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Abstract

Voting is a lottery in which an individual wins if she belongs to the majority or loses if she falls into the minority. The probabilities of winning and losing depend on the voting rules. The risk of losing can be reduced by increasing the majority threshold. This however has the negative effect of also lowering the chance to win. We compute the individual's preferred majority threshold, as a function of her risk attitudes, her voting power and her priors about how the other individuals will vote.

We find that the optimal threshold is higher when an individual is more risk averse, less powerful, and less optimistic about the chance that the others will vote like her. De facto, raising the threshold is a form of protection against the higher risk of being tyrannized by an unfavorable majority.

Keywords: optimal majority rule, super-majority, risk aversion, weighted votes, voter optimism.

JEL classification: D72, H11, D81.

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1 Introduction

Voting is probably the most common way to make collective decisions, not only in legislatures, but also in international bodies, corporations, associations, and even in condominiums. The decision made by the majority has impact on individual welfare. One's welfare lessens if an individual belongs to the minority rather than to the majority. Sometimes, adverse majorities make decisions that are even worse than the status quo. In this case, we say that the individual is expropriated or tyrannized by the majority. The probability of being tyrannized depends on the possibility of an adverse majority being formed.

We describe voting as a lottery with given gains and losses, and given probabilities of winning and losing. These probabilities depend on a variety of voting rules, such as the (super-)majority threshold, the way the votes are weighted, the distribution of the right to make proposals or amendments, etc.. Rules that favor decisiveness, by increasing the probability of forming a majority have the positive effect of increasing the chance of winning, and the negative effect of increasing the risk of losing.

We wonder if, from an individual viewpoint, there is any set of "optimal" rules that could maximize the difference between the expected gain and the expected loss of the voting lottery, and what this maximization depends on.

Intuitively, when the loss is very high compared to the gain, an individual prefers less decisive rules. For example, she could ask for more "checks and balances" or for higher super-majorities. This possibly explains why, when corporate boards vote on major actions (mergers and acquisitions, major capital expansions, etc.), high super-majorities are required, or why, in international treaties, the members can exercise vetoes when the decisions concern their crucial interests (the Council of the EU, the UN Security Council). The U.S. Federal Constitution requires a super-majority in cases where seeking a broad consensus rather than a bare majority is sought after. For example, a two-thirds vote is required to override a presidential veto, to ratify a treaty, or to expel a member of Congress. Three-fifths of the full Senate must approve any waiver regarding balanced budget provisions or points of order for the consideration of legislation that would violate a budget approved by Congress.

Moreover, the simple majority is less frequent than it may appear. Most countries have de facto super-majority requirements because of the two chambers and the executive veto. Thus it is not easy for a future majority to undo acts passed by a previous two-house majority and executive decision.

In the U.S., there is a lively debate about the proposed "Tax Limitation Constitutional Amendment" that would require a two-thirds majority in the House and Senate anytime a vote is taken on legislation that would result in a federal debt or tax increase. The supporters claim that when spending tends to result in a substantial number of individuals who are net losers, such spending should fail to command the support of a super-majority. Fifteen states, comprising one-third of the U.S. population, already have super-majority requirements for state tax increases.

In weighted voting contexts, such as the EU and the IMF, or even in legislatures where minorities are represented, protection from the tyranny of the majority is often claimed by the weaker voters. In these contexts, small members' weights are proportionally higher or super-majority thresholds occur more frequently. Most of the recent debate on reforming the EU decision making has concerned the reapportionment of voting weights and the super-majority lowering in the Council. Countries that are more reluctant to relinquish their sovereignty oppose majority threshold reductions, in order to keep their blocking ability against group decisions.

This paper investigates the optimal choice of institutions from the viewpoint of an individual who knows how much she will gain if she is in the majority, and how much she will lose if she is in the minority. She does not know, however, how the others will vote. We focus on the (super-)majority threshold that maximizes the expected value of a voting lottery. We show that if the number of voters is sufficiently high the individual's optimal threshold is unique. We find that the optimal threshold is higher when the loss is high (the gain is low) or when the individual has low voting power. We further consider the individual's priors about how the other players will vote: the more confident about the chance to win she is, the more she wants to facilitate the formation of the majority; thus her optimal threshold decreases.

We constrain the maximization within the simple majority and unanimity. The simple majority emerges as a corner solution when the gain is relatively high. Interestingly, we find that when an individual is non-confident about a favorable voting outcome, there are only corner solutions: unanimity is preferred when losses are high; the simple majority is favored when gains are high. In all the other cases, the optimal threshold is a super-majority.

