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Aggregation of State-Dependent Utilities

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Abstract

In an exchange economy under uncertainty populated by consumers having state-contingent utility functions, we analyze the nature of the efficient risk-sharing rules and the representative consumer's state-contingent utility function. We show that the representative consumer's responsiveness to state variables will typically depend on aggregate consumption levels even when the individual consumers' responsiveness do not depend on own consumptions. We also find that the heterogeneity in the individual consumers' responsiveness to state variables gives rise to a "convexifying effect" on the representative consumer's utility function, in a sense to be made precise. We also present applications of this result to the cases of heterogeneous beliefs and heterogeneous impatience.

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1 Introduction

In dynamic macroeconomics and finance, the use of representative-consumer models is prevalent. As in Mehra and Prescott (1985), the standard (and by now classical) representative-consumer model consists of a single consumer having a utility function U of the form

$$U(c) = E\left(\sum_{t=0}^{\infty} \delta^t \frac{c_t^{1-\beta}}{1-\beta}\right) \text{ or } U(c) = E\left(\int_0^{\infty} \exp(-\rho t) \frac{c_t^{1-\beta}}{1-\beta}\right),$$

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depending on whether the time span is discrete or continuous, and an initial endowment process $e = (e_t)_t$. Given that there is only one consumer, the equilibrium of such an economy must necessarily be the no-trade equilibrium, in which the consumer is induced to demand his own endowment process $e = (e_t)_t$. The equilibrium state price deflator $\pi = (\pi_t)_t$, which evaluates each consumption process $c = (c_t)_t$ via $E\left(\sum_{t=0}^{\infty} \pi_t c_t\right)$ or $E\left(\int_0^{\infty} \pi_t c_t\right)$, can then be written in the simple form of $\delta^t c_t^{-\beta}$ or $\exp(-\rho t) c_t^{-\beta}$. The task of identifying equilibrium asset price process with dividend process $d = (d_t)_t$ can therefore be reduced to one of calculating

$$E_t\left(\sum_{\tau=t}^{\infty} \delta^{t-\tau} \left(\frac{e_\tau}{e_t}\right)^{-\beta} d_\tau\right) \text{ or } E_t\left(\int_t^{\infty} \exp(-\rho(\tau - t)) \left(\frac{e_\tau}{e_t}\right)^{-\beta} d_\tau \, d\tau\right)$$

at each time t.

There are a couple of important assumptions embedded in this specification. First, the representative consumer has an expected utility function, thereby conforming the independence axiom. Second, the discount rate is deterministic, constant, and independent of consumption levels. Third, the representative consumer exhibits constant relative risk aversion.

When we take up any representative-consumer model, we are not really interested in the analysis of an economy consisting of a single consumer per se. Rather, we regard the representative consumer economy as a reduced economy consisting of multiple, heterogeneous individuals. Then a question arises: if we explicitly model an economy of multiple, heterogeneous individuals and derive the utility function for the representative consumer by aggregating their utility functions, are we likely to obtain an expected utility function, with the discount rate independent of time and consumption levels and the relative risk aversion constant? This paper is devoted to giving a negative answer to this question. We see that the expected utility function is unlikely to be obtained as the representative consumer's probabilistic belief is likely to depend on consumption levels; his discount rate is likely to depend on consumption levels and decrease over time; and his relative risk aversion is more likely to be decreasing rather than constant. These violations of standard properties can occur even when all individual consumers have an expected utility function with the discount rate deterministic, constant, and independent of consumption levels, and his relative risk aversion is constant.

Note that we are not arguing that the representative-consumer approach is logically incorrect or internally inconsistent. Rather, we are claiming that the specification of a utility function for the representative consumer needs some care and cannot be based on any justification that can typically given for individual consumers' utility functions, such as results of laboratory experiments. We are also aiming at general results, in the sense that the subsequent analysis does not depend on the number of individual consumers in the economy, the form of their utility functions, the wealth distribution across them, or the stochastic nature of their consumption processes. On the other hand, we maintain the assumption of complete asset markets and the assumption of state- and time-separability for utility functions. In particular, we cannot include recursive utility functions or utility functions of habit formation. These utility functions are interesting and important, but we would like to make full use of the existing results on aggregation of state- and time-separable utility functions.

This paper is organized as follows. Section 2 spells out our model and review some elementary and well known results. Section 3 establishes some general formulas relating the representative consumer's risk attitudes, discount rates, and probabilistic beliefs to the individual consumers' counterparts. Section 4 gives some economic interpretations to these formulas. Section 5 summarizes these results and suggests directions of future research.

2 Setup

2.1 Uncertainty and consumers

The setup of this paper is as follows. As usual, we represent the uncertainty surrounding the economy by a probability measure space $(\Omega, \mathscr{F}, P)$. In addition, to formulate state-dependent utility (felicity) functions, we use (Z, h), where Z is a nonempty, open subset of a finite-dimensional Euclidean space $\boldsymbol{R}^L$, and h is a measurable mapping of Ω into Z. Each element of Z is a *state variable*, which, as will be seen in the next paragraph, completely determines consumers' utility (felicity) functions. Thus, L denotes the dimension of a state variable and h specifies which state gives rise to which state variable. At this outset, the time span along which consumption can take place is not explicitly modeled and, as such, we assume for a moment that there is only one consumption period.

The economy consists of I consumers. Each consumer i has a possibly state-dependent felicity function $u_i : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$, which is at least twice continuously differentiable,[1] and satisfies $\partial u_i(x_i, z)/\partial x_i > 0 > \partial^2 u_i(x_i, z)/\partial (x_i)^2$ for every $(x_i, z) \in \boldsymbol{R}_{++} \times Z$ and the Inada condition, that is, for every $z \in Z$, $\partial u_i(x_i, z)/\partial x_i \to 0$ as $x_i \to \infty$, and $\partial u_i(x_i, z)/\partial x_i \to \infty$ as $x_i \to 0$. When the state variables are given by a measurable function $h : \Omega \to Z$, his utility function U_i over consumptions $c_i : \Omega \to \boldsymbol{R}_{++}$ are defined by requiring expected utility:

$$U_i(c_i) = E(u_i(c_i, h)) = \int_\Omega u_i\left(c_i(\omega), h(\omega)\right)\, dP(\omega), \tag{1}$$

To be exact, we need to impose some additional restrictions on c_i to make the integral well defined (finite). As such restrictions are irrelevant to the subsequent analysis, we shall not explicitly state or impose them.

The key parameters of the felicity function u_i (and thus of the utility function U_i) are *risk tolerance* and *responsiveness* to the state variables. The risk tolerance $s_i : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}_{++}$ is defined by

$$s_i(x_i, z) = -\frac{\partial u_i(x_i, z)/\partial x_i}{\partial^2 u_i(x_i, z)/\partial (x_i)^2}.$$

This is the reciprocal of the Arrow-Pratt measure of absolute risk aversion, with the dependence on z allowed for. The partial derivative with respect to x_i, $\partial s_i(x_i, t)/\partial x_i$, is called the

[1]The degree of continuous differentiability needed in each of the subsequent results will be made clear in its proof. All such conditions can be satisfied by imposing sufficiently high (finite) degrees of continuous differentiability on u_i.

cautiousness. The *responsiveness* to state variables, $q_i : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}^L$ is defined by

$$q_i(x_i, z) = \begin{pmatrix} q_i^1(x_i, z) \\ \vdots \\ q_i^L(x_i, z) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 u_i(x_i, z)/\partial z^1 \partial x_i}{\partial u_i(x_i, z)/\partial x_i} \\ \vdots \\ \dfrac{\partial^2 u_i(x_i, z)/\partial z^L \partial x_i}{\partial u_i(x_i, z)/\partial x_i} \end{pmatrix}.$$

Note then that

$$\frac{\dfrac{\partial u_i}{\partial x_i}\left(x_i, \underline{z}^1, \dots, \underline{z}^{\ell-1}, \overline{z}^\ell, \underline{z}^{\ell+1}, \dots, \underline{z}^L\right)}{\dfrac{\partial u_i}{\partial x_i}\left(x_i, \underline{z}^1, \dots, \underline{z}^{\ell-1}, \underline{z}^\ell, \underline{z}^{\ell+1}, \dots, \underline{z}^L\right)} = \exp\left(\int_{\underline{z}^\ell}^{\overline{z}^\ell} q_i^\ell\left(x_i, \underline{z}^1, \dots, \underline{z}^{\ell-1}, z^\ell, \underline{z}^{\ell+1}, \dots, \underline{z}^L\right) dz^\ell\right)$$

for all i, ℓ, and $\left(z^1, \dots, z^{\ell-1}, z^{\ell+1}, \dots, z^L\right) \in \boldsymbol{R}^{L-1}$, $\underline{z}^\ell \in \boldsymbol{R}$, and $\overline{z}^\ell \in \boldsymbol{R}$ whenever $\underline{z}^\ell \leq \overline{z}^\ell$ and $\left(z^1, \dots, z^{\ell-1}, z^\ell, z^{\ell+1}, \dots, z^L\right) \in Z$ for every $z^\ell \in \left[\underline{z}^\ell, \overline{z}^\ell\right]$.

An important class of utility functions is that of multiplicatively separable utility functions. A utility function u_i is *multiplicatively separable* if there are two functions $v_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$ and $p_i : Z \to \boldsymbol{R}_{++}$ such that $u_i(x_i, z) = p_i(z) v_i(x_i)$ for every $(x_i, z) \in \boldsymbol{R}_{++} \times Z$.[2] Then $s_i(x_i, z) = -v_i'(x_i)/v_i''(x_i)$ and

$$q_i(x_i, z) = \begin{pmatrix} \dfrac{\partial p_i(z)/\partial z_1}{p_i(z)} \\ \vdots \\ \dfrac{\partial p_i(z)/\partial z_L}{p_i(z)} \end{pmatrix},$$

or, more succinctly,[3]

$$(q_i(x_i, z))^\top = \frac{1}{p_i(z)} \nabla p_i(z) = \nabla \left(\ln p_i(z)\right).$$

We thus write $s_i(x_i)$ for $s_i(x_i, z)$ and $q_i(z)$ for $q_i(x_i, z)$ in this case. Then, $q_i^\ell(z)$ is equal to the percentage change in $p(z)$ when the ℓ-th state variable z^ℓ is changed by one unit, and $\ln p$ is a potential function of the vector field q.[4] Moreover,

$$\frac{p_i^\ell\left(z^1, \dots, z^{\ell-1}, \overline{z}^\ell, z^{\ell+1}, \dots, z^L\right)}{p_i^\ell\left(z^1, \dots, z^{\ell-1}, \underline{z}^\ell, z^{\ell+1}, \dots, z^L\right)} = \exp\left(\int_{\underline{z}^\ell}^{\overline{z}^\ell} q_i\left(z^1, \dots, z^{\ell-1}, z^\ell, z^{\ell+1}, \dots, z^L\right) dz^\ell\right)$$

for all i, ℓ, and $\left(z^1, \dots, z^{\ell-1}, z^{\ell+1}, \dots, z^L\right) \in \boldsymbol{R}^{L-1}$, $\underline{z}^\ell \in \boldsymbol{R}$, and $\overline{z}^\ell \in \boldsymbol{R}$ whenever $\underline{z}^\ell \leq \overline{z}^\ell$ and $\left(z^1, \dots, z^{\ell-1}, z^\ell, z^{\ell+1}, \dots, z^L\right) \in Z$ for every $z^\ell \in \left[\underline{z}^\ell, \overline{z}^\ell\right]$.

