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The Optimal and Efficient Auction Mechanisms with Complements

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“The Optimal and Efficient Auction Mechanisms with Complements”

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JAPAN

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1. Introduction

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We analyze the optimal auction in a generalized model based on Krishna and Rosenthal (1995). They analyze a problem of auction with synergies where two types of bidders compete against for complements. Selling items are complements for "a global bidder" and "a local bidder" is subject to a unit demand assumption. Then, they characterize the equilibrium bidding strategies for the simultaneous and sequential ascending-bid and sealed-bid second-price auctions. Further, they computationally show that they cannot obtain any general conclusion about which auction raises the higher expected revenue to the seller. Since their analysis is limited to particular auction procedures, the following problems remain open questions. How to characterize the optimal auction? How to implement the optimal auction? Can the optimal auction achieve efficient allocation? We answer these questions in this paper. Branco (1995) independently analyzed the similar problem to ours. He analyzed the model where the global bidder is interested only in a bundle of the two objects 1 and 2, but not interested in any single object at all, i.e., his value of any single object is 0. He shows a sufficient condition for the optimal auction to be efficient. In contrast to his model, we can conclude that the optimal auction “cannot” assure efficiency under any condition. Recently, Yokoo and et al. (2004) examine the effect of false-name bids on auctions and show that no mechanism can attain efficiency. We examine how our results are affected by false-name bid later in section 5. The rest of the paper is organized as follows. Section2 explains the model. Section3 analyzes the optimal auction mechanism. Section4 characterizes an efficient auction. Section5 shows several directions to extend the basic model in the text..

2. The Model

The seller auctions off two objects, say, 1 and 2. There are two types of bidders. One (following Krishna and Rosenthal, we call [1]him the “global” bidder) is interested in obtaining

[1] Without any intention of sexual discrimination, we use “he” for the global bidder and “she” for each local bidder.

both objects. When he obtains only one object, he values it v_0. When he obtains both objects together, he values them $2v_0 + g(v_0)$, where $g(v_0) \geq 0$ and $g'(v_0) \geq 0$ for all v_0. The other type of bidders (we call her a "local" bidder) is interested in only one object, 1 or 2. For simplicity, we assume that each local bidder's preference (i.e. whether she is interested in 1 or 2) is publicly known. When she is interested in object 1 (resp. 2) and obtains it, she values it v_1 (resp. v_2). We assume private independent value environment, that is, v_0 (resp. v_1, v_2) is drawn from a probability distribution function $F(v_0)$ (resp. $F_1(v_1), F_2(v_2)$) who has a density function $f(v_0)$ (resp. $f_1(v_1), f_2(v_2)$). The support for $F(v_0)$ (resp. $F_1(v_1), F_2(v_2)$) is $[\underline{v}_0, \bar{v}_0]$ (resp. $[\underline{v}_1, \bar{v}_1]$, $[\underline{v}_2, \bar{v}_2]$). Under these assumptions, we analyze the optimal auction by following the method introduced by Myerson (1981). Because of the revelation principle, we can restrict our attention on the direct revelation mechanism.

The following notions are similar to those in Myerson (1981).

Notation

$v = (v_0, v_1, v_2)$=true type vector of the global bidder, local bidder 1, and local bidder 2 respectively.

v_{-j} =true type vector of all the players except bidder j (j=0,1,2).

$w = (w_0, w_1, w_2)$=reported type vector of the global bidder, local bidder 1, and local bidder 2 respectively.

$p_j^i(v)$ = the probability that bidder j obtains object i when the reported type vector is $v = (v_0, v_1, v_2)$. (i,j)=(1,0), (2,0), (12,0), (1,1), or (2,2).

$Q_j^i(v) = E_{-v_j}[p_j^i(v)]$ = conditional probability (conditioned on bidder j's type being v_j) of $p_j^i(v)$. (i,j)=(1,0), (2,0), (12,0), (1,1), or (2,2).

$c_j(w_j)$ = the expected cost to be paid by bidder j when he/she reports his/her type as w_j

(j=0,1,2).

$U_0(w_0,v_0) = v_0(Q_0^1(w_0)+Q_0^2(w_0)+2Q_0^{12}(w_0)) + aQ_0^{12}(w_0) - c_0(w_0)$ = the expected utility of the global bidder when he reports his type as w_0 and his true type is v_0.

$U_j(w_j,v_j) = v_jQ_j(w_j) - c_j(w_j)$ = the expected utility of the local bidder j (j=1,2) when she reports her type as w_j and her true type is v_j.