We also consider individual attitudes towards risk. A risk averse individual dislikes losing more than she likes winning. In this case, she prefers a higher threshold: the status quo is more likely to remain, thus she feels protected from tyranny. In the phase of writing the constitution or the statutory rules of a corporation, there is a "veil of ignorance" about the gains and losses of future voting; in these cases, risk aversion is possibly the only variable that guides her choice of the best voting rules.

1.1 Novelty and related literature

Our work contributes to the vast literature on voting rules in a novel way. To the best of our knowledge, nobody has depicted voting as a lottery where outcome uncertainty can be controlled by setting up an appropriate threshold.

In our model, an individual faces a trade-off because reducing the threshold makes a favorable majority more likely, but it also makes tyranny more likely. Buchanan and Tullock (1962) analyze a different trade-off: a lower threshold reduces the costs of securing a majority agreement but it increases the losses suffered by the minority members. A higher threshold is justified when the minority's preferences are very different from those of the majority.

Voting uncertainty becomes central is Rae (1969). However, differently from our model, this uncertainty concerns the fact that the making of a law will generate gains or losses. Rae suggests (and Taylor, 1969, formally proves) that the simple majority is the only rule that minimizes the expected cost of being part of the minority. In Rae, costs and gains are not only equal but also equally likely to arise from a bill that is opposed to the status quo. Our model applies to a wider range of situations (asymmetric gains and losses, weighted votes, different degrees of risk aversion and confidence). It includes Rae's setting as a special case in which in fact it generates the same result.[1]

Aghion and Bolton (2002), show that the optimal qualified majority threshold increases in the expected cost of compensating the losing minority, when individuals do not know ex-ante if they will lose or gain from the provision of a public good. Aghion, Alesina and Trebbi (2004) analyze the level of insulation of political leaders. This is very much the constitutional choice of the majority threshold: an insulated (non-insulated) leader needs the support of a low (high) majority in order to pass legislation. The optimal degree of insulation depends on the cost of compensating the losers, the social benefits of policy reforms, the uncertainty about gains and losses, the

[1] The special case is the following: an individual has equal probability of ending up either in the majority or in the minority; i.e. her vote is negligible in determining the voting outcome and she thinks that the other players are equally likely either to vote for the alternative she likes or for the alternative she dislikes. The utilities of the two alternatives are symmetric with respect to the status quo. In this case, our model predicts that the preferred threshold is the simple majority.

degree of risk aversion. Their models are very related to ours. There are however several aspects which are different. Theirs are aggregate models, while ours is a strictly micro one: we do not consider the social efficiency of policy decisions and we do not compute the socially optimal threshold, but only an individual one. In those models, individuals have the same voting weight (one vote each); we consider asymmetries in voting power. In their models, individuals share ex-ante the same degree of uncertainty about gains and losses from policy decisions. In our model uncertainty concerns how the other player will vote; thus individuals can have different expectations about voting outcomes. Moreover, our model provides perhaps a more general treatment of risk aversion.

We relate the qualified majority to the degree of optimism about how the other players will vote. The idea that optimism can play a role in collective policy decisions was first proposed by Buchanan and Faith (1980,1981) and developed by Zorn and Martin (1986). Due to the great uncertainty typically attached to the outcome of government projects, the main conclusion of Zorn and Martin is that optimism about the net benefits of the projects heavily influences policy decision making. This idea of optimism is however different from ours: in our model an individual can be optimistic or pessimistic about the voting behavior of the other players, and she selects an optimal threshold accordingly.

Our model applies to weighted voting contexts. We show that powerful players prefer lower thresholds because this gives them higher leverage on the voting outcome. To our knowledge, no other work has analyzed such a relationship between weight and the preferred threshold. Usually, cooperative and non-cooperative models of legislative bargaining have explored the impact of voting weights on the division of the spoils.[2]

In our analysis, we show that voters who are more risk averse prefer higher thresholds. In our model, the uncertainty is about how the other people will vote. In the existing literature on risk aversion and voting, the uncertainty concerns the benefits of alternative policy proposals.[3]

We compute the individual optimal threshold, as a function of her gain,

[2] In general, for a given majority threshold cooperative analysis predicts a non-monotone relationship between weight and bargaining power(for a review see Owen (1995), Felsenthal and Machover (1998), or Benoit and Korhauser (2002)). Recently, Snyder, Ting, and Ansolabehere (2005) find that in a non-cooperative bargaining game weights and spoils are proportional.