[2] Wilson (1968) called v_i' a *surrogate marginal utility function* and p_i a *surrogate* probability assessment function in the context of heterogeneous beliefs.

[3] We regard the gradient vector $\nabla_z w(x, z)$ as a row vector. Since $q(x, z)$ is a column vector, we need to take the transpose $^\top$ to have an equality between the two.

[4] More generally, when u_i need not be multiplicatively separable, for each *fixed* $x \in \boldsymbol{R}_{++}$, $z \mapsto q_i(x, z)$ is a potential function of the vector field $z \mapsto \ln\left(\partial u_i(x, z)/\partial x\right)$.

2.2 Characterization of Pareto-efficient allocations

To find a Pareto efficient allocation of a given aggregate consumption $c : \Omega \to \boldsymbol{R}_{++}$ and its supporting (decentralizing) state-price deflator when the state variable is given by $h : \Omega \to Z$, it is sufficient to choose positive numbers $\lambda_1, \ldots, \lambda_I$ and consider the following maximization problem:

$$\begin{aligned} \max_{(c_1,\ldots,c_I)} \quad & \sum_i \lambda_i U_i(c_i) \\ \text{subject to} \quad & \sum_i c_i = c. \end{aligned} \tag{2}$$

Since the utility functions U_i are additive with respect to states and the expected utilities are calculated with respect to the common probability measure P, it can be rewritten as

$$\sum_i \lambda_i U_i(c_i) = E\left(\sum_i \lambda_i u_i(c_i, h)\right) = \int_\Omega \left(\sum_i \lambda_i u_i(c_i(\omega), h(\omega))\right) dP(\omega).$$

Hence, to solve the original maximization problem (2), it suffices to solve the simplified maximization problem

$$\begin{aligned} \max_{(x_1,\ldots,x_I)\in \boldsymbol{R}^I_{++}} \quad & \sum_i \lambda_i u_i(x_i, z) \\ \text{subject to} \quad & \sum_i x_i = x. \end{aligned} \tag{3}$$

for each pair of a realized aggregate consumption level $x \in \boldsymbol{R}_{++}$ and a state variable $z \in Z$. It can be easily proved that under the stated conditions, there is a unique solution, which we denote by $(f_1(x,z), \ldots, f_I(x,z))$. It can also be shown that for each f_i is continuously differentiable in both variables. We can define the value function of this problem $u : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ by

$$u(x,z) = \sum_i \lambda_i u_i \left(f_i(x,z), z\right).$$

This is the felicity function of the *representative consumer*. Then the solution to the original maximization problem is given by $(c^1, \ldots, c^I)$, where, for each i, $c^i : \Omega \to \boldsymbol{R}_{++}$ is defined by $c^i(\omega) = f_i\left(c(\omega), h(\omega)\right)$ for every $\omega \in \Omega$. The representative consumer's utility function is defined by $U(c) = E\left(u(c,h)\right)$.

Just as for an individual consumer's utility function, we define risk tolerance and responsiveness to state variables for the representative consumer as follows:

$$s(x,z) = -\frac{\partial u(x,z)/\partial x}{\partial^2 u_i(x,z)/\partial x^2},$$

$$q_i(x_i,z) = \begin{pmatrix} \dfrac{\partial^2 u_i(x_i,z)/\partial z^1 \partial x_i}{\partial u_i(x_i,z)/\partial x_i} \\ \vdots \\ \dfrac{\partial^2 u_i(x_i,z)/\partial z^L \partial x_i}{\partial u_i(x_i,z)/\partial x_i} \end{pmatrix}.$$

The cautiousness is defined as the partial derivative $\partial s(x,z)/\partial x$ with respect to the aggregate consumption level x. We will see that the representative consumer's felicity function u need not be multiplicatively separable between the aggregate consumption level x and the state variable z even when all individual consumers' felicity functions u_i are multiplicatively separable.

The representative consumer is, of course, not an "actual" consumer, who would trade on financial markets. Rather, he is a theoretical construct, whom we can use to identify asset prices. Specifically, if u is the representative consumer's felicity function and c is the aggregate consumption process, then his marginal utility process evaluated at the aggregate consumption, $(\partial u(c,h)/\partial x)$, is a state price density. This means that the price of an asset with dividend $\delta : \Omega \to \boldsymbol{R}$, relative to the risk-free bond (which pays off one unit of the commodity whichever state has been realized), is equal to

$$\frac{E\left(\dfrac{\partial u(c,h)}{\partial x}\delta\right)}{E\left(\dfrac{\partial u(c,h)}{\partial x}\right)}.$$

Although we analyze the Pareto efficient allocations and their supporting (decentralizing) prices, if the asset markets are complete, then our analysis is applicable to the equilibrium allocations and asset prices. This is because the first welfare theorem holds in complete markets, so that the equilibrium allocations are Pareto efficient and the equilibrium asset prices are given by the corresponding support prices. Since the $u_i(\cdot, z)$ are concave, the second welfare theorem also holds, so that every Pareto efficient allocation is an equilibrium allocation for some distribution of initial endowments. Hence an analysis of Pareto efficient allocations is also an analysis of equilibrium allocations.

When the solution to the maximization problem (2) is an equilibrium allocation, the individual consumers' wealth shares, evaluated by the equilibrium prices, determines the utility weights λ_i in (2). All the properties we shall explore in the subsequent analysis are valid regardless of the choice of utility weights. Hence, these properties are also valid for the equilibrium allocations regardless of wealth distributions.

3 General Formulas

We now present various formulas on the representative consumer's risk attitudes and responsiveness to state variables.

3.1 Formulas in the existing literature

We start by presenting some of the formulas in the existing literature to give the idea of what kind of formulas we aim at establishing in this paper. By Theorems 4 and 5 of Wilson (1968) to the $u_i(\cdot, z)$ for each $z \in Z$, we obtain

$$s(x,z) = \sum_i s_i(f_i(x,z),z), \tag{4}$$

$$\frac{\partial f_i}{\partial x}(x,z) = \frac{s_i(f_i(x,z),z)}{s(x,z)}. \tag{5}$$

By differentiating both sides of (4) with respect to x and applying (5), we obtain

$$\frac{\partial s}{\partial x}(x,z) = \sum_i \frac{\partial f_i}{\partial x}(x,z)\frac{\partial s_i}{\partial x_i}\left(f_i(x,z),z\right) = \sum_i \frac{s_i\left(f_i(x,z),z\right)}{s(x,z)}\frac{\partial s_i}{\partial x_i}\left(f_i(x,t),z\right) \tag{6}$$

This shows that the representative consumer's cautiousness is the weighted average of the individual consumers' counterparts, where the weights are proportional to their absolute risk tolerance.

Also, by Theorem 4 of Hara, Huang, and Kuzmics (2007),

$$\begin{aligned}\frac{\partial^2 s}{\partial x^2}(x,z) =& \sum_i \left(\frac{s_i\left(f_i(x,z),z\right)}{s(x,z)}\right)^2 \frac{\partial^2 s_i}{\partial\left(x_i\right)^2}\left(f_i(x,z),z\right) \\ &+ \frac{1}{s(x,z)}\sum_i \frac{s_i(f_i(x,z))}{s(x,z)}\left(\frac{\partial s_i}{\partial x_i}(f_i(x,z),z) - \frac{\partial s}{\partial x}(x,z)\right)^2.\end{aligned}$$

To appreciate this formula, note first that by (5), the first term on the right-hand side can be written as

$$\sum_i \frac{s_i\left(f_i(x,z),z\right)}{s(x,z)}\left(\frac{\partial^2 s_i}{\partial\left(x_i\right)^2}\left(f_i(x,z),z\right)\frac{\partial f_i}{\partial x}(x,z)\right)$$

Here, the term $\left(\partial^2 s_i\left(f_i(x,z),z\right)/\partial\left(x_i\right)^2\right)\left(\partial f_i(x,z)/\partial x\right)$ is the change in the cautiousness of consumer i arising from the increase in his consumption level, which is, in turn, caused by an increase in the aggregate consumption level. Thus the first term represents the direct effect on the representative consumer's cautiousness by an increase in aggregate consumption, hypothetically taking the weights $\left(\partial s_i\left(f_i(x,z),z\right)/\partial x_i\right)/s(x,z)$ as fixed. By (4), the second term on the right-hand side is the weighted variance of the individual consumers' cautiousness, divided by the representative consumer's absolute risk aversion. It can be shown that this term is equal to

$$\sum_i \frac{d}{dx}\left(\frac{s_i\left(f_i(x,z)\right)}{s(x,z)}\right)\frac{\partial s_i}{\partial x_i}\left(f_i(x,z),z\right).$$

Thus it represents the change in the representative consumer's cautiousness arising from the change in the weights $\left(\partial s_i\left(f_i(x,z),z\right)/\partial x_i\right)/s(x,z)$, hypothetically taking the individual consumers' cautiousness $\partial s_i\left(f_i(x,z),z\right)/\partial x_i$ as fixed. It shows that this indirect effect on the representative consumer's cautiousness is proportionally related to the weighted variance of the individual consumers' cautiousness; and the subsequent analysis is an attempt to capture this sort of indirect effects arising from heterogeneity. This formula therefore shows that the heterogeneity in the individual consumers' cautiousness increases the representative consumer's cautiousness, thereby making his risk tolerance, as a function of aggregate consumption levels, more convex.

3.2 Responsiveness to state variables

Our first proposition is concerned with the representative consumer's responsiveness to state variables and how the risk-sharing rules depend on state variables. They are analogous to equality (10) of Wison (1968), Theorem 1 of Amershi and Stoeckenius (1983), and equality (10) and Proposition 3 of Gollier and Zeckhauser (2005), who dealt with heterogeneous impatience.

Theorem 1 *For every* $(x, z) \in \boldsymbol{R}_{++} \times Z$*,*

$$\nabla_z f_i(x,z) = s_i(f_i(x,z),z)\left(q_i(f_i(x,z),z) - q(x,z)\right), \tag{7}$$

$$q(x,z) = \sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)} q_i(f_i(x,z),z). \tag{8}$$

(7) implies that if an individual consumer is more responsive to state variables than the representative consumer, his consumption level would be higher the higher the value the state variable takes. On the other hand, if he is less responsive to state variables than the representative consumer, his consumption level would be lower the higher the value the state variable takes. Moreover, the change in his consumption levels per unit change of the state variable is proportional to his absolute risk tolerance. The proportionality of the change with respect to the risk tolerance is quite intuitive: Even when he is more responsive to state variables than the representative consumer, if he were quite risk-averse ($s_i\left(f_i(x,z),z\right)$ being quite low), then his consumption would not be much affected by the state variable, making $\left|\partial f_i\left(x,z\right)/\partial z^\ell\right|$ almost zero for every ℓ.