$H_j(v_j) \equiv v_j - \dfrac{(1-F_j(v_j))}{f_j(v_j)}$ = virtual valuation of bidder j (j=0,1,2).

Assumption1 (linearlity)

$g(v_0) = kv_0 + a$, where $k > 0$ and $a > 0$.

Remark

We also assume $k = 0$ hereafter. Later, we mention how to extend all of our arguments to the case of $k > 0$. Further, the essential part of the results do not depend on the linearlity assumption. Our assumptions of the linearity and $k = 0$ are mainly for notational simplicity.

3. The optimal mechanism

We assume that the seller designs the selling mechanism to maximize her expected profit and commits to the mechanism. For simplicity, we assume the seller's valuations of the items are 0 and the seller does not incur any cost to sell the items. Using notation introduced in section2, the seller's problem (PS1) is expressed as follows.

(PS1)

$$\underset{p(.),c(.)}{Max} \quad E_v[c_0(v_0)+c_1(v_1)+c_2(v_2)]$$

s.t.

$U_j(v_j,v_j) \geq 0$ for any v_j, j=0,1, or 2

$U_0(v_0,v_0) \geq U_0(w_0,w_0) + (v_0-w_0)(Q_0^1(w_0)+Q_0^2(w_0)+2Q_0^{12}(w_0))$ for all w_0, $v_0 \in [\underline{v}_0,\bar{v}_0]$

..

.(1)

$U_j(v_j,v_j) \geq U_j(w_j,w_j) + (v_j - w_j)Q_j(w_j)$ for all w_j , v_j ...(2)

$p_0^1(v) + p_1^1(v) + p_0^{12}(v) \leq 1$ for all $v \in [\underline{v}_0, \overline{v}_0] \times [\underline{v}_1, \overline{v}_1] \times [\underline{v}_2, \overline{v}_2]$.

$p_0^2(v) + p_2^2(v) + p_0^{12}(v) \leq 1$ for all $v \in [\underline{v}_0, \overline{v}_0] \times [\underline{v}_1, \overline{v}_1] \times [\underline{v}_2, \overline{v}_2]$.

$p_j^i(v) \geq 0$ for (i,j)=(1,0), (2,0), (12,0), (1,1), or (2,2)

and for all $v \in [\underline{v}_0, \overline{v}_0] \times [\underline{v}_1, \overline{v}_1] \times [\underline{v}_2, \overline{v}_2]$.

We used the following lemma to replace (IC) by (1) and (2) in the above expression.

Lemma1.

(IC) and (1), (2) are equivalent, where,

(IC) $U_j(v_j,v_j) \geq U_j(w_j,v_j)$ for all w_j , v_j and j=0,1,2.

(1) $U_0(v_0,v_0) \geq U_0(w_0,w_0) + (v_0 - w_0)(Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))$ for all w_0 , v_0 .

(2) $U_j(v_j,v_j) \geq U_j(w_j,w_j) + (v_j - w_j)Q_j(w_j)$ for all w_j , v_j .

Proof.

First, we consider the global bidder.

By the definition of $U_0(v_0,v_0)$ and $U_0(w_0,v_0)$, we can obtain the following expresion.

(IC)

$$\Leftrightarrow (Q_0^1(v_0) + Q_0^2(v_0))v_0 + Q_0^{12}(v_0)(2v_0 + a) - c_0(v_0)$$

$$\square\ (Q_0^1(w_0) + Q_0^2(w_0))v_0 + Q_0^{12}(w_0)(2v_0 + a) - c_0(w_0) \qquad ...(3)$$

We can rewrite (3) as follows.

r.h.s. of (3)

$$= (Q_0^1(w_0) + Q_0^2(w_0))w_0 + Q_0^{12}(w_0)(2w_0 + a) - c_0(w_0)$$
$$+ (Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))(v_0 - w_0)$$

$= \; U_0(w_0, w_0) + (Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))(v_0 - w_0)$ ...(4)

Therefore, (3) is equivalent to

$U_0(v_0, v_0) \geq U_0(w_0, w_0) + (v_0 - w_0)(Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))$, which is (1).

We omit the proofs for the local bidders, since the arguments for them are similar. □

The next lemma is also useful to further rewrite seller's problem (PS1).

Lemma2.

(IC) and (IR) are equivalent to the following set of conditions.