Another branch of the literature has focused on the rationale of voting weights (e.g. Penrose (1946), Nitzan and Paroush (1982), Shapley and Grofman (1984).

[3] Recent contributions that also contain useful references are Aragones and Postlewaite (2002), Berinsky and Lewis (2005), and Harrington (1990).

loss, and degree of confidence. Despite the fact that our findings are relevant in constitutional design, we do not answer the question: "what voting rule will a country adopt?" Recent models of constitutional negotiations that answer this question are Aghion, Alesina, and Trebbi (2004), Messner and Polborn (2004), Barberà and Jackson (2004, 2006), Passarelli and Schure (2006).

This paper is structured as follows. In Section 1.1 we relate our work to the existing literature. Section 2 presents the setup of the voting model. In Section 2.1 we compute the optimal qualified majority. In Sections 2.1.2 and 2.1.1 we analyze how confidence about the voting outcomes affect the optimal threshold. In Sections 2.2 and 2.3 we explore the role of risk attitudes and voting weights, respectively. Section 3 concludes.

2 The model

Consider a set $N = \{1, ..., n\}$ of agents who play a voting game $\gamma(q; w_1, ..., w_n)$ where q is the majority threshold and w_i represents player i's ($i = 1, .., n$) number of votes. Let m denote the sum of votes, $\sum_N w_i$, and assume that the voting game is proper, i.e. $q > \frac{m}{2}$. Players can be of two types, either j or $\sim j$. Assume that in a legislature the players will have to vote on a policy that can generate losses and gains. If the policy is j then all the type-j players will gain and the type-$\sim j$ players will lose. If the policy is $\sim j$, the distribution of gains and losses will be reversed. If no policy passes, then the status quo, ς, remains. We assume that whenever a majority can be formed on a policy, either j or $\sim j$, that policy is put forward to be voted. All type-j vote in favor of j and type-$\sim j$ vote against, and vice versa. Abstention is not possible.

In order to save notation let j index not only the player's type, but also her name ($j = 1, .., n$), wherever this does not generate confusion. We say that player j "wins" if her preferred policy, j, passes; she "loses" if $\sim j$ passes. In the latter case she falls into the minority and she suffers the tyranny of an undesired majority. Thus "winning" is better than the status quo, and the status quo is better than "losing". Here player j's uncertainty concerns the type of the other players.[4]

[4]With a little variation in the interpretation of notation, our "legislative" framework applies to electoral competitions. In this case , j and $\sim j$ represent the electoral platforms proposed by two candidates. Type-j players lose from platform $\sim j$. Thus they have an incentive to reduce the ability of the undesirable candidate to pass reforms in $\sim j$ that change the status quo. This can be done by setting up a higher threshold.

Vice versa, they would like to facilitate their preferred candidte so as to implement

Formally, let ω denote the outcome of the voting game, and let $\Omega = \{\alpha, \beta, \varsigma\}$ be the set of all possible outcomes. Call $C = C_\alpha \times C_\beta$ the set of monetary consequences of any outcome, with each component referring to a specific player type. Let $g : \Omega \to C$ be a function that assigns each outcome a vector of monetary consequences. For each type-j player, call $v_j : C_j \to \mathbb{R}$ the utility evaluation of any monetary consequence. Let $u_j(\omega) := g \circ v_j$ be the Bernoulli utility function of the type-j player. By assumption, $u_j(j) > u_j(\varsigma) > u_j(\sim j)$.

We are interested in the probability of any outcome of the voting game. Take player j's viewpoint. Assume that she knows her type with certainty and she subjectively thinks that any other player in $N \setminus j$ is of type-j with probability p.[5] We can say that p represents player j's degree of optimism: p is high if she thinks that any other player is likely to vote for her most preferred policy. Conversely, $(1-p)$ is the probability that any other player votes for the least preferred policy, $\sim j$.