(8) means that the representative consumer's responsiveness is the weighted average of the individual consumers' counterparts where the weights are proportional to their absolute risk tolerance. This is analogous to (6), which shows that the representative consumer's cautiousness is the weighted average of the individual consumers' counterparts, with the same weights.

Proof of Theorem 1 The first-order condition for the maximization problem (2) is

$$\lambda_i \frac{\partial u_i}{\partial x_i}\left(f_i(x,z),z\right) = \frac{\partial u}{\partial x}(x,z) \tag{9}$$

for all i and $(x, z) \in \boldsymbol{R}_{++} \times Z$. By differentiating both sides with respect to z^ℓ, we obtain

$$\lambda_i\left(\frac{\partial^2 u_i}{\partial x_i^2}\left(f_i(x,z),z\right)\frac{\partial f_i}{\partial z^\ell}(x,z) + \frac{\partial^2 u_i}{\partial z^\ell \partial x_i}\left(f_i(x,z),z\right)\right) = \frac{\partial^2 u}{\partial z^\ell \partial x}(x,z). \tag{10}$$

By dividing both sides of (10) by those of (9), we obtain

$$\frac{\partial f_i}{\partial z^\ell}(x,z) = s_i(f_i(x,z),z)\left(q_i^\ell(f_i(x,z),z) - q^\ell(x,z)\right). \tag{11}$$

This implies, in the vector notation, (7). Also, by adding both sides of (11) and using $\sum_i \left(\partial f_i(x,z)/\partial z^\ell\right) = 0$, we obtain

$$q^\ell(x,z) = \sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)} q_i^\ell(f_i(x,z),z).$$

This implies, in the vector notation, (8). ///

3.3 Risk tolerance

We are interested in how the representative consumer's risk tolerance and responsiveness to state variables changes as the value that the state variables take varies, when the individual consumers' counterparts are heterogeneous.

The following theorem generalizes a formula presented in the proof of Theorem 3.3 of Malamud and Trubowitz (2006), in a sense to be made precise later.

Theorem 2 *For every ℓ and every $(x,t) \in \boldsymbol{R}_{++} \times \boldsymbol{R}_+$,*

$$\begin{aligned}\frac{\partial s}{\partial z^\ell}(x,z) =& \sum_i \frac{\partial s_i}{\partial z^\ell}\left(f_i(x,z),z\right) \\ &+ s(x,z) \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} \left(\frac{\partial s_i}{\partial x_i}(f_i(x,z),z) - \frac{\partial s}{\partial x}(x,z)\right)\left(q_i^\ell(f_i(x,z)) - q^\ell(x,z)\right).\end{aligned} \tag{12}$$

This theorem tells us that the change in the representative consumer's risk tolerance in response to a change in state variables can be decomposed into two terms. The first term is easy to grasp. As shown by (4), the representative consumer's risk tolerance is the sum of the individual consumers' counterparts. Thus the first term represents the direct effect on risk tolerance by the change in state variables. It is equal to zero when all individual consumers' felicity functions u_i are multiplicatively separable.

By (6) and (8), the second term of (12) is equal to the weighted covariance, multiplied by the representative consumer's risk tolerance, between the individual consumers' cautiousness and responsiveness to state variables, where the weights are proportional to the individual consumers' risk tolerance. Since the second term would be zero if all consumers' cautiousness or responsiveness are equal to one another, it captures the tendency of changes in the representative consumer's risk tolerance that arise from the heterogeneity in the individual consumers' cautiousness and responsiveness.

Proof of Theorem 2 By differentiating both sides of (4) with respect to z^ℓ, we obtain

$$\begin{aligned}\frac{\partial s}{\partial z^\ell}(x,z) &= \sum_i \left(\frac{\partial s_i}{\partial x_i}(f_i(x,z),)\frac{\partial f_i}{\partial z^\ell}(x,z) + \frac{\partial s_i}{\partial z^\ell}(f_i(x,z),z)\right) \\ &= \sum_i \frac{\partial s_i}{\partial z^\ell}(f_i(x,z),z) + \sum_i \frac{\partial s_i}{\partial x_i}(f_i(x,z),z) s_i(f_i(x,z),z)\left(q_i^\ell(f_i(x,t)) - q^\ell(x,t)\right),\end{aligned} \tag{13}$$

where the last equality follows from (11). Since

$$\sum_i s_i(f_i(x,z),z)\left(q_i^\ell(f_i(x,z),z)-q^\ell(x,z)\right)=0,$$

the second term of (13) can be written as

$$\sum_i \left(\frac{\partial s_i}{\partial x_i}(f_i(x,z),z)-\frac{\partial s}{\partial x}(x,z)\right) s_i(f_i(x,z),z)\left(q_i^\ell(f_i(x,z),z)-q^\ell(x,z)\right).$$

This is equal to the second term of the right-hand side of (12). ///

Corollary 1 *Suppose that u_i is multiplicatively separable for every i. Then*

$$\frac{\partial s(x,z)/\partial z^\ell}{s(x,z)}=\sum_i \frac{s_i(f_i(x))}{s(x,z)}\left(s_i'(f_i(x,z))-\frac{\partial s}{\partial x}(x,z)\right)\left(q_i^\ell(z)-q^\ell(x,z)\right)$$

for every ℓ and every $(x,t)\in \boldsymbol{R}_{++}\times\boldsymbol{R}_+$. Moreover,

1. *If $(s_1'(f_1(x,z)),\dots,s_I'(f_I(x,z)))$ and $\left(q_1^\ell(z),\dots,q_I^\ell(z)\right)$ are comonotone (that is, $\left(s_i'(f_i(x,z))-s_j'(f_j(x,z))\right)\left(q_i^\ell(z)-q_j^\ell(z)\right)\geq 0$ for every pair of consumers i and j), then $\partial s(x,z)/\partial z^\ell\leq 0$. This weak inequality holds as an equality if and only if $s_1'(f_1(x,z))=\cdots=s_I'(f_I(x,z))$ or $q_1^\ell(z)=\cdots=q_I^\ell(z)$.*

2. *If $(s_1'(f_1(x,z)),\dots,s_I'(f_I(x,z)))$ and $\left(q_1^\ell(z),\dots,q_I^\ell(z)\right)$ are anti-comonotone (that is, $\left(s_i'(f_i(x,z))-s_j'(f_j(x,z))\right)\left(q_i^\ell(z)-q_j^\ell(z)\right)\leq 0$ for every pair of consumers i and j), then $\partial s(x,z)/\partial z^\ell\leq 0$. This weak inequality holds as an equality if and only if $s_1'(f_1(x,z))=\cdots=s_I'(f_I(x,z))$ or $q_1^\ell(z)=\cdots=q_I^\ell(z)$.*

This corollary (and also subsequent results) can be best understood by considering the case where $L=1$, $Z=\boldsymbol{R}_+$, and for every i, there exist a $\gamma_i>0$ and an $\eta_i\in\boldsymbol{R}$ such that $s_i(x_i)=\gamma_i$ for every x_i and $q_i(z)=\eta_i$ for every z. In this case, the felicity functions can be written as $u_i(x_i,z)=p(z)v_i(x_i)$, where

$$v_i(x_i)=\frac{x_i^{1/\gamma_i}}{1-1/\gamma_i}\text{ and } p_i(z)=\exp(\eta_i z).$$

That is, v_i exhibits constant relative risk aversion and the multiplier p_i depends exponentially on the state variable z. The advantage of this case is that the (anti-)comonotonicity condition in Corollary 1 can be checked without knowing (x,z) or the f_i. Also, this case best illustrates the fact that it is implausible to impose multiplicative separability on the representative consumer's felicity function. Even if each individual consumer's felicity function is multiplicatively separable, the representative consumer's need not, depending on how the cautiousness and responsiveness are correlated with each other. In particular, if they are (anti-)comonotonically related, then it cannot be multiplicatively separable. As we will see later, this fact has an important implication on when the representative consumer's utility function can be written in the expected utility form.

3.4 Derivatives of the responsiveness

We turn our attention to how the representative consumer's responsiveness to state variables is affected by the change in aggregate consumption levels or in state variables.

Theorem 3 *For every ℓ and $(x,z) \in \boldsymbol{R}_{++} \times Z$,*

$$\begin{aligned}\frac{\partial q^\ell}{\partial x}(x,z) =& \sum_i \left(\frac{s_i\left(f_i(x,z),z\right)}{s(x,z)}\right)^2 \frac{\partial q_i^\ell}{\partial x_i}\left(f_i(x,z),z\right) \\ &+ \frac{1}{s(x,z)} \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} \left(\frac{\partial s_i}{\partial x_i}(f_i(x,z),z) - \frac{\partial s}{\partial x}(x,z)\right)\left(q_i^\ell(f_i(x,z)) - q^\ell(x,z)\right). \end{aligned} \tag{14}$$

Just as Theorem 2, this theorem tells us that the change in the representative consumer's impatience can be decomposed into two terms. The first term is easy to grasp. By (5), the first term can be rewritten as

$$\sum_i \frac{s_i\left(f_i(x,z),z\right)}{s(x,z)} \left(\frac{\partial q_i^\ell}{\partial x_i}\left(f_i(x,z),z\right) \frac{\partial f_i}{\partial x}(x,z)\right)$$

Here, the term $\left(\partial q_i\left(f_i(x,z),z\right)/\partial x_i\right)\left(\partial f_i(x,z)/\partial x\right)$ is the change in the responsiveness of consumer i arising from the increase in his consumption level, which, in turn, caused by an increase in the aggregate consumption level. Thus the first term represents the direct effect on the representative consumer's responsiveness by the change in aggregate consumption. It is equal to zero when all individual consumers' felicity functions u_i are multiplicatively separable.

The second term is equal to the weighted covariance, divided by the representative consumer's risk tolerance, between the individual consumers' cautiousness and responsiveness, where the weights are proportional to the individual consumers' risk tolerance. Since the second term would be zero if all consumers' cautiousness or responsiveness are equal to one another, it captures the tendency of changes in the representative consumer's responsiveness that arise from the heterogeneity in the individual consumers' cautiousness and responsiveness.

