}}/(\eta + \eta^{\mathscr{E}}) \in](1/\beta - 1)\beta/(3/\beta + 1), \beta[$, there exists a pair of values $1/\sigma^{\mathscr{PH}}$ and $1/\sigma^{\mathscr{F}}$ such that for $1/\sigma/ \in]0, 1/\sigma^{\mathscr{PH}}[$, the dynamical system assumes one modulus inside the unit circle and is locally unstable. It then typically undergoes a Poincaré-Hopf bifurcation for $1/\sigma = 1/\sigma^{\mathscr{PH}}$. For any $1/\sigma \in]1/\sigma^{\mathscr{PH}}, 1/\sigma^{\mathscr{F}}[$, the dynamical system assumes three modulis inside the unit circle and is locally undeterminate. It then typically undergoes a flip bifurcation for $1/\sigma = 1/\sigma^{\mathscr{F}}$, eventually assumes two moduli inside the unit circle and becomes locally unique for $1/\sigma > 1/\sigma^{\mathscr{F}}$.

Secondly and on the R.H.S. of Figure 8, considering the occurrence of $-\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}}) < -1$, it first appears from (15c) that the entailed occurrence of $\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) > \beta > (1-\beta)/(3+\beta)$ implies that the ordinate of $\Delta(0)$ cannot be located above the one of $B_{\mathscr{D}}$. For an origin of $\Delta(0)$ located between the ones of $A_{\mathscr{D}}$ and $B_{\mathscr{D}}$, *i.e*, for $\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) \in]\beta, 1[$, there exists a pair of values $1/\sigma^{\mathscr{F}}$ and $1/\sigma^{\mathscr{PH}}$ such that for $1/\sigma \in]0, 1/\sigma^{\mathscr{F}}[$, the dynamical system assumes one modulus inside the unit circle and is locally unstable. It then typically undergoes a flip bifurcation for $1/\sigma = 1/\sigma^{\mathscr{F}}$. For any $1/\sigma \in]1/\sigma^{\mathscr{F}}, 1/\sigma^{\mathscr{PH}}[$, the dynamical system assumes no moduli inside the unit circle and is locally fully unstable. It then typically undergoes a Poincaré-Hopf bifurcation for $1/\sigma = 1/\sigma^{\mathscr{PH}}$ and eventually assumes two moduli inside the unit circle and becomes locally unique for $1/\sigma > 1/\sigma^{\mathscr{PH}}$. Finally and for an origin of $\Delta(0)$ located between the ones of $C_{\mathscr{D}}$ and $A_{\mathscr{D}}$, *i.e.*, for $\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}}) > 1$, the dynamical system first assumes no moduli inside the unit circle and is locally fully unstable. It then typically undergoes a Poincaré-Hopf bifurcation for $1/\sigma = 1/\sigma^{\mathscr{PH}}$ and eventually assumes two moduli inside the unit circle and becomes locally determinate for $1/\sigma > 1/\sigma^{\mathscr{PH}}$.

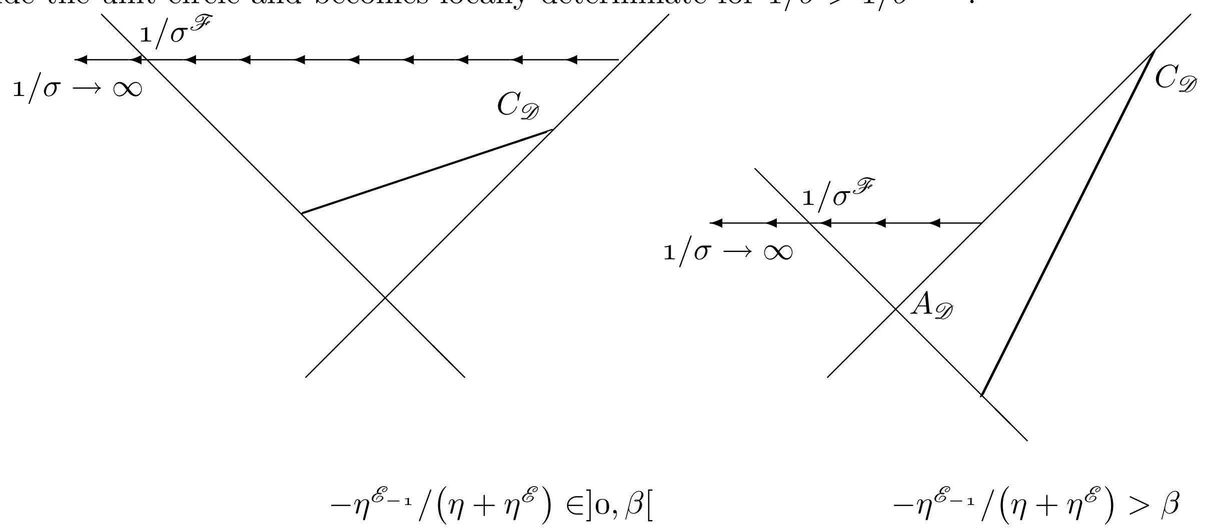


Figure 9: Typologies for $\eta+\eta^{\mathscr{E}} > 0$ and $\eta^{\mathscr{E}_{-1}} < 0$.

Finally focusing on the occurrence of a a lagged running-away from the Joneses specification $\eta^{\mathscr{E}_{-1}} < 0$, Figure 9 builds upon the satisfaction of $\mathscr{D} > 0$ and displays the two admissible configurations according to the sign of $\mathscr{D}+1 = -\eta^{\mathscr{E}_{-1}}/(\eta+\eta^{\mathscr{E}})+1$. On the L.H.S. of Figure 9, $-\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}}) \in]0, 1[$: from (15b), the ordinate of $\Delta(0)$ is located above the one of $C_{\mathscr{D}}$ and there exists a $1/\sigma^{\mathscr{F}}$ such that for $1/\sigma \in]0, 1/\sigma^{\mathscr{F}}[$, the dynamical system assumes one modulus inside the unit circle and is locally unstable. It then typically undergoes a flip bifurcation for $1/\sigma = 1/\sigma^{\mathscr{F}}$ and eventually assumes two moduli inside the unit circle and becomes locally unique for any $1/\sigma > 1/\sigma^{\mathscr{F}}$. Secondly and on the R.H.S. of Figure 9, $-\eta^{\mathscr{E}_{-1}}/\beta(\eta+\eta^{\mathscr{E}}) > 1$: whilst the ordinate of $\Delta(0)$ locates above the one of $A_{\mathscr{D}}$ but may be located above or below the ordinate of $C_{\mathscr{D}}$. Interestingly, this has no implications on the bifurcation scenarios: there indeed always exists a $1/\sigma^{\mathscr{F}}$ such that for $1/\sigma/ \in]0, 1/\sigma^{\mathscr{F}}[$, the dynamical system assumes one modulus inside the unit circle and is locally unstable. It then typically undergoes a flip bifurcation for $1/\sigma = 1/\sigma^{\mathscr{F}}$ and eventually assumes two moduli inside the unit circle and becomes locally unique for any $1/\sigma > 1/\sigma^{\mathscr{F}}$.