Call $S_j \subseteq N \setminus j$ the coalition of the other players who vote for policy j. The probability of winning, $\Pr\{j\}$, is given by the probability that S_j gets at least $q - w_j$ votes. Then player j adds her own w_j votes, and the majority forms. Given the uncertainty regarding the voting behavior of the other players, the sum of votes in S_j is a random event that behaves like the sum of $n-1$ independent random variables, Z_i, $(i = 1, .., j-1, j+1, ..n)$; where $Z_i = w_i$ with probability p, and $Z_i = 0$ with probability $(1-p)$. Suppose the number of players is sufficiently large and let the Central Limit Theorem to apply. Thus, the sum of votes gotten by S_j is distributed like a normal with parameters $\mu_j = \sum_{N \setminus j} w_i p$, and $\sigma_j^2 = \sum_{N \setminus j} w_i^2 p(1-p)$. Let $f^j(.)$ be its density function.

Therefore, the probability of the most preferred policy j winning is

$$\Pr\{j\} = \int\limits_{q-w_j}^{m-w_j} f^j(x)dx. \tag{1}$$

Conversely, the probability of falling into the minority (the probability of a winning coalition on policy $\sim j$ forming) is

$$\Pr\{\sim j\} = \int\limits_{q}^{m-w_j} f^{\sim j}(x)dx \tag{2}$$

policies in j, via lower thresholds. The legislative trade-off arises.

[5]Because p represents player j's subjective beliefs, it should be indexed with a "j". Since we are considering only the individual j, we omit this index for simplicity.

where $f^{\sim j}$ is a normal density function with parameters: $\mu_{\sim j} = \sum_{N \setminus j} w_i(1-p)$, and $\sigma^2_{\sim j} = \sigma^2_j = \sum_{N \setminus j} w_i^2 p(1-p)$.

The probability for the status quo remaining (the probability that neither policy j nor $\sim j$ reach the majority) is

$$\Pr\{\varsigma\} = 1 - \Pr\{j\} - \Pr\{\sim j\} \tag{3}$$

Of course, with the simple majority, the status quo probability $\Pr\{\varsigma\}$ is always zero. With unanimity, if p is not "too close" to one or to zero, the status quo is "almost" certain.[6]

Thus, voting can be described as a lottery L_j, with three outcomes in Ω and their probability distribution in (1-3). Player j's expected utility from the voting lottery is

$$EU_j(L_j) = \Pr\{j\} \cdot u_j(j) + \Pr\{\sim j\} \cdot u_j(\sim j) + \Pr\{\varsigma\} \cdot u_j(\varsigma) \tag{4}$$

Observe that the outcome probabilities (1-3) depend, among other things, on the characteristics of the voting game; namely, on the threshold q and on the vote distribution. They also depend on player j's degree of optimism, p. Moreover, her degree of risk aversion affect the outcome utilities in (4). In section 2.1 below we will show that player j has an optimal threshold that maximizes the expected utility of her voting lottery. Below we will explore the relationships between the optimal threshold and player j's degree of optimism, p, her number of votes, w_j, and her degree of risk aversion.

2.1 Optimal thresholds

In this section we compute the threshold q_0 that maximizes player j's expected utility in (4). The first-order condition (FOC) for maximization is

$$\left.\frac{\partial EU_j(L)}{\partial q}\right|_{q_0} = -f^j(q-w_j) \cdot [u_j(j) - u_j(\varsigma)] + f^{\sim j}(q) \cdot [u_j(\varsigma) - u_j(\sim j)] = 0 \tag{5}$$

or

$$f^j(q-w_j) \cdot [u_j(j) - u_j(\varsigma)] = f^{\sim j}(q) \cdot [u_j(\varsigma) - u_j(\sim j)] \tag{6}$$

[6] Recall that the number of players is high. For example, with 40 players who vote for j with probability $p = 0.9$, the probability of forming a unanimity regarding j is less than 1.5%, whereas the the status quo probability is more than 98.5%.

From (6) it is clear that the individual balances the negative marginal impact of the threshold on the expected benefit of belonging to the majority (left-hand side) with the marginal impact of the threshold on reducing the expected loss of falling into the minority (a lower expected tyranny - the right-hand side).