Proof of Theorem 3 By (8),

$$s(x,z)q^\ell(x,z) = \sum_i s_i(f_i(x,z),z)q_i^\ell(f_i(x,z),z). \tag{15}$$

By differentiate both sides of (15) with respect to x, we obtain

$$\begin{aligned}
&\frac{\partial s}{\partial x}(x,z)q^\ell(x,z) + s(x,z)\frac{\partial q^\ell}{\partial x}(x,z) \\
=&\sum_i \left(\frac{\partial s_i}{\partial x_i}(f_i(x,z),z)\frac{\partial f_i}{\partial x}(x,z)q_i^\ell\left(f_i(x,z),z\right) + s_i\left(f_i(x,z),z\right)\frac{\partial q_i^\ell}{\partial x_i}(f_i(x,z),z)\frac{\partial f_i}{\partial x}(x,z)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial q^\ell}{\partial x}(x,z) =&\frac{1}{s(x,z)}\sum_i s_i\left(f_i(x,z),z\right)\frac{\partial q_i^\ell}{\partial x_i}(f_i(x,z),z)\frac{\partial f_i}{\partial x}(x,z) \\
&+\frac{1}{s(x,z)}\left(\sum_i \frac{\partial s_i}{\partial x_i}(f_i(x,z),z)\frac{\partial f_i}{\partial x}(x,z)q_i^\ell\left(f_i(x,z),z\right) - \frac{\partial s}{\partial x}(x,z)q^\ell(x,z)\right) \\
=&\sum_i\left(\frac{s_i\left(f_i(x,z),z\right)}{s(x,z)}\right)^2\frac{\partial q_i^\ell}{\partial x_i}(f_i(x,z),z) \\
&+\frac{1}{s(x,z)}\left(\sum_i \frac{s_i\left(f_i(x,z),z\right)}{s(x,z)}\frac{\partial s_i}{\partial x_i}(f_i(x,z),z)q_i^\ell\left(f_i(x,z),z\right) - \frac{\partial s}{\partial x}(x,z)q^\ell(x,z)\right),
\end{aligned}$$

where the last equality follows from (5). By (4), (6), and (8),

$$\begin{aligned}
&\sum_i \frac{s_i\left(f_i(x,z),z\right)}{s(x,z)}\frac{\partial s_i}{\partial x_i}(f_i(x,z),z)q_i^\ell\left(f_i(x,z),z\right) - \frac{\partial s}{\partial x}(x,z)q^\ell(x,z) \\
=&\sum_i \frac{s_i(f_i(x,z))}{s(x,z)}\left(\frac{\partial s_i}{\partial x_i}(f_i(x,z),z) - \frac{\partial s}{\partial x}(x,z)\right)\left(q_i^\ell(f_i(x,z)) - q^\ell(x,z)\right).
\end{aligned}$$

The proof is thus completed. ///

If all the u_i are multiplicatively separable, then

$$\frac{\partial q^\ell}{\partial x}(x,z) = \frac{1}{s(x,z)}\sum_i\frac{s_i(f_i(x))}{s(x,z)}\left(s_i'(f_i(x,z)) - \frac{\partial s}{\partial x}(x,z)\right)\left(q_i^\ell(z) - q^\ell(x,z)\right) \tag{16}$$

for every ℓ and $(x,z) \in \boldsymbol{R}_{++} \times Z$. Just as Corollary 1, this shows that the representative consumer's utility function is unlikely to be multiplicatively separable even the individual counterparts are, unless their cautiousness and responsiveness has exactly zero covariance.

We next give a formula for $\partial q^\ell(x,z)/\partial z^k$, which shows how the representative consumer's responsiveness to one state variable z^ℓ is affected by a change in another state variable z^k.

Theorem 4 *For all ℓ, k, and $(x,z) \in \boldsymbol{R}_{++} \times \boldsymbol{R}_{+}$,*

$$
\begin{aligned}
&\frac{\partial q^\ell}{\partial z^k}(x,z) \\
=&\sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)} \frac{\partial q_i^\ell}{\partial z^k}(f_i(x,z),z) \\
&+\sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)} \frac{\partial s_i}{\partial x_i}(f_i(x,z),z) \left(q_i^\ell(f_i(x,z),z) - q^\ell(x,z)\right) \left(q_i^k(f_i(x,z),z) - q^k(x,z)\right) \\
&+\sum_i \frac{(s_i(f_i(x,z),z))^2}{s(x,z)} \frac{\partial q_i^\ell}{\partial x_i}(f_i(x,z),z) \left(q_i^k(f_i(x,z),z) - q^k(x,z)\right) \\
&+\sum_i \frac{\frac{\partial s_i}{\partial z^k}(f_i(x,z),z)}{s(x,z)} \left(q_i^\ell(f_i(x,z),z) - q^\ell(x,z)\right). \qquad (17)
\end{aligned}
$$

This theorem tells us that the changes in the representative consumer's responsiveness to state variables can be decomposed into four terms. The first term is easy to grasp. As shown by (8), the representative consumer's responsiveness is equal to the weighted average of the individual consumers' counterparts, where the weights are proportional to their risk tolerance. Thus the first term represents the direct effect, by the change in state variables, on the representative consumer's responsiveness, while the weights are hypothetically fixed.

The third and fourth terms represent the change in the representative consumer's responsiveness caused by the impact on the individual consumers' responsiveness by the change in consumption levels and the impact on the individual consumers' risk tolerance by state variables. These terms are equal to zero if all consumers' felicity functions are multiplicatively separable.

The second term is most interesting. It represents the impact on the representative consumer's responsiveness when the individual consumers have differing responsiveness. As mentioned above, the representative consumer's responsiveness is equal to the weighted average of the individual consumers' counterparts, and the weights are proportional to their risk tolerance. If their responsiveness is different, then the risk-sharing rules f_i would depend on the state variable z^k; that is, the partial derivative $\partial f_i(x,z)/\partial z^k$ would be different from zero. Unless the cautiousness, $\partial s_i(f_i(x,z),z)/\partial x_i$, is zero (which would be the case if u_i exhibited constant absolute, rather than relative, risk aversion), the change in consumption levels has an impact on the individual consumers' risk tolerance, and thus on the representative consumer's responsiveness, which is the weighted average of the individual consumers' responsiveness, with the weights given by their risk tolerance. The second term, therefore, captures the change in the representative consumer's responsiveness arising from the heterogeneity in the individual consumers' responsiveness.

Proof of Theorem 4 By (8) and differentiation for a product,

$$\frac{\partial q^\ell}{\partial z^k}(x,z) = \sum_i \frac{d}{dz^k}\left(\frac{s_i(f_i(x,z),z)}{s(x,z)}\right) q_i^\ell\left(f_i(x,z),z\right) + \sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)} \frac{d}{dz^k}\left(q_i^\ell\left(f_i(x,z),z\right)\right). \tag{18}$$

By (7),

$$\begin{aligned}
&\frac{d}{dz^k}\left(\frac{s_i(f_i(x,z),z)}{s(x,z)}\right) \\
=&\frac{\frac{\partial s_i}{\partial x_i}(f_i(x,z),z)}{s(x,z)} \frac{\partial f_i}{\partial z^k}(x,z) + \frac{\frac{\partial s_i}{\partial z^k}(f_i(x,z),z)}{s(x,z)} - \frac{s_i(f_i(x,z),z)}{(s(x,z))^2}\frac{\partial s}{\partial z^k}(x,z) \\
=&\frac{s_i(f_i(x,z),z)}{s(x,z)}\left(\frac{\frac{\partial s_i}{\partial z^k}(f_i(x,z),z)}{s_i(f_i(x,z),z)} - \frac{\frac{\partial s}{\partial z^k}(x,z)}{s(x,z)} + \frac{\partial s_i}{\partial x_i}(f_i(x,z),z)\left(q_i^k(f_i(x,z),z) - q^k(x,z)\right)\right).
\end{aligned}$$

By (4),

$$\sum_i \frac{d}{dz^k}\left(\frac{s_i(f_i(x,z),z)}{s(x,z)}\right) = 0.$$

Thus,

$$\begin{aligned}
&\sum_i \frac{d}{dz^k}\left(\frac{s_i(f_i(x,z),z)}{s(x,z)}\right) q_i^\ell\left(f_i(x,z),z\right) \\
=&\sum_i \frac{d}{dz^k}\left(\frac{s_i(f_i(x,z),z)}{s(x,z)}\right)\left(q_i^\ell\left(f_i(x,z),z\right) - q^\ell(x,z)\right) \\
=&\sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)}\frac{\partial s_i}{\partial x_i}(f_i(x,z),z)\left(q_i^k\left(f_i(x,z),z\right) - q^k(x,z)\right)\left(q_i^\ell\left(f_i(x,z),z\right) - q^\ell(x,z)\right) \\
&+\sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)}\left(\frac{\frac{\partial s_i}{\partial z^k}(f_i(x,z),z)}{s_i(f_i(x,z),z)} - \frac{\frac{\partial s}{\partial z^k}(x,z)}{s(x,z)}\right)\left(q_i^\ell\left(f_i(x,z),z\right) - q^\ell(x,z)\right) \\
=&\sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)}\frac{\partial s_i}{\partial x_i}(f_i(x,z),z)\left(q_i^\ell\left(f_i(x,z),z\right) - q^\ell(x,z)\right)\left(q_i^k\left(f_i(x,z),z\right) - q^k(x,z)\right) \\
&+\sum_i \frac{\frac{\partial s_i}{\partial z^k}(f_i(x,z),z)}{s(x,z)}\left(q_i^\ell\left(f_i(x,z),z\right) - q^\ell(x,z)\right),
\end{aligned} \tag{19}$$

where we used (8) to obtain the second term on the far right hand side.

Again by (7),

$$
\begin{aligned}
&\frac{d}{dz^k}\left(q_i^\ell\left(f_i(x,z),z\right)\right)\\
=&\frac{\partial q_i^\ell}{\partial x_i}\left(f_i(x,z),z\right)\frac{\partial f_i}{\partial z^k}(x,z)+\frac{\partial q_i^\ell}{\partial z^k}\left(f_i(x,z),z\right)\\
=&\frac{\partial q_i^\ell}{\partial x_i}\left(f_i(x,z),z\right)s_i\left(f_i(x,z),z\right)\left(q_i^k\left(f_i(x,z),z\right)-q^k(x,z)\right)+\frac{\partial q_i^\ell}{\partial z^k}\left(f_i(x,z),z\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_i\frac{s_i(f_i(x,z),z)}{s(x,z)}\frac{d}{dz^k}\left(q_i^\ell\left(f_i(x,z),z\right)\right)\\
=&\sum_i\frac{s_i(f_i(x,z),z)}{s(x,z)}\frac{\partial q_i^\ell}{\partial z^k}\left(f_i(x,z),z\right)\\
&+\sum_i\frac{s_i(f_i(x,z),z)}{s(x,z)}\frac{\partial q_i^\ell}{\partial x_i}\left(f_i(x,z),z\right)s_i\left(f_i(x,z),z\right)\left(q_i^k\left(f_i(x,z),z\right)-q^k(x,z)\right).
\end{aligned}
\tag{20}
$$

Thus, by (29), (19), and (20), we obtain (17). ///

The right-hand side of (17) in Theorem 4 can be much simplified if we concentrate on the case of multiplicatively separable felicity functions. Moreover, it is much more illustrative to represent the resultant equalities in the matrix notation. We regard $q_i(x_i)$ and $q(x,z)$ as L-dimensional column vectors and define the partial Jacobian matrix $D_zq(x,z)$ by

$$
D_zq(x,z)=\begin{pmatrix}
\frac{\partial q^1}{\partial z^1}(x,z) & \cdots & \frac{\partial q^1}{\partial z^L}(x,z)\\
\vdots & \ddots & \vdots\\
\frac{\partial q^L}{\partial z^1}(x,z) & \cdots & \frac{\partial q^L}{\partial z^L}(x,z)
\end{pmatrix}\in\boldsymbol{R}^{L\times L},
$$

and similarly $Dq_i(z)\in\boldsymbol{R}^{L\times L}$.