To sum up and from Figures 6 and 7, as long as the contemporaneous spillover effects stemming from aggregate consumption do not question the concavity of utility, the implications of spillover effects stemming from earlier aggregate consumption will not question the local uniqueness of the steady state: there will at most exist a two-dimensional stable manifold leading to the steady state. Interestingly however, such a memory of past consumption choices may bend the economy towards the uprise of unstability through the emerge of periodic or quasi-periodic equilibria. As this clearly appears from Figures 6b and Figures 7b with scenarios associated with $|\mathscr{D}| > 1$ that such complicated attractors are admissible, whatever the sign of these lagged aggregate consumption spillover effects. One may finally further notice that there is a strong link between local unstability are arbitrarily large values for the elasticity of substitution between the technological factors. In strong opposition with this first range of results, the case for strong contemporaneous spillover effects stemming from aggregate consumption that ultimately overcome the concavity properties of the utility function, namely the keeping-up with the Joneses configurations available through Figures 8 and 9, assumes complex and contradictory implications on the stability properties. First, when, as for Figure 9, such a strong positive mimetism dimension is introduced jointly with a negative one stemming from past consumption, though local uniqueness is not questioned, local stability may be lost and unstability occur through the emergence of periodic equilibria. Second and more interestingly on Figure 8, it is the very conjunction of positive spillover effects stemming from current consumption and past consumption that has the more dramatic implications on the stability properties of the economy. Highly complicated scenarios with strong sensibilities with respect to factors substituability but also with respect to the size of these spillovers then indicate a natural area for expectations-driven fluctuations.

IV– Keeping-up with the Joneses: Golden Rule Equilibria in the Model of Overlapping Generations

This section will consider a second application based upon a version of the basic characterisation by Diamond [12] of the capital accumulation process in the model of overlapping generations that is augmented to an explicit account for labour supply and by an extra useless asset and is due to Benhabib & Laroque [7]. Though its augmentation through a keeping-up with the Joneses assumption on preferences just mimics the previous section, the ensued characteristic polynomial exhibits a richer structure that translates into a more advanced references to the tools introduced by Section II.

IV.1 – The Setup

The economy is populated by generations of agents who live for two periods, the population size being constant accross generations. At any date $t \geq 1$, the total population summarises to a *young* agent of generation t and an *old* agent of generation $t-1$. At date $t = 1$, an agent was born *old*. An agent of generation $t \geq 1$ chooses a labour supply $\ell_t^t \in [0, \bar{\ell}]$, where $\bar{\ell}$ denotes a maximum physical bound on his labour supply, $\bar{\ell} > 0$ for his young age t and a consumption level c_{t+1}^t for his old age $t+1$. His preferences are featured by an intertemporal utility function $U(\ell_t^t, c_{t+1}^t, C_{t+1}^t) = u(c_{t+1}^t, C_{t+1}^t) - v(\ell_t^t)$, for $u \in C^k(\mathbb{R}_+^* \times \mathbb{R}_+^*, \mathbb{R}_+)$, $k \geq 4$, $\partial u(c_{t+1}^t, C_t)/\partial c_{t+1}^t > 0$, $\partial^2 u(c_{t+1}^t, C_t)/\partial (c_{t+1}^t)^2 < 0$ for all $c_t^t \in \mathbb{R}_+^*$, $\lim_{c_{t+1}^t \to 0} \partial u(c_{t+1}^t, C_{t+1})/\partial c_{t+1}^t = \infty$, $\lim_{c_{t+1}^t \to \infty} \partial u(c_{t+1}^t, C_{t+1})/\partial c_{t+1}^t = 0$

and $v \in C^k(]0,\bar{\ell}],\mathbb{R}_+)$, $k \geq 4$, $\partial v(\ell_t^t)/\partial \ell_t^t > 0$, $\partial^2 v(\ell_t^t)/\partial(\ell_t^t)^2 > 0$ for all $\ell_t^t \in]0,\bar{\ell}]$, $\lim_{\ell_t^t \to 0} \partial v(\ell_t^t)/\partial \ell_t^t = 0$, $\lim_{\ell_t^t \to \bar{\ell}} \partial v(\ell_t^t)/\partial \ell_t^t = +\infty$.

A representative agent of generation $t \geq 1$ is allowed, when he is young, to save through a capital asset in amount s_t^t or through a monetary asset m_t the relative price of which in terms of the numeraire consumption good denotes as B_t. Taking $\{\omega_t, B_t, \mathscr{R}_{t+1}\}$ as given, for ω_t that denotes the real wage rate and $\mathscr{R}_{t+1}$ that denotes the gross return rate of the capital stock, an individual would then select a 4-uple $\{\ell_t^t, c_{t+1}^t, s_t^t, m_t\} \in \mathbb{R}_+^4$ in order to maximise $U(\ell_t^t, c_{t+1}^t, C_{t+1})$ subject to a pair of instantaneous budget constraints:

$$B_t m_t + s_t^t = \omega_t \ell_t^t,$$
$$c_{t+1}^t = \mathscr{R}_{t+1} s_t^t + B_{t+1} m_t.$$

Arbitrage implies $B_{t+1}/B_t = \mathscr{R}_{t+1}$. Rearranging, the constraint set of the above program can hence be simplified through the definition of an intertemporal budget constraint as $c_{t+1}^t = \mathscr{R}_{t+1}\omega_t \ell_t^t$. The first-order condition of this program is given by:

$$(16) \quad \frac{\partial v}{\partial \ell_t^t}(\ell_t^t) - \mathscr{R}_{t+1}\omega_t \frac{\partial u}{\partial c_{t+1}^t}(\mathscr{R}_{t+1}\omega_t \ell_t^t, C_t) = 0.$$