$U_0(v_0, v_0) = U_0(\underline{v}_0, \bar{v}_0) + \int_{\underline{v}_0}^{v_0} (Q_0^1(w_0) + Q_0^2(v_0) + 2Q_0^{12}(v_0)) dw_0$ ...(4)

$U_j(v_j, v_j) = U_j(\underline{v}_j, \underline{v}_j) + \int_{\underline{v}_j}^{v_j} Q_j^j(w_j) dw_j$ for j=1,2. ...(5)

$Q_0^1(v_0) + Q_0^2(v_0) + 2Q_0^{12}(v_0)$ is a monotonically increasing function w.r.t. v_0. ...(6)

$Q_j^j(v_j)$ are monotonically increasing functions w.r.t. v_j for j=1,2. ...(7)

$U_0(\underline{v}_0, \underline{v}_0) \geq 0$. ...(8)

$U_j(\underline{v}_j, \underline{v}_j) \geq 0$ for j=1,2. ...(9)

Proof.

First, we would like to show that, if(IC) and (IR) hold for the global bidder, then the set of conditions, (4), (6), and (8) should hold. By using (2) twice, we can obtain

$$(v_0 - w_0)(Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0)) \leq U_0(v_0, v_0) - U_0(w_0, w_0)$$
$$\leq \; (v_0 - w_0)(Q_0^1(v_0) + Q_0^2(v_0) + 2Q_0^{12}(v_0)) \qquad \text{...(10)}$$

Dividing (10) by $(v_0 - w_o)$ and taking $w_0 \to v_0$ yields

$$\frac{dU_0(v_0)}{dv_0} = Q_0^1(v_0) + Q_0^2(v_0) + 2Q_0^{12}(v_0) \qquad \text{...(11)}$$

Integrating (11) becomes

$U_0(v_0,v_0) = U_0(\underline{v}_0,\underline{v}_0) + \int_{\underline{v}_0}^{v_0}(Q_0^1(w_0)+Q_0^2(w_0)+Q_0^{12}(w_0))dw_0$, which is (4). So, (IC) is equivalent to (4). From this, it is obvious that (IR) is equivalent to (6) and (8).

It is obvious that, if (4), (6), and (8) hold for the global bidder, then (IR) and (IC) should hold. So, we omit the proof for this direction.

We also omit the proofs for the local bidders since, again, the arguments for them are very similar to those for the global bidder. □

Now, we can characterize the optimal auction mechanism, $p_0^i(v)$ (i=1,2,12), $p_j^j(v)$ (j=1,2), and $c_j(v_j)$ (j=0,1,2) as follows.

Proposition1.

The optimal auction mechanism is characterized as follows.

$p_0^i(v)$ (i=1,2,12) and $p_j^j(v)$ (j=1,2) are the solutions for

$$\underset{p(.)}{Max} \quad E_v[H_0(v_0)(p_0^1(v)+p_0^2(v)+p_0^{12}(v))+ap_0^{12}(v)+H_1(v_1)p_1^1(v)+H_2(v_2)p_2^2(v)]$$

s.t. (4), (5), (6), (7), equality versions of (8) and (9), and the same constraints on the probabilities as those in (PS1).

$$c_0(v_0) = E_{-v_0}[(Q_0^1(v_0)+Q_0^2(v_0)+2Q_0^{12}(v_0))v_0 + aQ_0^{12}(v_0) - \int_{\underline{v}_0}^{v_0}(Q_0^1(w_0)+Q_0^2(w_0)+2Q_0^{12}(w_0))dw_0 \quad \text{...(12)}$$

$$c_j(v_j) = E_{-v_j}[Q_j^j(v_j)v_j - \int_{\underline{v}_j}^{v_j} Q_j^j(w_j)dw_j] \text{ for j=1,2.} \quad \text{...(13)}$$

Proof.

From lemma2, the first half part of the statement is obtained. To show the latter half part, we use (4) for the global bidder and (5) for the local bidders. First, we consider the global bidder.