The following corollary follows from Theorem 4 and (8)

Corollary 2 *Suppose that u_i is multiplicatively separable for every i. Then*

$$
\begin{aligned}
\frac{\partial q^\ell}{\partial z^k}(x,z)=&\sum_i\frac{s_i(f_i(x,z))}{s(x,z)}\frac{\partial q_i^\ell}{\partial z^k}(z)\\
&+\sum_i\frac{s_i(f_i(x,z))}{s(x,z)}s_i'(f_i(x,z))\left(q_i^\ell(z)-q^\ell(x,z)\right)\left(q_i^k(z)-q^k(x,z)\right)
\end{aligned}
$$

for all ℓ, k, and $(x,z)\in\boldsymbol{R}_{++}\times Z$. In the matrix notation,

$$
D_zq(x,z)=\sum_i\frac{s_i(f_i(x,z))}{s(x,z)}Dq_i(z)+\sum_i\frac{s_i(f_i(x,z))}{s(x,z)}s_i'(f_i(x,z))\left(q_i(z)-q(x,z)\right)\left(q_i(z)-q(x,z)\right)^\top.
\tag{21}
$$

The formula (21) is rich in qualitative implications, as the following corollary shows.

Corollary 3 *Suppose that u_i is multiplicatively separable for every i.*

1. *If $s_i'(f_i(x,z)) \geq 0$ for every i, then*

$$D_z q(x,z) - \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} Dq_i(z) \tag{22}$$

is positive semi-definite. If $s_i'(f_i(x,z)) > 0$ for every i and there is no hyperplane of $\boldsymbol{R}^L$ that contains all the $q_i(z)$, then it is positive definite.

2. *If $s_i'(f_i(x,z)) \leq 0$ for every i, then (22) is negative semi-definite. If $s_i'(f_i(x,z)) < 0$ for every i and there is no hyperplane of $\boldsymbol{R}^L$ that contains all the $q_i(z)$, then it is negative definite.*

3. *If $Dq_i(z)$ is positive semi-definite and $s_i'(f_i(x,z)) \geq 0$ for every i, then $D_z q(x,z)$ is positive semi-definite. It is positive definite if $Dq_i(z)$ is positive definite for some i, or if $s_i'(f_i(x,z)) > 0$ for every i and there is no hyperplane of $\boldsymbol{R}^L$ that contains all the $q_i(z)$.*

4. *If $Dq_i(z)$ is negative semi-definite and $s_i'(f_i(x,z)) \leq 0$ for every i, then $D_z q(x,z)$ is negative semi-definite. It is negative definite if $Dq_i(z)$ is negative definite for some i, or if $s_i'(f_i(x,z)) < 0$ for every i and there is no hyperplane of $\boldsymbol{R}^L$ that contains all the $q_i(z)$.*

Part 1 of Corollary 3 establishes an impact of the individual consumers' heterogeneous responsiveness on the representative consumer's responsiveness. To see this, recall that by (8), the representative consumer's responsiveness is equal to the weighted average of the individual consumers' counterparts, where the weights $s_i(f_i(x,z))/s(x,z)$ are proportional to risk tolerance. Then, by (21), (22) is the difference between the derivative of the representative consumer's responsiveness and the "fictitious" derivative of his responsiveness when the weights are hypothetically fixed. It is equal to zero if and only if the responsiveness is identical across consumers, that is, $q_1(z) = \cdots = q_I(z)$. Thus, (22) represents the bias in the estimation of the derivatives of the representative consumer's responsiveness, $D_z q(x,z)$, when the heterogeneity in the individual consumers' responsiveness is erroneously ignored.

In the appendix, we show that if $s_i'(f_i(x,z)) \geq 0$ for every i, then this impact on the representative consumer's responsiveness can be formalized as a *convexifying* effect. Specifically, we show that if we construct a fictitious economy consisting of consumers having the same risk tolerance as the true one (represented by the s_i) but different responsiveness from the true ones (represented by the q_i) in such a way that the heterogeneity in the individual consumers' responsiveness is absent, then the logarithmic transformation of the marginal utility function of the (true) representative consumer is more convex than the fictitious representative consumer's counterpart, and the difference between the Hessian matrices of these two is equal to (22), which is positive semi-definite.

4 Applications

The argument in the preceding sections are fairly general. We shall now discuss how the results in the general argument can be applied to more specific settings. We will see that many results in the existing literature can be derived from our results.

4.1 Heterogeneous beliefs

Suppose now that for each $i = 1, \ldots, I$, consumer i has a subjective probability measure P_i on $(\Omega, \mathscr{F})$ and a state-independent felicity function $v_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$. His utility function U_i is defined as the expected utility function

$$U_i(c_i) = \int_\Omega v_i(c_i(\omega))\, dP_i(\omega),$$

where $c_i : \Omega \to \boldsymbol{R}_{++}$.

We impose two conditions on the P_i. First, for every i, P_i and P are mutually absolutely continuous. Let $dP_i/dP : \Omega \to \boldsymbol{R}_{++}$ be its Radon-Nikodym derivative. Second, there are a positive integer L, an open subset Z of $\boldsymbol{R}^L$, a measurable mapping $h : \Omega \to Z$, and, for each i, a twice continuously differentiable function $p_i : Z \to \boldsymbol{R}_{++}$ such that $P_i/dP = p_i \circ h$. This assumption means roughly that although the subjective probabilistic belief of consumer i may be different from the objective (natural or physical) probability measure P, the likelihood ratio for any given state between the two is a function of finite-dimensional state variables; and the dependence can be made twice continuously differentiable. Define $u_i : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ by $u_i(x_i, z) = p_i(z)v_i(x_i)$. Then

$$U_i(c_i) = \int_\Omega v_i(c_i(\omega))\frac{dP_i}{dP}(\omega)\, dP(\omega) = \int_\Omega v_i(c_i(\omega))p_i(h(\omega))\, dP(\omega) = \int_\Omega u_i(c_i(\omega), h(\omega))\, dP(\omega).$$

This shows that the expected utility function generated by a felicity function v_i and a subjective probabilistic belief P_i satisfies the assumptions for the state-contingent utility function u_i stated in Section 2.

For these u_i and λ_i, solve the simplified maximization problem (3), denote the solution by $(f_1(x, z), \ldots, f_I(x, z))$, and define the value function by $u(x, z) = \sum_i \lambda_i u_i(f_i(x, z))$. This is the representative consumer's felicity function, and his overall utility function U is defined by $U(c) = E\left(u(c, h)\right)$. In general, u is not multiplicatively separable and, as has been studied by Wilson (1968), there is may not be any probability measure on $(\Omega, \mathscr{F})$ with respect to which U can be written in the form of expected utility. However, following Gollier (2007), we define his probabilistic belief via its Radon-Nikodym derivative, which in general depends on the aggregate consumption level x.

Specifically, assume that $\omega \mapsto \partial u(x, h(\omega))/\partial x$ is P-integrable for every x.[5] Define p :

[5] We will later see that this assumption is irrelevant, because we will only be interested in the percentage change in probabilities caused by a change in state variables, and any constant multiple has no effect on the value of any percentage change. Moreover, the following definition of p involves h, but, as far as the percentage change

$\boldsymbol{R}_{++} \times Z \to \boldsymbol{R}_{++}$ by

$$p(x,z) = \frac{\dfrac{\partial u}{\partial x}(x,z)}{E\left(\dfrac{\partial u}{\partial x}(x,h)\right)}$$

If u were multiplicatively separable, so that $u(x,z) = \hat{p}(z)v(x)$ for some functions v and $\hat{p}$ with $E(\hat{p}(h)) = 1$, then

$$p(x,z) = \frac{\hat{p}(z)v'(x)}{E(\hat{p}(h))v'(x)} = \hat{p}(z).$$

Thus $p(x,\cdot) = \hat{p}$ for every x. Therefore, p (or, more precisely, $p(x, h(\cdot))$) is a natural generalization of the Radon-Nikodym derivative of the representative consumer's subjective probabilistic belief. Since multiplying a positive constant does not affect q,

$$q(x,z) = \frac{1}{p(x,z)} \nabla_z p(x,z), \text{ that is, } q^{\ell}(x,z) = \frac{\dfrac{\partial p}{\partial z^{\ell}}(x,z)}{p(x,z)} = \frac{d}{dz^{\ell}} \left(\ln p(x,z)\right). \tag{23}$$

Hence $q^{\ell}(x,z)$ is the percentage change in the likelihood ratio of the representative consumer's subjective probability relative to the objective probability due to a unit change in the ℓ-th state variable z^{ℓ}, when the aggregate consumption level is x. Similarly, $q_i^{\ell}(z)$ is the percentage change in the likelihood ratio of consumer i's subjective probability relative to the objective probability due to a unit change in z^{ℓ}.

We shall now apply the results in Section 3 to obtain some results regarding the representative consumer's probabilistic belief. First, by (23) and an analogous equality for each $q_i(z)$, (8) can be rewritten as

$$\frac{\dfrac{\partial p}{\partial z^{\ell}}(x,z)}{p(x,z)} = \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} \frac{\dfrac{\partial p_i}{\partial z^{\ell}}(z)}{p_i(z)} \tag{24}$$

for every ℓ and (x,z). That is, the percentage change in the likelihood ratio of the representative consumer's subjective probability relative to the objective probability due to a unit change in a state variable is equal to the weighted average of the individual consumers' counterparts, where the weights are proportional to their risk tolerance. (24) implies (8) in Proposition 3 of Gollier (2007). In particular, the percentage change in the likelihood ratio of the representative consumer's subjective probability is more biased towards those of more risk-tolerant consumers. Note however that (24) does not claim that the likelihood ratio of the representative consumer's subjective probability relative to the objective probability is equal to the weighted average of the individual consumers' counterparts, with the weights proportional to risk tolerance, which would be expressed as

$$p(x,z) = \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} p_i(z). \tag{25}$$

This equality, in general, does not hold. One of the reasons for this is that if we define $\hat{p}$:

is concerned, it does not matter.

$\boldsymbol{R}_{++} \times Z \to \boldsymbol{R}_{++}$ by

$$\hat{p}(x,z) = \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} p_i(z), \tag{26}$$

then its expected value $E\left(\hat{p}(x,h)\right)$ need not be equal to one. But this is not an essential cause for discrepancy between $p(x,z)$ and $\hat{p}(x,z)$, because we can divide $\hat{p}$ by a positive constant $\left(E\left(\hat{p}(x,h)\right)\right)$ without affecting affecting its responsiveness

$$\hat{q}(x,z) = \frac{1}{\hat{p}(x,z)} \nabla \hat{p}(x,z), \tag{27}$$

and a discrepancy may well remain between $q(x,z)$ and $\hat{q}(x,z)$. The following proposition shows that this discrepancy is related to the correlation between the $p_i(z)$ and the $q_i(z)$.