Reintegrating the intertemporal budget constraint, this restates, letting $\mathscr{V}(\ell) \triangleq \ell \cdot \partial v(\ell)/\partial \ell$ for $\ell \in]0,\bar{\ell}]$ and $\mathscr{U}(c,C) \triangleq c \cdot \partial u(c,C)/\partial c$, for $c > 0$, as $\mathscr{U}(c_{t+1}^t, C_{t+1}^t) = \mathscr{V}(\ell_t^t)$. Note that, under the above assumptions, $\partial \mathscr{V}(\ell)/\partial \ell > 0$ and that $\mathscr{V} \in \mathscr{C}^k(]0,\bar{\ell}],\mathbb{R}_+)$, so that the *reflected generational offer curve* $\varphi(\cdot, C)$ of generation t boils down to $\ell_t^t = \varphi(c_{t+1}^t, C_{t+1}^t) \triangleq (\mathscr{V}^{-1} \circ \mathscr{U})(c_{t+1}^t, C_{t+1})$. Imposing further $\mathscr{V}(0) = 0$, it derives that $\varphi(0,C) = 0$ and $\partial\varphi(0,C)/\partial c > 0$. Letting then $\eta(c,C) \triangleq c \cdot [\partial^2 u(c,C)/\partial c^2]/[\partial u(c,C)/\partial c]$, $\eta_\ell(\ell) \triangleq \ell \cdot [\partial^2 v(\ell)/\partial \ell^2]/[\partial v(\ell)/\partial \ell]$ and $\xi_\varphi(c,C) \triangleq c \cdot [\partial\varphi(c,C)/\partial c]/\varphi(c,C)$, one obtains, for any $c > 0$ and a given C:

$$(17) \quad \xi_\varphi(c,C) = [1 + \eta(c,C)]/[1 + \eta_\ell(\ell)].$$

Finally noticing that, for any $c > 0$, $\xi_\varphi(c,C) < 1$, it is further assumed that $1 + \eta(c,C) > 0$, *i.e.*, $\xi_\varphi(c,C) \in]0,1[$, a *gross substitutability* assumption being retained on preferences.

Letting the production facet be described by the same standard technology used in Section III and M denote the fixed money supply, an extensive definition of a symmetric equilibrium with externalities would then build from a triple $\{\ell_t^t, c_{t+1}^t, s_t^t\}$ that describes an optimal action of the individual of generation t given $\{\omega_t, \mathscr{R}_{t+1}\}$, and the holding of $\ell_t^t = L_t$, $K_{t+1} = s_t^t$, $K_{t+1} = Y_t + (1-\mu)K_t$, $c_{t+1}^t = -Y_t + F(K_t, L_t)$ and $m_t = M$ that would respectively depict the clearings of the labour market, the market of the productive asset, the consumption good market and the money market. Finally a *side* condition $c_t^t = C_t$ singles out a symmetric competitive equilibrium with externalities the existence of which shall henceforth be assumed. For clarification purposes, it is first convenient to reassess the status of the first-order condition (16) in the course of this competitive equilibrium with externalities:

$$(16') \quad \frac{\partial v}{\partial \ell_t^t}(\ell_t^t) - \mathscr{R}_{t+1}\omega_t \frac{\partial u}{\partial c_{t+1}^t}(\mathscr{R}_{t+1}\omega_t \ell_t^t, \mathscr{R}_{t+1}\omega_t \ell_t^t) = 0.$$

Incorporating then the intertemporal budget constraint, this restates, introducing $\mathscr{U}^{\mathscr{E}}(c) \triangleq c \cdot [\partial u(c,c)/\partial c]$, $c = C$, for $c > 0$, as $\mathscr{U}^{\mathscr{E}}(c_{t+1}^t) = \mathscr{V}(\ell_t^t)$. The equilibrium formulation of the reflected generational offer curve $\varphi^{\mathscr{E}}(\cdot)$ of generation t is then available as $\ell_t^t = \varphi^{\mathscr{E}}(c_{t+1}^t) \triangleq (\mathscr{V}^{-1} \circ \mathscr{U}^{\mathscr{E}})(c_{t+1}^t)$. Letting also $\eta^{\mathscr{E}}(c,C) \triangleq c \cdot [\partial^2 u(c,C)/\partial c \partial C]/[\partial u(c,C)/\partial c]$ and $\xi_{\varphi}^{\mathscr{E}}(c) \triangleq c \cdot [\partial \varphi^{\mathscr{E}}(c)/\partial c]/\varphi(c,c)$, one obtains, for any $c > 0$:

(18) $\quad \xi_{\varphi}^{\mathscr{E}}(c) = \left[1 + \eta(c,c) + \eta^{\mathscr{E}}(c,c)\right]/\left[1 + \eta_{\ell}(\ell)\right].$

where it is noted that $\xi_{\varphi}^{\mathscr{E}}(c) \gtreqless 0$ according to whether $\eta(c,c) + \eta^{\mathscr{E}}(c,c) \gtreqless -1$ and that the holding of $\xi_{\varphi}^{\mathscr{E}}(c) \geq 1$ cannot any longer be discarded.
More generally, having completed substitutions, any competitive equilibrium with externalities will be associated with a sequence which satisfies:

(19a) $\quad L_t - \varphi^{\mathscr{E}}\left(\left[\frac{\partial F}{\partial K}(K_{t+1}, L_{t+1}) + (1-\mu)\right]\frac{\partial F}{\partial L}(K_t, L_t)L_t\right) = 0,$

(19b) $\quad K_{t+1} + MB_t - \frac{\partial F}{\partial L}(K_t, L_t)L_t = 0,$

(19c) $\quad B_{t+1} - \left[\frac{\partial F}{\partial K}(L_{t+1}, L_{t+1}) + (1-\mu)\right]B_t = 0.$

Under the previous assumption, an competitive equilibrium with externalities such as (19) defines a three-dimensional dynamical system $[\,K_{t+1} \quad L_{t+1} \quad B_{t+1}\,]' = \Upsilon(K_t, L_t, B_t)$. Whilst its three-dimensional feature is a direct and unsurprising characteristic of monetary equilibria in the model of overlapping generations with an explicitly described labour supply, it may be worth noticing that $\Upsilon(\cdot)$ fundamentally differs from the system $\Psi(\cdot)$ that was considered in Section III: it indeed builds from a unique predetermined variable, namely the capital stock. A benchmark long-run golden rule steady state is then defined as a triple $\{K^\star, L^\star, B^\star\} \in \mathbb{R}_+^* \times \,]0, \bar{\ell}[\times \mathbb{R}_+^*$ which solves:

(20a) $\quad L^\star - \varphi^{\mathscr{E}}\left[\frac{\partial F}{\partial L}(K^\star, L^\star)L^\star\right] = 0,$

(20b) $\quad K^\star + MB^\star - \frac{\partial F}{\partial L}(K^\star, L^\star)L^\star = 0,$

(20c) $\quad \frac{\partial F}{\partial K}(K^\star, L^\star) + (1-\mu) = 1.$

Under the above range of assumptions on preferences and the technology, it is a standard argument to show that, whilst, from the concavity assumptions and the Inada conditions on $F(\cdot,\cdot)$, (20c) defines a unique $K^\star/L^\star$, that can be associated to a unique $B^\star/L^\star$ from (20c), recalling that a monetary steady state is associated with the holding of $c^\star = \mathscr{R}^\star\omega^\star\ell^\star = \omega^\star\ell^\star$, the existence of at least one $L^\star$ from (20a) will be ensured by letting

$$\lim_{\ell \to 0}\left[\frac{\partial u}{\partial c}(c,c) \Big/ \frac{\partial v}{\partial \ell}(\ell)\right] > \frac{1}{\mathscr{R}\omega} > \lim_{\ell \to \bar{\ell}}\left[\frac{\partial u}{\partial c}(c,c) \Big/ \frac{\partial v}{\partial \ell}(\ell)\right]$$

further prevail. To the latter $L^\star$ can eventually be associated a unique $K^\star$ and a unique $B^\star$, that establishes the existence of at least one golden rule steady state position. Though existence is not a difficult issue, the sought for uniqueness is more involving: as a matter of fact, it can be shown that the case for multiplicity would unequivocally stem from the existence of a root to