From (4) and the definition of $U_0(v_0,v_0)$, we can obtain

$$c_0(v_0) = (Q_0^1(v_0) + Q_0^2(v_0) + 2Q_0^{12}(v_0))v_0 + aQ_0^{12}(v_0) - U(\underline{v}_0, \underline{v}_0)$$
$$- \int_{\underline{v}_0}^{v_0} (Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))dw_0$$

Since the seller would like to maximize the expected payment from bidders, $U_0(\underline{v}_0, \underline{v}_0) = 0$ should hold (this satisfies (IR)). So,

$$c_0(v_0) = (Q_0^1(v_0) + Q_0^2(v_0) + 2Q_0^{12}(v_0))v_0 + aQ_0^{12}(v_0)$$
$$- \int_{\underline{v}_0}^{v_0} (Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))dw_0$$

Further,

$$E_{v_0}[\int_{\underline{v}_0}^{v_0} (Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))dw_0$$
$$= \int_{\underline{v}_0}^{\bar{v}_0} \int_{\underline{v}_0}^{v_0} (Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))dw_0 f(v_0)dv_0$$
$$= \int_{\underline{v}_0}^{\bar{v}_0} \int_{w_0}^{\bar{v}_0} (Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))f(v_0)dv_0 dw_0$$
$$= \int_{\underline{v}_0}^{\bar{v}_0} (1 - F(w_0))(Q_0^1(w_0) + Q_0^2(w_0) + 2Q_0^{12}(w_0))dw_0$$
$$= E_v[\frac{1 - F(v_0)}{f(v_0)}(p_0^1(v) + p_0^2(v) + 2p_0^{12}(v))]$$

So,

$$E_{v_0}[c_0(v_0)]$$
$$= E_v\left[\left(\frac{1 - F(v_0)}{f(v_0)} + v_0\right)(p_0^1(v_0) + p_0^2(v_0) + 2p_0^{12}(v_0)) + ap_0^{12}(v_0)\right]$$
$$= E_v[H_0(v_0)(p_0^1(v_0) + p_0^2(v_0) + 2p_0^{12}(v_0)) + ap_0^{12}(v_0)] \qquad ...(14)$$

Similar arguments for the local bidders give us the following results.

$$E_{v_j}[c_j(v_j)] = E_v[H_j(v_j)p_j^j(v_j)] \text{ for j=1,2.} \qquad ...(15)$$

We can rewrite the seller's problem (P1) by using (14) and (15) as follows.

$$\underset{p(.)}{Max} \quad E_v[H_0(v_0)(p_0^1(v) + p_0^2(v) + p_0^{12}(v)) + ap_0^{12}(v) + H_1(v_1)p_1^1(v) + H_2(v_2)p_2^2(v)]$$

s.t. (4), (5), (6), (7), equality versions of (8) and (9), and the constraints on the probabilities in (PS1).

This is the desired result. □

If we impose the regularity condition being held, we can further characterize the optimal auction mechanism.

Asumption2 (regurality)

The regularity condition holds if $H_j^{'}(v_j) \geq 0$.

Proposition2.

Under the regularity condition, the optimal auction mechanism is characterized as follows.

$$p_0^1(v) = \begin{cases} 1 \quad if \quad J_0(v_0) + J_2(v_2) > 2J_0(v_0) + a \quad and \quad J_0(v_0) > J_1(v_1) \\ 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad otherwise \end{cases}$$

$$p_0^2(v) = \begin{cases} 1 \quad if \quad J_0(v_0) + J_1(v_1) > 2J_0(v_0) + a \quad and \quad J_0(v_0) > J_2(v_2) \\ 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad otherwise \end{cases}$$

$$p_0^{12}(v) = \begin{cases} 1 \quad if \quad 2J_0(v_0) + a > J_1(v_1) + J_2(v_2) \\ 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad otherwise \end{cases}$$

$$p_1^1(v) = \begin{cases} 1 \quad if \quad p_0^2(v) = 0 \\ 0 \quad\quad\quad otherwise \end{cases}$$

$$p_2^2(v) = \begin{cases} 1 \quad if \quad p_0^1(v) = 0 \\ 0 \quad\quad\quad otherwise \end{cases}$$

Further, $c_j^j(v_j)$ (j=0,1,2) are given by the same expression as those in Proposition1.

Proof.

Since it is obvious from the seller's maximization problem under the regularity condition, We omit the proof. □

Remark.

First of all, when k>0, we only need to replace $2H_0(v_0)$ by $(2+k)H_0(v_0)$ everywhere. Then, all the proofs work.

Secondly, extending our model to an arbitrary number of bidders is also easy. In this case, we only need to use first order statistic instead of the a probability distribution function itself for each type of bidders. Then, again, all the proofs work.

Thirdly, extending our model to an arbitrary number of objects is also easy as long as all the objects are homogeneous, since the problem has still one dimension type representation.

Efficiency.

Unfortunately, the next proposition demonstrates that the optimal auction mechanism is never efficient if $F(.) = F_1(.) = F_2(.)$.

Prosposition3.

The optimal auction mechanism is not efficient if $F(.) = F_1(.) = F_2(.)$.