Proposition 1 *Suppose that u_i is multiplicatively separable for every i. If we define $\hat{p} : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}_{++}$ by (26) and $\hat{q} : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}^L$ by (27), then, for every ℓ,*

$$\hat{q}^\ell(x,z) - q^\ell(x,z) = \frac{1}{\hat{p}(x,z)} \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} \left(s_i'\left(f_i(x,z)\right) + 1\right) \left(p_i(z) - \hat{p}(x,z)\right) \left(q_i^\ell(z) - q^\ell(x,z)\right). \tag{28}$$

Proof of Proposition 1 By (26) and differentiation for a product,

$$\frac{\partial \hat{p}}{\partial z^\ell}(x,z) = \sum_i \frac{d}{dz^\ell} \left(\frac{s_i(f_i(x,z),z)}{s(x,z)} \right) p_i\left(z\right) + \sum_i \frac{s_i(f_i(x,z),z)}{s(x,z)} \frac{\partial p_i}{\partial z^\ell}\left(z\right). \tag{29}$$

By (7),

$$\begin{aligned} &\frac{d}{dz^\ell} \left(\frac{s_i(f_i(x,z))}{s(x,z)} \right) \\ =& \frac{s_i'(f_i(x,z))}{s(x,z)} \frac{\partial f_i}{\partial z^\ell}(x,z) - \frac{s_i(f_i(x,z))}{(s(x,z))^2} \frac{\partial s}{\partial z^\ell}(x,z) \\ =& \frac{s_i(f_i(x,z))}{s(x,z)} \left(s_i'(f_i(x,z)) \left(q_i^\ell(f_i(x,z)) - q^\ell(x,z) \right) - \frac{\frac{\partial s}{\partial z^\ell}(x,z)}{s(x,z)} \right). \end{aligned}$$

By (4),

$$\sum_i \frac{d}{dz^\ell} \left(\frac{s_i(f_i(x,z))}{s(x,z)} \right) = 0.$$

Thus, by (26),

$$
\begin{aligned}
&\sum_i \frac{d}{dz^\ell}\left(\frac{s_i(f_i(x,z))}{s(x,z)}\right) p_i(z) \\
=&\sum_i \frac{d}{dz^\ell}\left(\frac{s_i(f_i(x,z))}{s(x,z)}\right) (p_i(z) - \hat{p}(x,z)) \\
=&\sum_i \frac{s_i(f_i(x,z))}{s(x,z)} s_i'(f_i(x,z)) \left(q_i^\ell(z) - q^\ell(x,z)\right) (p_i(z) - \hat{p}(x,z)) \\
&- \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} \frac{\frac{\partial s}{\partial z^k}(x,z)}{s(x,z)} (p_i(z) - \hat{p}(x,z)) \\
=&\sum_i \frac{s_i(f_i(x,z))}{s(x,z)} s_i'(f_i(x,z)) (p_i(z) - \hat{p}(x,z)) \left(q_i^\ell(z) - q^\ell(x,z)\right).
\end{aligned} \tag{30}
$$

On the other hand, since $\partial p_i(z)/\partial z^\ell = p_i(z) q_i^\ell(z)$,

$$
\begin{aligned}
&\frac{1}{\hat{p}(x,z)} \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} \frac{\partial p_i}{\partial z^\ell}(z) - q^\ell(x,z) \\
=&\frac{1}{\hat{p}(x,z)} \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} \left(p_i(z) q_i^\ell(z) - \hat{p}(x,z) q^\ell(x,z)\right) \\
=&\frac{1}{\hat{p}(x,z)} \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} (p_i(z) - \hat{p}(z)) \left(q_i^\ell(z) - q^\ell(x,z)\right).
\end{aligned} \tag{31}
$$

Thus, by (29), (30), and (31), we obtain (28). ///

The following example shows that in a two-consumer economy, when the two consumers have constant and equal relative risk aversion and the beliefs are given by gamma distributions, then the representative consumer's belief predicted by $\hat{p}$ puts higher probabilities on upper or lower tails than the true representative consumer's belief.

Example 1 Let $\Omega = Z = \boldsymbol{R}_{++}$, $\mathscr{F} = \mathscr{B}(\boldsymbol{R}_{++})$, $h(\omega) = \omega$ for every ω, and P follow the gamma distribution with parameters (κ, θ), so that its density function (Radon-Nikodym derivative with respect to the Lebesgue measure) is

$$
\frac{\theta^\kappa}{\Gamma(\kappa)} z^{\kappa-1} \exp(-\theta z).
$$

Note that this is the exponential distribution with parameter θ if $\kappa = 1$. Let $I = 2$ and both consumers have constant and equal relative risk aversion β. As for their beliefs, P_i follows the gamma distribution with parameters (κ, θ_i), where $\theta_1 < \theta_2$. Note that the parameter κ is common across P, P_1, and P_2.

Then

$$
p_i(z) = \left(\frac{\theta_i}{\theta_0}\right)^\kappa \exp(-(\theta_i - \theta_0) z) \text{ and } q_i(z) = -(\theta_i - \theta_0).
$$

The right-hand side of (28) is equal to[6]

$$\frac{1+1/\beta}{\hat{p}(x,z)} \sum_i \frac{s_i(f_i(x,z))}{s(x,z)} \left(p_i(z) - \hat{p}(x,z)\right)\left(q_i(z) - q(x,z)\right).$$

Thus $\hat{q}(x,z) - q(x,z) \gtreqless 0$ if and only if $p_1(z) \gtreqless p_2(z)$, which is equivalent to

$$z \gtreqless \kappa \frac{\ln\theta_2 - \ln\theta_1}{\theta_2 - \theta_1}.$$

Denote the right-hand side by z^*. Then

$$\hat{q}(x,z) - q(x,z) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \text{ if } z \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} z^*. \tag{32}$$

We shall now explore an implication of (32) on the discrepancy between p and $\hat{p}$. Recall first that $E\left(p(x,h)\right) = 1$. Let P_0 be the probability measure on $\boldsymbol{R}_{++}$ such that $dP_0/dP = p(x,h)$. By dividing $\hat{p}(x,\cdot)$ by $E\left(\hat{p}(x,h)\right)$, we can assume that $E\left(\hat{p}(x,h)\right) = 1$. Then

$$\int_{\boldsymbol{R}_{++}} \frac{\hat{p}(x,h(\omega))}{p(x,h(\omega))}\, dP_0(\omega) = \int_{\boldsymbol{R}_{++}} \frac{\hat{p}(x,h(\omega))}{p(x,h(\omega))} p(x,h(\omega))\, dP(\omega) = \int_{\boldsymbol{R}_{++}} \hat{p}(x,h(\omega))\, dP(\omega) = 1.$$

Since

$$\hat{q}(x,z) - q(x,z) = \frac{d}{dz}\left(\ln \frac{\hat{p}(x,z)}{p(x,z)}\right),$$

(32) implies that the likelihood ratio $\hat{p}(x,z)/p(x,z)$ is a strictly decreasing function of z on $(0,z^*)$, is a strictly increasing function of z on (z^*,∞), and attains its unique minimum at z^*. Thus, $\hat{p}$ puts more probabilities on at least one of the upper and lower tails than p.

Remark 1 The quadratic utility functions, $v(x) = (-1/2)(x-d)^2$ with a constant d, have been used in the Capital Asset Pricing Model. These functions do not conform to the setting of this paper, because $v'(x) < 0$ for every $x > d$ and the Inada condition is not satisfied on $\boldsymbol{R}_{++}$. However, if we take the open interval $(-\infty, d)$, which is bounded from above but not bounded from below, as the domain of v, then $v''(x) < 0 < v'(x)$ for every $x \in (-\infty,d)$ and the Inada condition is satisfied, that is, $v'(x) \to 0$ as $x \to d$ and $v'(x) \to \infty$ as $x \to -\infty$. Moreover, let $s : (-\infty, d)$ be its risk tolerance, then $s(x) = d - x$ and hence $s'(x) = -1$ for every x. Hence v exhibits increasing absolute risk aversion and its cautiousness is constantly equal to zero. We shall now prove, as suggested by (28), that if every consumer has a quadratic utility function, then p and $\hat{p}$ coincide with each other up to a scalar multiplication.

Indeed, let $u_i : (-\infty, d_i) \times Z \to \boldsymbol{R}$ be the felicity function of consumer i, where $u_i(x_i,z) =$

[6] Given the constant and equal relative risk aversion β, it can be shown that neither $p(x,z)$ nor $\hat{p}(x,z)$ depends on x. Thus, we could write $p(z)$ for $p(x,z)$, $\hat{p}(z)$ for $\hat{p}(x,z)$, $q(z)$ for $q(x,z)$, $\hat{q}(z)$ for $\hat{q}(x,z)$, and $s(x)$ for $s(x,z)$. In particular, the representative consumer's felicity function u is multiplicatively separable and his overall utility function U_i can be written in the expected-utility form.

$p_i(z)v_i(x_i)$ and $v_i(x_i) = (-1/2)\,(x_i - d_i)^2$. For the maximization problem (3) with these felicity functions, we can define the representative consumer's utility function $u : (-\infty, d) \times Z \to \boldsymbol{R}$, where $d = \sum d_i$. It can be easily shown that

$$u(x,z) = \frac{1}{\sum\limits_i (\lambda_i p_i(z))^{-1}} \left(-\frac{1}{2}\right) (d-x)^2.$$

Thus, the representative consumer also has a multiplicatively separable, quadratic utility function and we can take

$$p(z) = \frac{1}{\sum\limits_i (\lambda_i p_i(z))^{-1}}.$$

Moreover, the efficient risk-sharing rules $f_1, \ldots, f_I$ are given by

$$d_i - f_i(x,z) = \frac{(\lambda_i p_i(z))^{-1}}{\sum\limits_j (\lambda_j p_j(z))^{-1}}(x,z).$$

Thus

$$\frac{s_i(f_i(x,z))}{s(x)} = \frac{(\lambda_i p_i(z))^{-1}}{\sum\limits_j (\lambda_j p_j(z))^{-1}}.$$

Hence

$$\hat{p}(z) = \sum_i \frac{(\lambda_i p_i(z))^{-1}}{\sum\limits_j (\lambda_j p_j(z))^{-1}} p_i(z) = \left(\sum_i \lambda_i^{-1}\right) p(z).$$

Therefore, p and $\hat{p}$ coincide with each other up to scalar multiplication, and q and $\hat{q}$ coincide with each other.

Next, we show, within the same parametric family as Example 1, if we ignore the heterogeneity in belief, then we underestimate the fatness of the upper and lower tails of the density function of the representative consumer's belief, and hence the prices of the call and put options written on the aggregate consumption.

Example 2 Let $\Omega = Z = \boldsymbol{R}_{++}$, $\mathscr{F} = \mathscr{B}(\boldsymbol{R}_{++})$, $h(\omega) = \omega$ for every ω, and P follow the gamma distribution with parameters (κ, θ), so that its density function (Radon-Nikodym derivative with respect to the Lebesgue measure) is

$$\frac{\theta^\kappa}{\Gamma(\kappa)} \omega^{\kappa-1} \exp(-\theta z).$$

Note that this is the exponential distribution with parameter θ if $\kappa = 1$. All consumers have constant and equal relative risk aversion β. Assume that $\beta < \kappa$. As for their beliefs, P_i follows the gamma distribution with parameters (κ, θ_i). Note that the parameter κ is common across P and the P_i, but θ_i need not. The Radon-Nikodym derivative of P_i with respect to P is given by

$$p_i(\omega) = \left(\frac{\theta_i}{\theta_0}\right)^{\kappa} \exp(-(\theta_i - \theta_0)\omega) \text{ and hence } q_i(\omega) = -(\theta_i - \theta_0).$$

for every i and ω. Assume that the θ_i are not completely equal. Then, by (21), $\partial q(x,\omega)/\partial\omega > 0$ for every ω.