$1/\xi_\varphi^{\mathscr{E}} = 1$ and thus, from (18), of the rather exceptional occurrence of $\eta(c,c)+\eta^{\mathscr{E}}(c,c) = \eta_\ell(\ell)$. As shall however soon appear, such an area is entirely disconnected from the assessment of the local stability issue over a given section $(\mathscr{T},\mathscr{M})$ on Figures 4 and 5: otherwise stated, for any of the latters, there always exists a unique steady state, the multiplicity issue being therefore not discussed further. Considering then a linearisation of the dynamical system in the neighbourhood of a steady state and relying on the system of notations (11), it is established in Appendix 2 that the coefficients of the associated characteristic polynomial $\mathscr{P}(z) = -z^3 + \mathscr{T}z^2 - \mathscr{M}z + \mathscr{D} = 0$ that features a linearisation of the dynamical system in a neighbourhood of the steady state list as:

(21a) $$\mathscr{T} = 1 + \frac{s}{(1-s)\mu} + \frac{(1-s)\mu}{s} - \left(1 - \frac{1}{\xi_\varphi^{\mathscr{E}}}\right)\frac{\sigma}{(1-s)\mu},$$

(21b) $$\mathscr{M} = 1 + \frac{s}{(1-s)\mu} + \frac{(1-s)\mu}{s} - \left(1 - \frac{1}{\xi_\varphi^{\mathscr{E}}}\right)\frac{\sigma}{(1-s)\mu} - \left(1 - \frac{1}{\xi_\varphi^{\mathscr{E}}}\right)\frac{(1-s)\mu}{s},$$

(21c) $$\mathscr{D} = \frac{1}{\xi_\varphi^{\mathscr{E}}}$$

IV.2 – The Analysis

Again, it is noticed that the coefficient $\mathscr{D}$ does not depend on the elasticity of substitution between capital and labour, *i.e.*, σ, the latter being selected as a bifurcation parameter for the representations over the plane $(\mathscr{T},\mathscr{M})$ that are parameterised by $\mathscr{D}$, namely by the inverse of the elasticity of the equilibrium offer curve $1/\varphi^{\mathscr{E}}$. Further noticing that the equation of the parameterised curve $\Delta(\sigma)$ is available from

(22) $$\begin{aligned}\mathscr{M} &= \mathscr{T} - \left(1 - 1/\xi_\varphi^{\mathscr{E}}\right)/\alpha \\ &= \mathscr{T} - (1-\mathscr{D})/\alpha,\end{aligned}$$

for $\alpha \triangleq s/(1-s)\mu$, where, for future reference, it is noted that $\Delta(\sigma)$ happens to be respectively upper-bounded and lower bounded by a straight-line of equation $\mathscr{M} = \mathscr{T}$ for respectively $\mathscr{D} < 1$ and $\mathscr{D} > 1$, such a straight-line being associated with the boundary case $1/\alpha = 0$. The components of the directional vector of the straight-line (22) are as for themselves given by:

(23) $$\frac{\partial\mathscr{T}}{\partial\sigma}(\sigma) = \frac{\partial\mathscr{M}}{\partial\sigma}(\sigma) = -\left(1 - 1/\xi_\varphi^{\mathscr{E}}\right)\frac{\alpha}{s} = -(1-\mathscr{D})\frac{\alpha}{s}\cdot$$

Interestingly, (22) indicates that the graph of $\Delta(\sigma)$ is parallel to the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$. Furthermore, substituting (22) into the equation of $(A_{\mathscr{D}}C_{\mathscr{D}})$, namely and from (2) $-1 + \mathscr{T} - \mathscr{M} + \mathscr{D} = 0$, it is obtained that the respective position of the parameterised line $\Delta(\sigma)$ with respect to the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$ is available from:

(24) $$(1-\mathscr{D})\left(1/\alpha - 1\right).$$

Hence and from (24), for $\alpha < 1$, $\Delta(\sigma)$ is respectively located in the above or in the underneath of the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$ according to whether $\mathscr{D} > 1$ or $\mathscr{D} < 1$ prevails. Unsurpisingly, reverse conjunctions hold for $\alpha > 1$; namely, respective locations of $\Delta(\sigma)$ in the underneath or in the above of critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$ according to whether $\mathscr{D} > 1$ or $\mathscr{D} < 1$ prevails. Furthermore, and from (23), for $\mathscr{D} > 1$, $\Delta(\sigma)$ happens to be upward-orientated whereas and for $\mathscr{D} \in (-\infty, 1[$, it is downward-orientated.

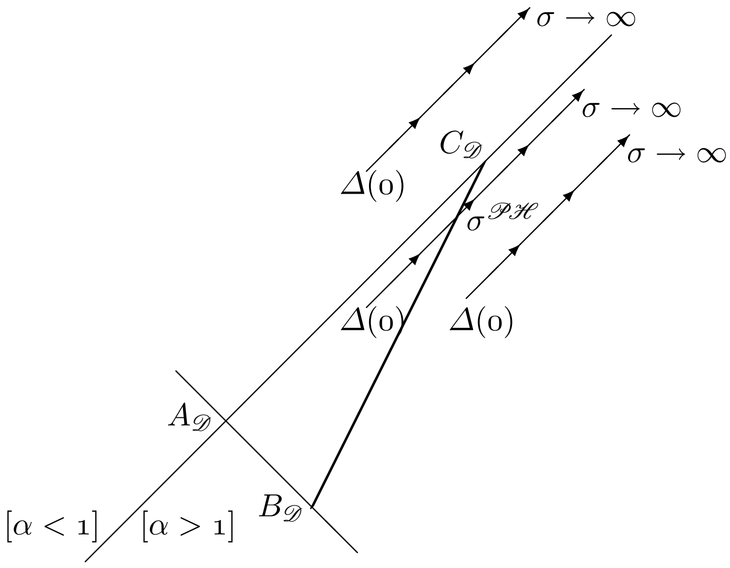


Figure 10: Bifurcation schemes for $\xi_{\varphi}^{\mathscr{E}} \in]0, 1[$.