Proof.

First of all, notice that the following statement should hold if the optimal auction mechanism is efficient.

$$(2+k)H_0(v_0)+a = H_1(v_1)+H_2(v_2) \quad \text{iff} \quad (2+k)v_0 + a = v_1 + v_2 \qquad \text{...(17)}$$

where, $H_0(.) = H_1(.) = H_2(.)$ from the assumption of $F(.) = F_1(.) = F_2(.)$.

It suffices to show that there exists no $F(.)$ which satisfies (17). From the r.h.s. of (17), we obtain

$$v_0 = \frac{v_1+v_2}{2+k} \qquad \text{...(18)}$$

Rewriting l.h.s. of (17) by using (18) and the definition of $H_0(.), H_1(.)$, and $H_2(v_2)$ becomes

$$(2+k)\left(\frac{v_1+v_2}{2+k}\right) - (2+k)\frac{1-F\left(\frac{v_1+v_2}{2+k}\right)}{f\left(\frac{v_1+v_2}{2+k}\right)} = v_1 - \frac{1-F(v_1)}{f(v_1)} + v_2 - \frac{1-F(v_2)}{f(v_2)} \qquad \text{...(19)}$$

Arranging (19) becomes

$$\frac{1-F\left(\frac{v_1+v_2}{2+k}\right)}{f\left(\frac{v_1+v_2}{2+k}\right)}=\left(\frac{1}{2+k}\right)\left(\frac{1-F(v_1)}{f(v_1)}+\frac{1-F(v_2)}{f(v_2)}\right) \qquad ...(20)$$

Lemma3.

bf(x)+bf(y)=f(b(x+y)) $\Leftrightarrow$ f(.) is homogeneous of degree one ...(21)

$\Leftarrow$ is obvious. To show, $\Rightarrow$, let x=y. Then, l.h.s. of (21) becomes

2bf(x)=f(2bx) ...(22)

Let denote 2b as z. Then, (22) is

zf(x)=f(zx) ...(23)

(23) is the definition for f(x) to be homogeneous of degree one.

From (20) and lemma 3, $G(.)=\frac{1-F(.)}{f(.)}$ should be homogeneous of degree one for the proposition to hold. However, (20) tells us that it is not the case, since $G(.)$ does not go through the origin. So, the optimal auction mechanism is not efficient.

Next, we would like to analyze the direction of bias of the optimal auction mechanism, i.e., whether it favors the global bidder or the local bidders. The following two examples tell us that we cannot determine the direction of the bias generally and it depends on the shape of the probability distribution function.

Example1.

In this example, the optimal auction mechanism favors local bidders.

Assume that a=0 and $F(.)=F_1(.)=F_2(.)=U[0,1]$. First of all, note that $G(v)=1-v$ in this case and the regularity condition is satisfied, since $G^{'}(v)<0$. At $(2+k)v_0=v_1+v_2$,

$$(2+k)G(v_0)=(2+k)\left(1-\frac{v_1+v_2}{2+k}\right)=(2+k)-(v_1+v_2)>2-(v_1+v_2)=G_1+G_2 \qquad ...(24)$$

Since $H(v) \equiv v - G(v)$, (24) means

$$(2+k)H(v_0) < H(v_1) + H(v_2) \text{ at } (2+k)v_0 = v_1 + v_2 \qquad ...(25)$$

(25) means that the virtual valuation for the global bidder is strictly less than the sum of the ones for the two local bidders when the true valuation of the global bidder is same as the sum of the ones for the two local bidders. This is the desired result.

Example2.

In this example, the optimal auction mechanism favors the global bidder.

Assume that a=0 and $F(v) = F_1(v) = F_2(v) = \sqrt{2}\sin v, \quad v \in [0, \pi/4]$. Then,

$f(v) = f_1(v) = f_2(v) = \sqrt{2}\cos v$ and $G(v) = \dfrac{1-F(v)}{f(v)} = \dfrac{1-\sqrt{2}\sin v}{\sqrt{2}\cos v}$. The regularity condition holds in this case, since

$$G'(v) = \frac{-2\cos^2 v + (1-\sqrt{2}\sin v)\sqrt{2}\sin v}{2\cos^2 v} = \frac{-2(\cos^2 v + \sin^2 v) + \sqrt{2}\sin v}{2\cos^2 v} = \frac{-2+\sqrt{2}\sin v}{2\cos^2 v} < 0 \qquad ...(26)$$