Since $s_i'(x^i) = 1/\beta$ for every i and x^i, $\partial q(x,\omega)/\partial x = 0$ by (16). Hence u is also multiplicatively separable. Moreover, although the proof is omitted here, it is possible to show that u also exhibits constant relative risk aversion β. We can thus write $u(x,\omega) = p(\omega)v(x)$, with $s'(x) = 1/\beta$ for every x and $q'(\omega) > 0$ for every ω.

The aggregate consumption is given by $c : \Omega \to \boldsymbol{R}_{++}$ such that $c(\omega) = \omega$. Thus the heterogeneous probabilistic beliefs can all be interpreted as those regarding the aggregate consumption levels. We assume that both $E\left(pv'(c)\right)$ and $E\left(pv'(c)c\right)$ are finite.

Let's now imagine that a modeler erroneously assumed that all consumers have the same belief, which is given by the Gamma distribution with parameters $(\kappa, \hat{\theta})$, where

$$\hat{\theta} = \frac{E\left(pv'(c)\right)}{E\left(pv'(c)c\right)}(\kappa - \beta). \tag{33}$$

Thus, the representative consumer's belief is also given by the Gamma distribution with parameters $(\kappa, \hat{\theta})$. The fictitious representative consumer's utility function $\hat{u}$ is given by $\hat{u}(x,\omega) = \hat{p}(\omega)v(x)$, where $\hat{q}(\omega) = \hat{p}'(\omega)/p(\omega) = -(\hat{\theta} - \theta_0)$.

Define $\pi : \Omega \to \boldsymbol{R}_{++}$ and $\hat{\pi} : \Omega \to \boldsymbol{R}_{++}$ by

$$\pi(\omega) = \frac{p(\omega)v'(c(\omega))}{E\left(pv'(c)\right)} \text{ and } \hat{\pi}(\omega) = \frac{\hat{p}(\omega)v'(c(\omega))}{E\left(\hat{p}v'(c)\right)}.$$

Then π and $\hat{\pi}$ are the state-price deflators, with the risk-free bond being the numeraire, when the representative consumer has the subjective probability densities p and $\hat{p}$, respectively.[7] In fact, $E(\pi) = E(\hat{\pi}) = 1$. Note that $p(\omega)/\hat{p}(\omega) = \pi(\omega)/\hat{\pi}(\omega)$ for every ω. It can be derived from (33) that $E(\pi c) = E(\hat{\pi} c)$, that is, π and $\hat{\pi}$ give the same price for the aggregate consumption c.

By this property and $q'(\omega) > 0 = \hat{q}'(\omega)$, we can show that $p(\omega)/\hat{p}(\omega)$ is a strictly convex function of ω, taking value 1 at exactly two points in Ω. This means that the density p, which correctly recognizes the heterogeneity in beliefs, is more dispersed (puts higher probabilities on both upper and lower tails) than the density $\hat{p}$, which incorrectly ignores the heterogeneity in beliefs. Following the argument of Proposition 1 of Franke, Stapleton, and Subrahmanyam (1999), we can derive from this fact that $E(\hat{\pi} d) < E(\pi d)$ for every convex and nonlinear $d : \boldsymbol{R}_{++} \to \boldsymbol{R}$. This means that any derivative asset, whose payoff is a convex and nonlinear function of the aggregate consumption, is underpriced if the heterogeneity in beliefs is ignored. In particular, since both call and put options have convex and nonlinear payoffs, the option prices are underestimated by erroneously assuming homogeneous beliefs.

[7] In other words, for each $d : \Omega \to \boldsymbol{R}$, $E(\pi d)$ and $E(\hat{\pi} d)$ are the prices for the forward contract promising to deliver d.

4.2 Heterogeneous impatience

Although we assumed in Section 2 that there is only one consumption period, the present formulation of state-dependent utilities can accommodate utility functions in a continuous-time setting. In this subsection, we show how this can be done by following Hara (2006).

Let $\boldsymbol{R}_+ = [0, \infty)$ be the time span and $\boldsymbol{F} = (\mathscr{F}_t)_{t \in \boldsymbol{R}_+}$ be a filtration, describing the way in which the information on Ω is gradually revealed. Let $\mathscr{K}$ be the set of all $\boldsymbol{R}_{++}$-valued progressively measurable stochastic processes, that is, the set of those $c = (c_t)_{t \in \boldsymbol{R}_+}$ such that the restriction of c on $\Omega \times [0, t]$ is $(\mathscr{F}_t \otimes \mathscr{B}([0,t]))$-measurable for every $t \in \boldsymbol{R}_+$. Denote by η the Lebesgue measure on $\boldsymbol{R}_+$.

We assume that each consumer i has a utility function U_i over consumption processes in $\mathscr{K}$ of the form $E\left(\int_{\boldsymbol{R}_+} \exp(-\rho^i t) v_i(c^i_t)\, d\eta(t)\right)$, where ρ_i is a strictly positive number representing his impatience, $v_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$ be a felicity function satisfying the assumptions in Section 2, and $c^i = \left(c^i_t\right)_{t \in \boldsymbol{R}_+}$.[8]

We now show how this class of utility function can be written in the form (1) of state-dependent utilities in Section 2. Let $\mathscr{M}$ be the set of all subsets M of $\Omega \times \boldsymbol{R}_+$ such that $M \cap (\Omega \times [0, T]) \in \mathscr{F}_t \otimes \mathscr{B}([0,t])$ for every $t \in \boldsymbol{R}_+$. Then $\mathscr{M}$ is a σ-field of $\Omega \times \boldsymbol{R}_+$ and $\mathscr{K}$ coincides with the set of all $\boldsymbol{R}_{++}$-valued $\mathscr{M}$-measurable stochastic processes. For each $\rho > 0$, denote by η^ρ be the probability measure on $\boldsymbol{R}_+$ following the exponential distribution with parameter ρ, that is, its density function is given by $t \mapsto \rho \exp(-\rho t)$. Then the product measure space $(\Omega \times \boldsymbol{R}_+, \mathscr{M}, P \otimes \eta^\rho)$ is a probability measure space. Then, by Fubini's Theorem,

$$U_i(c^i) = \rho_i^{-1} \int_{\Omega \times \boldsymbol{R}_+} v_i\left(c^i_t(\omega)\right)\, d\left(P \otimes \eta^{\rho_i}\right).$$

Since

$$\frac{d\left(P \otimes \eta^{\rho_i}\right)}{d\left(P \otimes \eta^{\rho}\right)}(\omega, t) = \frac{\rho_i}{\rho} \exp\left(-\left(\rho_i - \rho\right) t\right),$$

if we define $p_i(t) = \rho^{-1} \exp\left(-\left(\rho_i - \rho\right)\right)$ and $u_i(x^i, t) = v_i(x^i) p_i(t)$, then

$$\begin{aligned} U_i(c^i) &= \rho_i^{-1} \int_{\Omega \times \boldsymbol{R}_+} v_i\left(c^i_t(\omega)\right) \frac{\rho_i}{\rho} \exp\left(-\left(\rho_i - \rho\right) t\right)\, d\left(P \otimes \eta^\rho\right) \\ &= \int_{\Omega \times \boldsymbol{R}_+} v_i\left(c^i_t(\omega)\right) p_i(t)\, d\left(P \otimes \eta^\rho\right) \\ &= \int_{\Omega \times \boldsymbol{R}_+} u_i(c^i_t(\omega), t)\, d\left(P \otimes \eta^\rho\right). \end{aligned}$$

[8] To be exact, the utility function U_i is defined only on the set of those c^i for which the expected utility is finite. The details on how to modify the domain of U_i taking this fact into consideration can be found in Hara (2006).

Thus U_i can be written in the form of (1).[9] Note that

$$q_i(t) = -(\rho_i - \rho)$$

and hence the subjective discount rate of consumer i is equal to $\rho - q_i(z)$.

Let $u : \boldsymbol{R}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}$ be the representative consumer's felicity function and $s : \boldsymbol{R}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}$ and $q : \boldsymbol{R}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}$ be his risk tolerance and responsiveness to time (state variable). Then, just as in the case of individual consumers, $\rho - q(x,t)$ is the discount rate for the representative consumer. We now look into how the representative consumer's discount rate is affected by the heterogeneity of the individual consumers' discount rates and cautiousness.

First, by (8),

$$\rho - q(x,t) = \sum_i \frac{s_i\left(f_i(x,t)\right)}{s(x,t)} \rho_i. \tag{34}$$

This means that the representative consumer's discount rate is the weighted sum of the individual consumers' discount rates, where the weights are proportional to their risk tolerance. It has been already established by Gollier and Zeckhauser (2006, Proposition 3).

Second, by (16),

$$\frac{d}{dx}\left(\rho - q(x,t)\right) = \frac{1}{s(x,t)} \sum_i \frac{s_i(f_i(x,t))}{s(x,t)} \left(s_i'(f_i(x,t)) - \frac{\partial s}{\partial x}(x,t)\right) \left(\rho_i - \left(\rho - q(x,t)\right)\right).$$

This means that the representative consumer's discount rate is higher at higher consumption levels if and only if the weighted covariance between the individual consumers' cautiousness and discount rates is positive; and his discount rate is lower at higher consumption levels if the covariance is negative. Since it is quite possible that the covariance is different from zero, this formula case doubts on the prevalent use of consumption-independent discount rates in the representative-consumer model.[10]

Third, by (21),

$$\frac{d}{dt}\left(\rho - q(x,t)\right) = -\frac{1}{s(x,t)} \sum_i \frac{s_i(f_i(x,t))}{s(x,t)} s_i'\left(f_i(x,t)\right) \left(\rho_i - \left(\rho - q(x,t)\right)\right)^2.$$

This shows that the if the individual consumers risk tolerance is increasing (equivalently, their absolute risk aversion is decreasing) as in the case of constant relative risk aversion, then the representative consumer's discount rates decreases over time. Gollier and Zeckhauser (2006, Proposition 5) established the same conclusion, but without obtaining any formula relating the time derivative of the representative consumer's discount rates to the the individual consumers'

[9] To be exact, since the time span $\boldsymbol{R}_{+}$ is not an open subset of $\boldsymbol{R}$, u_i does not satisfy all the conditions of Section 2. However, u_i is twice continuously differentiable at the boundary point 0 of $\boldsymbol{R}_{+}$, all the general results in 3 are applicable to u_i.

[10] For example, if all individual consumers exhibit constant relative risk aversion and the coefficients of constant relative risk aversion and the subjective discount rates are comonononically or anti-comonotonically related, then the covariance is strictly positive or strictly negative, unless either of the relative risk aversion coefficients or the discount rates are completely equal across all individual consumers.

counterparts. The above formula case doubts on the prevalent use of constant discount rates in the representative-consumer model.