Such dimensions are illustrated through Figures 10-13 where the localisation of $\Delta(\sigma)$ with respect to the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$ unequivocally associates the crossing of the critical segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$ with the equilibrium occurrence of $\alpha > 1$. Scrutinising further the set of Figures 10-13, it is worth recalling that, on any of these figures, all the admissible straight-lines $\Delta(\sigma)$ will either be upper-bounded or lower-bounded by a straight-line of equation $\mathscr{M} = \mathscr{T}$ according to the position of $\mathscr{D}$ with respect to 1 under which they are drawn. But it is then immediate that the latter, *i.e.*, another straight-line of slope 1 that is parallel to the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$, encompasses the origin $(0, 0)$ on any of the figures. Though there is no need to explicitly state the lattter, by definition and from Section II.2, it is to be located in the interior of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$. Otherwise stated, another useful partition becomes available: the straight-line $\mathscr{M} = \mathscr{T}$ that passes through the origin will itself be upper-bounded and lower-bounded by another parallel to $(A_{\mathscr{D}}C_{\mathscr{D}})$ that passes through the point $B_{\mathscr{D}}$ for respectively $\mathscr{D} < 1$ and $\mathscr{D} > 1$. This implies that the whole collection of parameterised straight-lines $\Delta(\sigma)$ will respectively be located below and above the latter for $\mathscr{D} < 1$ and $\mathscr{D} > 1$.

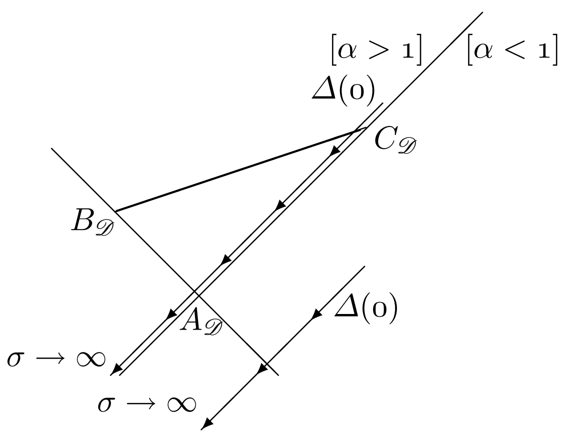


Figure 11: Bifurcation schemes for $\xi_{\varphi}^{\mathscr{E}} > 1$.

In order to complete the aforementioned pictures, it then just remains to locate $\Delta(0)$, namely the origin of the parameterised line $\Delta(\sigma)$. More specifically, the issue is to locate $\Delta(0)$ with respect to the flip critical line $(A_{\mathscr{D}}B_{\mathscr{D}})$ and the Poincaré-Hopf critical segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$. This information, combined with the one brought by the features of the directional vector (23), allows for concluding about the path followed by the economy as σ varies, and thus to reach conclusions about the determinacy properties as well as the scope for local bifurcations. Firstly considering the position of $\Delta(0)$ with respect to $(A_{\mathscr{D}}B_{\mathscr{D}})$, the latter rests upon the sign of :

$$(25) \quad 1 + \mathscr{T}(0) + \mathscr{M}(0) + \mathscr{D} = 1 + 2\mathscr{T}(0) - 1/\alpha + (1 + 1/\alpha)\mathscr{D}.$$

The R.H.S. of (25) is unambiguously positive as long as $\mathscr{D}$ belongs to $]-1,\infty)$. It follows that in such case, $\Delta(0)$ is located above the flip critical line $(A_{\mathscr{D}}B_{\mathscr{D}})$. Remembering that for $\mathscr{D} < 1$, $\Delta(\sigma)$ is downward-orientated, it emerges that a flip bifurcation, ensuring the existence of cycles of period two in the neighbourhood of the steady state, is then, as this is illustrated on Figures 11-13, bound to occur when $\sigma = \sigma_{\mathscr{F}}$, for:

$$(26) \quad \sigma^{\mathscr{F}} = \frac{-2\big[1 + s/(1-s)\mu + (1-s)\mu/s\big] + \big(1 - 1/\xi_{\varphi}^{\mathscr{E}}\big)(1-s)\mu/s - 1 - 1/\xi_{\varphi}^{\mathscr{E}}}{\big(1 - 1/\xi_{\varphi}^{\mathscr{E}}\big)/(1-s)\mu}$$

Conversely and on Figure 10, in the sub-case $\mathscr{D} > 1$ the occurrence of a flip bifurcation is ruled out, the parameterised line $\Delta(\sigma)$ being upward-orientated. Finally considering the localisation of $\Delta(0)$ with respect to $[B_{\mathscr{D}}C_{\mathscr{D}}]$, the latter is available from the sign of

$$(27) \quad \mathscr{M}(0) - 1 - [\mathscr{T}(0) - \mathscr{D}]\mathscr{D} = (1 - \mathscr{D})(\alpha - \mathscr{D}).$$

A short glance at Figures 10-13 indicates that the sign (27), i.e., the localisation of $\Delta(0)$ with respect to $[B_{\mathscr{D}}C_{\mathscr{D}}]$, is only relevant in the case $\alpha > 1$ under which the holding of a Poincaré-Hopf bifurcation cannot be discarded on a *a priori* basis. It may then be assessed that, for $\mathscr{D} < 1$, as the R.H.S. of (27) is positive, $\Delta(0)$ happens to be located above $[B_{\mathscr{D}}C_{\mathscr{D}}]$. The parameterised half-line $\Delta(\sigma)$ being further downward-orientated, it follows that a Poincaré-Hopf bifurcation, indicating the existence of quasi-periodic equilibria around the steady state, is bound to occur for $\sigma = \sigma_{\mathscr{PH}}$, where

$$(28) \quad \sigma^{\mathscr{PH}} = (1-s)\mu\frac{s/(1-s)\mu - 1/\xi_{\varphi}^{\mathscr{E}}}{1 - 1/\xi_{\varphi}^{\mathscr{E}}}$$

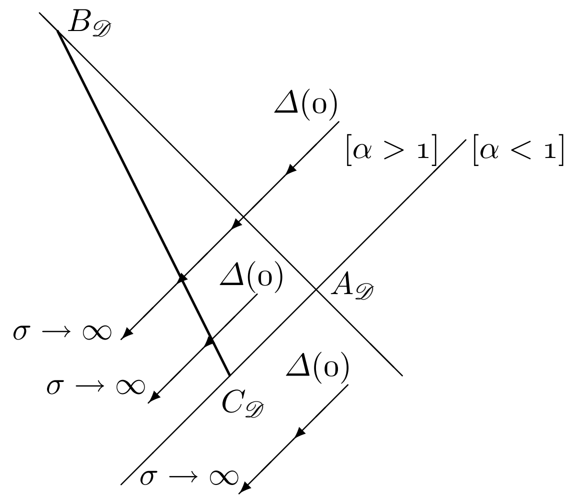


Figure 12: Bifurcation schemes for $\xi_{\varphi}^{\mathscr{E}} \in]-1, 0[$.