Further, $G''(v) = \dfrac{2\sqrt{2}\cos^3 v + 16\cos v \sin v(2-\sqrt{2}\sin v)}{4\cos^4 v} > 0 \quad \mathit{for} \quad \forall v \in [0.\pi/4]$ ...(27)

From (26) and (27),

$$G\left(\frac{v_1+v_2}{2}\right) < \frac{1}{2}G(v_1) + \frac{1}{2}G(v_2) \quad \mathit{for} \quad \forall v_1, v_2 \in [0, \pi/4] \qquad ...(28)$$

For small enough k, still

$$G\left(\frac{v_1+v_2}{2+k}\right) < \frac{1}{2+k}G(v_1) + \frac{1}{2+k}G(v_2) \quad \mathit{for} \quad \forall v_1, v_2 \in [0, \pi/4] \qquad ...(29)$$

Since $H(v) \equiv v - G(v)$, (29) means

$$(2+k)H(v_0) < H(v_1) + H(v_2) \text{ at } (2+k)v_0 = v_1 + v_2 \qquad ...(30)$$

(30) means that the virtual valuation for the global bidder is strictly larger than the sum of the ones for the two local bidders when the true valuation of the global bidder is same as the sum of the ones for the two local bidders. This is the desired result.

Remark.

If two objects are independent, surely, the optimal auction mechanism is efficient.

4. An efficient mechanism.

Now, we are ready to characterize efficient auction mechanisms. First of all, the Clarke-Groves-Vickrey mechanism (the CGV mechanism) can implement efficient allocation in our context. It is expressed as follows.

Proposition4.

Suppose $F(.) = F_1(.) = F_2(.)$ holds. The CGV mechanism implements efficient allocation, which is expressed as follows in our context.

1. The global bidder submits three bids. The first one is for object 1, b_0^1, the second one is for object 2, b_0^2, and the last one is for a bundle of the object 1 and 2, b_0^{12}.

2. The local bidder j (j=1,2) submits a bid for object j, b_j^j.

3. The global bidder obtains only object 1 when $b_0^1 > b_1^1$ and $b_0^{12} < b_1^1 + b_2^2$, and pays b_1^1. In this case, local bidder 2 obtains object 2 and pays b_0^2. Similarly, the global bidder obtains only object 2 when $b_0^2 > b_2^2$ and $b_0^{12} < b_1^1 + b_2^2$, and pays b_2^2. In this case, the local bidder 1 obtains object 1 and pays b_0^1. Finally, the global bidder obtains both object 1 and 2 when $b_0^{12} > b_1^1 + b_2^2$ and pays $b_1^1 + b_2^2$. In this case, the local bidders do not obtain anything and do not pay anything.

Proof.

Since truth-telling is always the equilibrium strategy in the CGV mechanism in our setting, the result follows immediately.

Remark.

In the CGV auction, allowing the global bidder to submit a combinatorial bid is the key to assure the efficiency.

5. Extensions.

We analyze the simplest possible model with two types of bidders (one global bidder and two local bidders) and two items in the text. However, our results hold for a more general model with arbitrary number of bidders with arbitrary number of items. Further, none of linearlity assumption on bidders' valuation functions, the assumptions on the seller's cost and reserve valuation changes the results. Moreover, the regularity condition on bidders' virtual valuation is imposed only for simplicity. We can use the ironing technique to deal with non regular case and can obtain similar results. The only crucial assumption here is that bidders' types are common knowledge in the sense that whether a bidder is a global bidder or one of two types of local bidders is known to all players. In more general model with arbitrary numbers of bidders and items, who is interested in which set of items is common knowledge in our model. Once we give up this assumption, things become much worse. Yokoo et. al. (2004) show that even VCG mechanism cannot attain efficiency if there is false-name bids. In our context, it means that if a global bidder can hide his identity and can pretend to be two local bidders, then, VCG mechanism we proposed in section4 is not any more efficient. In addition, we have to analyze auctions with multi-dimensional type to characterize the optimal auction mechanism. Still, in this case, our result of the optimal auction never achieving efficiency holds. We believe that this paper sheds light on important problem of auction design, namely, who designs auction for what purpose? If the government is the designer, it puts more weight on efficiency than revenue. On the other hand, if a private company is the seller, it maximizes its expected revenue. Especially when the government is the seller and would like to raise money to reduce their financial deficit such as FCC spectrum auctions, it has to be careful when they choose an auction procedure. To raise money, or to achieve efficiency, that is the question and they cannot have both.

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