Note that the discount rates can be independent of aggregate consumption levels and yet decreasing over time. In particular, this is the case if all consumers exhibit constant and equal relative risk aversion. In such a case, Gollier and Zeckhauser (2006, Section IV) showed that the representative consumer's discount rate may be a hyperbolic function of time. Hara (2007) identified the class of discount factors that can arise from the consumers of constant and equal relative risk aversion.

5 Conclusion

We have investigated implications of heterogeneous responsiveness to state variables in an economy populated by multiple consumers who have state-dependent expected utility functions. We have established some formulas showing how the representative consumer's risk attitudes will be changed by state variables and how the change in aggregate consumption levels affect his responsiveness to state variables. These formulas clarify when and, if so, how, his felicity function fails to be multiplicatively separable. We have also found a formula showing how his responsiveness to one state variable is affected by a change in another state variable.

We have applied these formals to analyze the consequences of heterogeneity in probabilistic beliefs and time discount rates. Among other results, we have shown that the representative consumer may not have any constant discount rate or any probability measure with respect to which his expected utility is calculated. We have also shown that his discount rate is likely to be decreasing over time and his probabilistic belief is likely to be more dispersed than individual consumers' beliefs.

There are many interesting directions of future research. First, given that our specification of state variables can be multi-dimensional, we should look into the consequence of the joint heterogeneity in probabilistic beliefs and time discount rates. Second, the case of multiple goods should be investigated: although it is possible to set up the welfare maximization problem (3) taking x as multi-dimensional, such multi-dimensionality makes the characterization of the risk-sharing rules more complicated. It may be more promising to apply the results here to the indirect functions taking the spot prices as state variables. Third, we should explore implications of our results to asset pricing in continuous time. Such implications may also help us to judge which, say, term structure models are deemed as more plausible than others based on equilibrium considerations in a heterogeneous economy.

A Formalization of the Convexifying Effect

We claimed, after stating Corollary 2, that the heterogeneity in individual consumers' responsiveness to state variables gives rise to a convexifying effect. But we did not specify exactly which function is made more convex by the presence of such heterogeneity. In this appendix, we define a function that is to be made more convex, thereby giving a precise meaning to the convexifying effect.

A.1 Logged marginal utility functions

First, for the subsequent analysis, we introduce the concept of a logged marginal utility function. Let $u : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ be a state-contingent felicity function having the properties stated in Section 2, such as (at least) twice continuous differentiability, positive first partial derivatives and negative second partial derivatives with respect to the consumption level, and Inada condition. Then define $w : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ by

$$w(x,z) = \ln\left(\frac{\partial u}{\partial x}(x,z)\right)$$

for every (x, z). We call this the *logged marginal utility function* derived from u. It is easy to show that w is (at least once) continuously differentiable and satisfies $\partial w(x, z)/\partial x < 0$ for every (x, z), and, for every z, $w(x, z) \to -\infty$ as $x \to \infty$ and $w(x, z) \to \infty$ as $x \to 0$. Conversely, for any such function $w : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$, if we define $u : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ by

$$u(x,z) = \int_1^x \exp\left(w(y,z)\right)\, dy, \tag{35}$$

then u satisfies the properties stated in Section 2. Moreover, the logged marginal utility function derived from u coincides with the given w. Furthermore, if we start with a felicity function u, derive the logged marginal utility function w, and then define a new function $\hat{u} : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ by the right-hand side of (35), then the difference $\hat{u}(x, z) - u(x, z)$ depends only on z. Since adding to a felicity function a function that depends only on z does not change the solution to (3), we can conclude that there is an essentially bijective correspondence between felicity functions and logged marginal utility functions.

Let w be the logged marginal utility function derived from u and q be the responsiveness to state variables of u. It is easy to show that $q(x, z) = (\nabla_z w(x, z))^\top$ for every (x, z). That is, for each x, $w(x, \cdot) : Z \to \boldsymbol{R}$ is a potential function of the vector field $q(x, \cdot) : Z \to \boldsymbol{R}^L$. Thus, $D_z q(x, z) = \nabla_z^2 w(x, z)$ and, in particular, $D_z q(x, z)$ is a symmetric matrix for every (x, z).

A.2 Fictitious representative consumer

In this subsection, we construct a fictitious representative consumer. A fictitious representative consumer is derived from a group of individual consumers who have the same risk attitudes as the true ones but different responsiveness in such a way that they all share the same responsiveness at some given value z^* of state variables.

For each i, let u_i be the felicity function of consumer i, which is multiplicatively separable. Write $u_i(x_i, z) = p_i(z)v_i(x_i)$. By solving (3), we obtain the risk-sharing rules $f_i : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}_{++}$, one for each consumer, and the representative consumer's felicity function $u : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$. Let $w : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ be the logged marginal utility function derived from u. Let

$(x^*, z^*) \in \boldsymbol{R}_{++} \times Z$.

For each i, define $\varphi_i : Z \to \boldsymbol{R}$ by $\varphi_i(z) = \ln p_i(z)$. Then define $\hat{\varphi}_i : Z \to \boldsymbol{R}$ by $\hat{\varphi}_i(z) = \varphi_i(z) + (q(x^*, z^*) - q_i(z^*))^\top (z - z^*)$ for every z. Then $\nabla\hat{\varphi}_i(z) = \nabla\varphi_i(z) + (q(x^*, z^*) - q_i(z^*))^\top$ and $\nabla^2\hat{\varphi}_i(z) = \nabla^2\varphi_i(z)$ for every z, and $\hat{\varphi}_i(z^*) = \varphi_i(z^*)$ and $\nabla\hat{\varphi}_i(z^*) = q(x^*, z^*)$.

Then define $\hat{u}_i : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ by $\hat{u}_i(x_i, z) = \exp(\hat{\varphi}(z))\, v_i(x_i)$ for every (x_i, z). Let $\hat{w}_i$ be the logged marginal utility function derived from $\hat{u}_i$, $\hat{s}_i$ be its risk tolerance, and $\hat{q}_i$ be its responsiveness to state variables. Then,

$$\begin{aligned}
\hat{w}_i(x_i, z) &= \ln v_i'(x_i) + \hat{\varphi}(z), \\
\hat{s}_i(x_i, z) &= s_i(x_i, z), \\
\hat{q}_i(x_i, z) &= q_i(z) + (q(x^*, z^*) - q_i(z^*)), \\
D_z\hat{q}_i(x_i, z) &= Dq_i(z)
\end{aligned}$$

for every (x_i, z). We can thus write $\hat{q}_i(z)$ for $\hat{q}_i(x_i, z)$. Then $\hat{q}_i(z^*) = q(x^*, z^*)$ and $D\hat{q}_i(z^*) = Dq_i(z^*)$. In words, by constructing $\hat{u}_i$ out of u_i, each individual consumer's risk tolerance s_i is kept as before, while his responsiveness to state variables, q_i, is shifted so that all individual consumers share the same responsiveness $q(x^*, z^*)$ at $z = z^*$.

By solving (3) where the u_i are replaced by the $\hat{u}_i$ but the original utility weights λ_i are retained, we obtain the risk-sharing rules $\hat{f}_i : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}_{++}$, one for each consumer, and the representative consumer's felicity function $\hat{u} : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$. We call $\hat{u}$ the *fictitious* representative consumer's utility function. Let $\hat{w}$ be the logged marginal utility function derived from $\hat{u}$.

A.3 Proof of the convexifying effect

In this section, we prove that the (true) representative consumer's logged marginal utility function w is more convex than the fictitious representative consumer's logged utility function $\hat{w}$ around (x^*, z^*). Since the only difference between the w and the $\hat{w}$ is that the former involve heterogeneous responsiveness at z^* but the latter do not, the additional convexity of w around (x^*, z^*) can be attributed to heterogeneous responsiveness involved in the construction of w.

Let $\hat{s}$ be the risk tolerance of $\hat{u}$ and $\hat{q}$ be the responsiveness to state variables of $\hat{u}$. Using the first-order condition, it is easy to show that $\hat{f}_i(x, z^*) = f_i(x, z^*)$ for all i and x. This implies that $\hat{u}(x, z^*) = u(x, z^*)$, $\hat{w}(x, z^*) = w(x, z^*)$, and $\hat{s}(x, z^*) = s(x, z^*)$ for every x. Thus, $\hat{u}$ represents the same risk attitudes as u whenever $z = z^*$. Moreover, since $\hat{q}_i(z^*) = q(x^*, z^*)$ for every i, (8) implies that

$$\hat{q}(x^*, z^*) = q(x^*, z^*).$$

Thus $\hat{u}$ represents the same responsiveness as u at (x^*, z^*). Since $\hat{q}(x^*, z^*) = D_z\hat{w}(x^*, z^*)$ and

$q(x^*, z^*) = D_z w(x^*, z^*)$,

$$\hat{w}(x^*, z^*) = w(x^*, z^*),$$
$$\nabla_z \hat{w}(x^*, z^*) = \nabla_z w(x^*, z^*).$$

That is, $\hat{w}$ coincides with w around (x^*, z^*) up to the first order. Yet, we shall now prove that $\nabla_z^2 \hat{w}(x^*, z^*)$ is, in general, not equal to $\nabla_z^2 w(x^*, z^*)$. Rather, the difference $\nabla_z^2 \hat{w}(x^*, z^*) - \nabla_z^2 w(x^*, z^*)$ is positive semi-definite if $s_i'(x_i) \geq 0$ for every x_i, that is, the u_i exhibit decreasing absolute risk aversion. This means that w is more convex than $\hat{w}$ around (x^*, z^*).[11]

The proof goes as follows. Since $\hat{q}_1(z^*) = \cdots = \hat{q}_I(z^*)$, by applying Corollary 2 to $\hat{q}$, we obtain

$$D_z \hat{q}(x^*, z^*) = \sum_i \frac{\hat{s}_i\left(\hat{f}_i(x^*, z^*)\right)}{\hat{s}(x^*, z^*)} D\hat{q}_i\left(z^*\right) = \sum_i \frac{s_i(f_i(x^*, z^*))}{s(x^*, z^*)} Dq_i\left(z^*\right).$$

Thus, by applying Corollary 2 to q, we obtain

$$D_z q(x^*, z^*) - D_z \hat{q}(x^*, z^*) = \sum_i \frac{s_i(f_i(x^*, z^*))}{s(x^*, z^*)} s_i'(f_i(x^*, z^*)) \left(q_i(z^*) - q(x^*, z^*)\right) \left(q_i(z^*) - q(x^*, z^*)\right)^\top.$$

Since the left-hand side is equal to $\nabla_z^2 \hat{w}(x^*, z^*) - \nabla_z^2 w(x^*, z^*)$ and the right-hand side is positive semi-definite, this completes the proof.

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[11] Strictly speaking, to conclude that w is more convex than $\hat{w}$ around (x^*, z^*), we need to guarantee that $\nabla_z^2 \hat{w}(x^*, z^*) - \nabla_z^2 w(x^*, z^*)$ is positive definite. The sufficient condition in parts 1 and 2 of Corollary 3 for (22) to be positive definite is sufficient for this as well.

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