Then specialising the argument on the sub-case $\mathscr{D} \in]-1,1[$ on Figures 11 and 12, the complete portrait — the path followed by the economy as σ is increased from zero — is finally available: the equilibrium is initially locally determinate since it assumes one eigenvalue within the unit circle, then a Poincaré-Hopf bifurcation occurs giving rises to a two-dimensional local indeterminacy with three moduli inside the unit circle. Finally and after the occurrence of a flip bifurcation for $\sigma = \sigma_{\mathscr{F}}$, the equilibrium exhibits one degree of indeterminacy with two moduli within the unit circle.
Facing finally with the case $\mathscr{D} > 1$ and as this is illustrated on Figure 10, the sign of (27) is no longer unambiguous. Nonetheless, neither the Poincaré-Hopf bifurcation nor the existence of locally indeterminate equilibria are precluded.

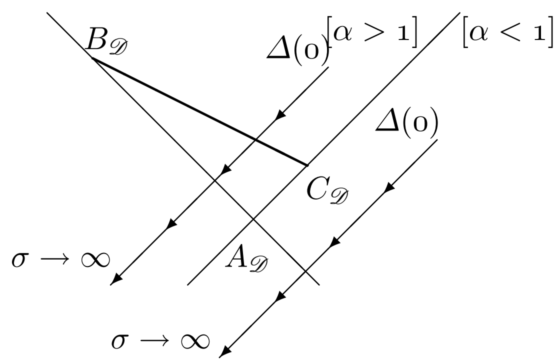


Figure 13: Bifurcation schemes for $\xi_{\varphi}^{\mathscr{E}} < -1$

To sum up, recalling the interpretation of the equilibrium offer curve and from Figure 10, it is confirmed that, up to weak for consumption complementarities that would not question the gross substitutability assumption retained on his preferences, it is only for arbitrarily low order of the elasticity of subsitution that local unstability and an explicit scope for a Poincaré-Hopf bifurcation emerge as admissible configurations. For larger orders, the steady state is locally indeterminate. Second and on Figure 11, under an attraction dimension the preferences of the individual that stems from the consumption of the others, namely a *keeping-up with the Joneses* configuration, complicated bifurcation schemes with successive bifurcations for different parameter values become admissible. The most interesting facet of these occurrences formulates as the scope for periodic cycles that was canonically associated with the violation of the gross subsitutability axiom on preferences. The latter is indirectly recovered through figures 12 and 13 where it is the extra-concavity brought on the marginal utility of consumption by the level of consumption of the others that results in an equilibrium violation of this axiom. In opposition to this, the configuration 11 illustrates the richness of the keeping-up with the Joneses configuration that unequivocally bends the economy towards a widened area for expectations-driven fluctuations.

VI – References

[1]. Abel, A.B., Asset Pricing under Habit Formation and Catching up with the Joneses. American Economic Review 90: 38–42, 1990.

[2]. Allais, M. .*Economie et intérêt*. Paris, 1947.

[3]. Alonso-Carrera, J., J. Caballé & X. Raurich, Growth, Habit Formation and Catching up with the Joneses, Working Paper 497.01 UAB-IAE (CSIC), to appear in the European Economic Review, 2 0 0 1.

[4]. Alvarez-Cuadrado, F., G. Monteiro & S.J. Turnovsky, Habit Formation, Catching up with the Joneses and Economic Growth. Journal of Economic Growth 9: 47–80, 2 0 0 4.

[5]. ——, Can Consumption Spillovers be a Source of Equilibrium Indeterminacy?, Journal of Economic Dynamics & Control, .

[6]. ——, Welfare Implications of the Interaction between Habits and Consumption Externalities, International Economic Review 47, 557-71, 2 0 0 6.

[7]. Benhabib, J. & R. Farmer Indeterminacy and Increasing Returns. Journal of Economic Theory 63: 19–41, 1 9 9 4.

[8]. —— & G. Laroque On Competitive Cycles in Productive Economies. Journal of Economic Theory 45: 145–70, 1 9 8 8.

[9]. Caroll, C.D., J. Overland & D. Weil, Comparison Utility in a Growth Model. Journal of Economic Growth 2: 339-67, 1 9 9 7.

[10]. Chen, B. & M. Hsu, Admiration as a Source of Indeterminacy. *Economic Letters* 95: 96-103, 2 0 0 7.

[11]. Devaney, R. Introduction to Chaotic Dynamical Systems. Addison-Wesley, 1 9 8 6.

[12]. Diamond, P. National Debt in a Neoclassical Growth Model. *American Economic Review* 35: 1126-50, 1 9 6 5.

[13]. Dupor B.& W.-F. Liu, Jealousy and Equilibrium Overconsumption. American Economic Review 93: 423-8, 2 0 0 3.

[14]. Duesenberry, J.S., *Income, Savings and the Theory of Consumer Behaviour*. Harvard University Press, 1 9 4 9.

[15]. Gali, J., Keeping up with the Joneses. Consumption Externalities, Portfolio Choice and Asset Prices. Journal of Money, Credit and Banking 26: 1–8, 1 9 9 4.

[16]. Garcia-Penalosa, C. & S. J. Turnovsky, Consumption Externalities: A Representative Consumer Model when Agents are Heterogeneous. To appear in *Economic Theory*. 2 0 0 8.

[17]. Grandmont, J.-M. Market Psychology and Business Cycles. *Journal of Economic Theory* 83-0: 1-13, 1 9 9 8.

[18]. —— Nonlinear Difference Equations, Bifurcations and Chaos: An Introduction. C.e.p.r.e.m.a.p. Working Paper, 1 9 8 8.

[19]. ——, P. Pintus & R. de Vilder Capital-Labour Substitution and Competitive Endogenous Business Cycles. Journal of Economic Theory 80: 14-59, 1 9 9 8.

[20]. Kehoe,T. & D. Levine & P. Romer. On Characterising Equilibria with Externalities & Taxes as Solutions to Optimization Problems. *Economic Theory* 2: 43-68, 1 9 9 2.

[21]. Liu, W. & S. Turnovsky. Consumption Externalities, Production Externalities & Long-Run Macroeconomic Efficiency. Journal of Public Economics 89: 1117-29, 2 0 0 5.

[22]. Luttmer, E. Neighbours as Negatives: Relative Earnings and Well-Being. *Quarterly Journal of Economics* 89: 1117-29, 2 0 0 5.

[23]. Ok, E.A. & L. Kockesen. Negatively Interdependent Preferences. *Social Choice & Welfare* 17: 553-8, 2000.

[24]. Pollak, R. Interdependent Preferences. *American Economic Review* 66: 309-20, 1976.

[25]. Ramsey, F.P. A Mathematical Theory of Savings. Economic Journal 40: 128-37, 1928.

[26]. Ravina, E. Habit Persistence and Keeping-Up with the Joneses: Evidence from Micro Data. Manuscript. New York University. , 2007.

[27]. Ryder, H. & G. Heal. Optimal Growth with Intertemporally Dependent Preferences. *Review of Economic Studies* 40: 1- 31, 1973.

[28]. Samuelson, P. An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money. *Journal of Political Economy* 66: 467-82, 1958.

[29]. Turnovsky, S.J. & G. Monteiro. Consumption Externalities, Production Externalities & Efficient Capital Accumulation under Time Non-Separable Preferences. *European Economic Review* 51: 479-54, 2007.

V – Proofs

V.1 – Derivation of the Coefficients of the Characteristic Polynomial in the Benchmark Case

Linearising the characteristic polynomial in the neighbourhood of the steady state leads to:

$$\begin{bmatrix} \dfrac{dc_{t+1}}{c^\star} \\ \dfrac{dK_{t+1}}{K^\star} \\ \dfrac{dX_{t+1}}{X^\star} \end{bmatrix} = \begin{bmatrix} 1-\dfrac{\eta^{\mathscr{E}-1}}{\eta+\eta^{\mathscr{E}}}-\dfrac{(1-s)}{\sigma}\dfrac{c^\star}{K^\star} & \dfrac{1}{\eta}\dfrac{(1-s)}{\sigma}\left(\dfrac{c^\star}{K^\star}+1\right) & \dfrac{\eta^{\mathscr{E}-1}}{\eta+\eta^{\mathscr{E}}} \\ -\dfrac{c^\star}{K^\star} & \left(\dfrac{c^\star}{K^\star}+1\right)s & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{dc_t}{c^\star} \\ \dfrac{dK_t}{K^\star} \\ \dfrac{dX_t}{X^\star} \end{bmatrix}$$

Further noticing that $c^\star/K^\star + 1 = 1/\beta s$ and letting, *e.g.* J_{11} denote the first element of the first row in the above Jacobian Matrix, the expressions of $\mathscr{T}$, $\mathscr{M}$ and $\mathscr{D}$ in the main text are straighforwardly derived by noticing that they respectively correspond to $J_{11}+J_{22}$, $J_{11}J_{22}-J_{12}J_{21}-J_{13}$ and $-J_{13}J_{22}$. △

VI.2 – The Characteristic Polynomial associated to Golden Rule Equilibria

The linearised form of the dynamical system is available as:

$$\begin{bmatrix} \xi_\varphi^{\mathscr{E}} s/(1-s) & \xi_\varphi^{\mathscr{E}} & \xi_\varphi^{\mathscr{E}} s/(1-s) \\ -1/(1-s) & [1/(1-s)-\eta/\sigma](1-s) & [1/(1-s)+\eta(1-s)/s\sigma](1-s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathscr{Y}_{t+1} \\ \mathscr{L}_{t+1} \\ \mathscr{K}_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & [-s/(1-s)\sigma+1/(1-\sigma)](1-s) & s/\sigma \\ -\eta & 0 & -(1-\eta) \end{bmatrix} \begin{bmatrix} \mathscr{Y}_t \\ \mathscr{L}_t \\ \mathscr{K}_t \end{bmatrix} = 0.$$

Then assuming that $\mathscr{D} \neq 0$, for $\mathscr{D} = -(1/\xi_\varphi^{\mathscr{E}})(\eta s/\sigma)$, the components of the Jacobian Matrix $[\mathscr{J}]$ list as:

$$
\begin{aligned}
\mathscr{J}_{11} &= -\mathscr{D}^{-1}\Big\{\xi_\varphi^{\mathscr{E}} s\big[-\big(-1/\sigma - (1-s)/s\sigma\big)\eta\big](-\eta)\Big\},\\
\mathscr{J}_{12} &= -\mathscr{D}^{-1}\Big\{\big[1/(1-s) - \eta/\sigma\big](1-s)\\
&\qquad\qquad + \xi_\varphi^{\mathscr{E}}(1-s)\big[-\big(-s/\sigma(1-s) + 1/(1-s)\big)\big]\Big\},\\
\mathscr{J}_{13} &= -\mathscr{D}^{-1}\Big\{\xi_\varphi^{\mathscr{E}}\big((1-s)s/(1-s)\big)\big[-\big(1/\sigma\big)\big]\\
&\qquad + \xi_\varphi^{\mathscr{E}}\big((1-s)s/(1-s)\big)\big[\big(-(1/\sigma) + [-(1-s)/s\sigma]\big)\eta\big][-(1-\eta)]\Big\},\\
\mathscr{J}_{21} &= -\mathscr{D}^{-1}\Big\{\xi_\varphi^{\mathscr{E}}\big(ss/(1-s)\big)[-(1-s)/s\sigma]\eta(-\eta)\Big\},\\
\mathscr{J}_{22} &= -\mathscr{D}^{-1}\Big\{\Big(-1/(1-s)\Big)s + \xi_\varphi^{\mathscr{E}}\big((1-s)s/(1-s)\big)\big[-\big(-s/(1-s)\sigma + 1/(1-s)\big)\big]\Big\},\\
\mathscr{J}_{23} &= -\mathscr{D}^{-1}\Big\{\xi_\varphi^{\mathscr{E}}\big(ss/(1-s)\big)\big[-\big(1/\sigma\big)\big] + \xi_\varphi^{\mathscr{E}}\big(ss/(1-s)\big)[-(1-s)/s\sigma]\eta[-(1-\eta)]\Big\},\\
\mathscr{J}_{31} &= -\mathscr{D}^{-1}\Big\{\xi_\varphi^{\mathscr{E}}\big((1-s)s/(1-s)\big)\big[(1/\sigma)\eta\big](-\eta)\Big\},\\
\mathscr{J}_{32} &= 0,\\
\mathscr{J}_{33} &= -\mathscr{D}^{-1}\Big\{\xi_\varphi^{\mathscr{E}}\big((1-s)s/(1-s)\big)\big[(1/\sigma)\eta\big]\big[-(1-\eta)\big]\Big\}.
\end{aligned}
$$

The determinant of the Jacobian Matrix is first available as:

$$
\begin{aligned}
\det(\mathscr{J}) &= \mathscr{J}_{33}\big(\mathscr{J}_{11}\mathscr{J}_{22} - \mathscr{J}_{12}\mathscr{J}_{21}\big) + \mathscr{J}_{31}\big(\mathscr{J}_{12}\mathscr{J}_{23} - \mathscr{J}_{22}\mathscr{J}_{13}\big)\\
&= [-(1-\eta)]\big(-\mathscr{D}^{-1}\big)^2\Big\{(-\eta)\xi_\varphi^{\mathscr{E}}\big((1-s)(1-s)s/(1-s)\big)\\
&\qquad\times\Big(\big[\big(-(1/\sigma) - (1-s)/s\sigma\big)\eta\big]\Big\{-1/(1-s)\Big\} - [[-(1-s)/s\sigma]\eta]\\
&\qquad\qquad\times\Big\{1/(1-s) - (1/\sigma)\eta\Big\}\Big)\\
&\qquad + (-\eta)\big(\xi_\varphi^{\mathscr{E}}\big)^2\big((1-s)ss/(1-s)\big)\big[-\big(-s/(1-s)\sigma + 1/(1-s)\big)\big]\Big(-\big[(1/\sigma)\eta\big]\Big)\\
&\quad + (-\eta)\big(-\mathscr{D}^{-1}\big)^2\Big\{[-(1-\eta)]\xi_\varphi^{\mathscr{E}}\big((1-s)ss/(1-s)\big)\\
&\qquad\times\Big(-\big[\big(-(1/\sigma) + [-(1-s)/s\sigma]\big)\eta\big]\Big\{-1/(1-s)\Big\} + ([-(1-s)/s\sigma])\eta\\
&\qquad\qquad\times\Big\{1/(1-s) - (1/\sigma)\eta\Big\}\Big)\\
&\qquad + \xi_\varphi^{\mathscr{E}}\big((1-s)ss/(1-s)\big)\big[-\big(1/\sigma\big)\big]\Big(-(1/\sigma)\eta\Big)\\
&\qquad + \xi_\varphi^{\mathscr{E}}\big((1-s)ss/(1-s)\big)\big[-\big(-s/(1-s)\sigma + 1/(1-s)\big)\big]\big[(1/\sigma)\eta\big]\big[-(1-\eta)\big]\Big\}\\
&= \big(-\mathscr{D}^{-1}\big)^2\xi_\varphi^{\mathscr{E}}\big((1-s)ss/(1-s)\big)\Big(-(1/\sigma)\eta\Big)\Big(-\big(1/\sigma\big)(-\eta)\Big)\\
&= \frac{1}{\xi_\varphi^{\mathscr{E}}}.
\end{aligned}
$$

Completing related lines of computations, it is first obtained that the trace of the Jacobian

Matrix is available along:

$$\begin{aligned}\mathrm{tr}(\mathscr{J}) &= \mathscr{J}_{11} + \mathscr{J}_{22} \\ &= \frac{1/(1-s)}{\xi_\varphi^{\mathscr{E}}\eta/\sigma} + (1-\eta) + \frac{\xi_\varphi^{\mathscr{E}}\{[(1/\sigma + (1-s)/s\sigma)\eta]\eta - (1/\sigma + 1/(1-s))\}}{\xi_\varphi^{\mathscr{E}}\eta/\sigma}.\end{aligned}$$

It is then noticed that the third term in the above expression can be reformulated according to:

$$= \eta + \frac{(\eta/\sigma)\big((1-s)\eta/s\big) - \big(1/\sigma + 1/(1-s)\big)}{\eta/\sigma}.$$

Merging with the previous expression, it is obtained that:

$$\mathrm{tr}(\mathscr{J}) = 1 - \left(1 - \frac{1}{\xi_\varphi^{\mathscr{E}}}\right)\frac{1/(1-s)}{(1/\sigma)\eta} + \frac{(1/\sigma)\eta\big((1-s)\eta/s\big) + s/(1-s)}{(1/\sigma)\eta},$$

Finally facing with the sum of the principal minors of the Jacobian Matrix:

$$\mathrm{spm}(\mathscr{J}) = \mathscr{J}_{11}\mathscr{J}_{22} - \mathscr{J}_{12}\mathscr{J}_{21} + \mathscr{J}_{11}\mathscr{J}_{23} - \mathscr{J}_{13}\mathscr{J}_{31} + \mathscr{J}_{22}\mathscr{J}_{33},$$

where:

$$\begin{aligned}&\mathscr{J}_{11}\mathscr{J}_{22} - \mathscr{J}_{12}\mathscr{J}_{21} \\ &\quad = \big(-\mathscr{D}^{-1}\big)^2\Big\{\xi_\varphi^{\mathscr{E}}\big[(1/\sigma)\eta\big](1-s)ss/(1-s)\big[s/(1-s)\sigma\big] \\ &\qquad\qquad - \eta\xi_\varphi^{\mathscr{E}}\big[\eta/\sigma\big]\big[-(1-s)\eta/s\sigma\big](1-s)ss/(1-s) \\ &\qquad\qquad - \eta\big(\xi_\varphi^{\mathscr{E}}\big)^2\big(1/\sigma + 1/(1-s)\big)\big((1/\sigma)\big)\eta(1-s)ss/(1-s)\Big\};\end{aligned}$$

$$\begin{aligned}&\mathscr{J}_{11}\mathscr{J}_{33} - \mathscr{J}_{13}\mathscr{J}_{31} \\ &\quad = \big(-\mathscr{D}^{-1}\big)^{-1}\Big\{\xi_\varphi^{\mathscr{E}}\big[\eta/\sigma\big](1-s)s/(1-s)\Big\} \\ &\quad = 1;\end{aligned}$$

$$\mathscr{J}_{22}\mathscr{J}_{33} = \big(-\mathscr{D}^{-1}\big)^{-1}\Big\{-\big(-1/(1-s)\big) + \xi_\varphi^{\mathscr{E}}\big[-\big(1/\sigma + 1/(1-s)\big)\big](1-s)s/(1-s)\Big\};$$

Finally, gathering terms, it is obtained that :

$$\mathrm{spm}(\mathscr{J}) = 1 - \left(1 - \frac{1}{\varphi}^{\mathscr{E}}\right)\frac{1/(1-s)}{(1/\sigma)\eta} + \frac{1}{\xi_\varphi^{\mathscr{E}}}\frac{(1/\sigma)\eta\big((1-s)\eta/s\big)}{(1/\sigma)\eta} - \frac{-s/(1-s)\sigma}{(1/\sigma)\eta}.$$

To sum up, the trace, the sum of the principal minors and the determinant of the Jacobian Matrix considered in the neighbourhood of a steady state derive as

$$\begin{aligned}\mathrm{tr}(\mathscr{J}) &= 1 - \left(1 - \frac{1}{\xi_\varphi^{\mathscr{E}}}\right)\frac{1/(1-s)}{(1/\sigma)\eta} + \frac{(1/\sigma)\eta\big((1-s)\eta/s\big) + s/(1-s)\sigma}{(1/\sigma)\eta}, \\ \mathrm{spm}(\mathscr{J}) &= 1 - \left(1 - \frac{1}{\xi_\varphi^{\mathscr{E}}}\right)\frac{1/(1-s)}{(1/\sigma)\eta} + \frac{1}{\xi_\varphi^{\mathscr{E}}}\frac{(1/\sigma)\eta\big((1-s)\eta/s\big)}{(1/\sigma)\eta} - \frac{-s/(1-s)\sigma}{(1/\sigma)\eta}, \\ \det(\mathscr{J}) &= \frac{1}{\xi_\varphi^{\mathscr{E}}}.\end{aligned}$$

The statement follows. △

Ce quatrième jour de février de l'an deux mille et huit.