The threshold that satisfies the FOC is

$$q_0 = \frac{m}{2} + \frac{\sigma_j^2 \ln RASQ_j}{w_j + \mu_j - \mu_{\sim j}} \tag{7}$$

where

$$RASQ_j = \frac{u_j(\varsigma) - u_j(\sim j)}{u_j(j) - u_j(\varsigma)} \tag{8}$$

The second-order condition (SOC) is

$$\left.\frac{\partial^2 EU_j(L)}{\partial q^2}\right|_{q_0} = -f_q^{j\prime}(q-w_j)\cdot[u_j(j) - u_j(\varsigma)] + f_q^{\sim j\prime}(q)\cdot[u_j(\varsigma) - u_j(\sim j)] < 0 \tag{9}$$

with $f_q^{j\prime}(q - w_j) = \frac{\partial f^j(q-w_j)}{\partial q}$ and $f_q^{\sim j\prime}(q) = \frac{\partial f^{\sim j}(q)}{\partial q}$; thus,

$$f_q^{\sim j\prime}(q_0) \cdot RASQ_j < f_q^{j\prime}(q_0 - w_j) \tag{10}$$

$RASQ_j$ in (8) represents the *Relative Advantage of the Status Quo*. It represents the ratio between the benefits of not being tyrannized by an undesired majority, $u_j(\varsigma) - u_j(\sim j)$, and the benefits of being part of a desired winning majority, $u_j(j) - u_j(\varsigma)$. In voting situations, a trade-off between these two benefits occurs. In particular, by increasing q, any majority is less likely to form. As a result, the tyranny of an undesired majority is less likely, but winning becomes more difficult. High values of $RASQ_j$ reveal high sensitivity to not being tyrannized. Intuitively, we expect that players with high $RASQ$ tend to prefer high super-majorities. We prove this intuition below.

Because of the assumption that γ is proper, the maximization of (4) is constrained in $[q_s, m]$. Let us see graphically how this constraint works. Let ε be a positive arbitrary number lower than $\min\{w_1, .., w_n\}$. Call $q_s = \frac{m}{2} + \varepsilon$ the simple majority threshold of the voting game γ.

- If q_0 in (7) is internal and (10) is satisfied, then we have an internal solution (graph (a) in figure 1): player j prefers a qualified majority.

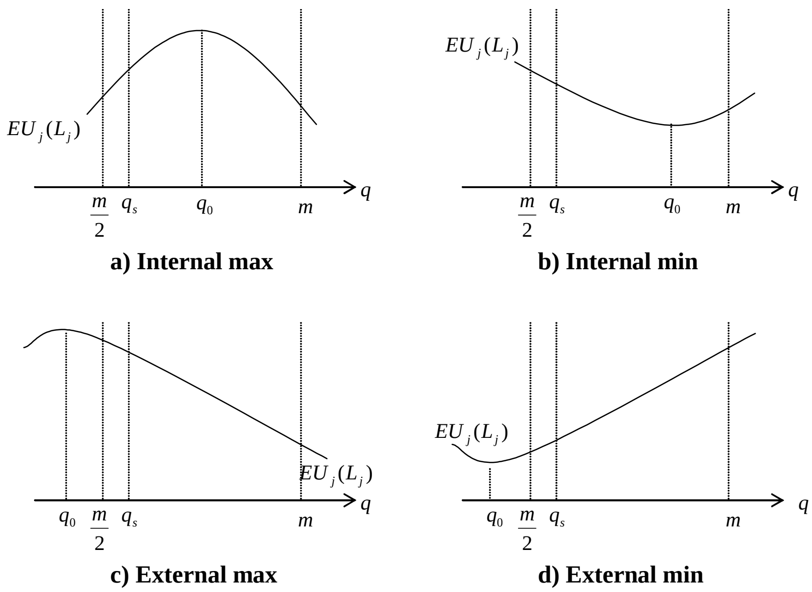

Figure 1: Optimal thresholds

- If $q_0 < q_s$, then we have a corner solution: player j's optimal threshold is the simple majority if q_0 is a maximum (graph (c) in figure 1) or unanimity if q_0 is a minimum (graph and (d) in figure 1).
- We have a corner solution also when q_0 is an internal minimum (in graph (b) the player prefers the simple majority).

To go more into the details of player j's maximization, let us take advantage of figure 2 as well. Recall that f^j and $f^{\sim j}$ are two normal densities with the same variance, and "centered" in $\mu_j = \sum_{N \setminus j} w_i p$ and $\mu_{\sim j} = \sum_{N \setminus j} w_i (1-p)$, respectively. Thus they are "far apart", and their distance reflects the degree of optimism, p. Call $A = \mu_j + w_j$ and $B = \mu_{\sim j}$. It is easy to see that

$$\begin{aligned} p &< \frac{1}{2} - \frac{w_j}{2\sum_{N \setminus j} w_i} \Leftrightarrow B > A \\ p &> \frac{1}{2} - \frac{w_j}{2\sum_{N \setminus j} w_i} \Leftrightarrow A > B \end{aligned}$$

We can say that player j is:

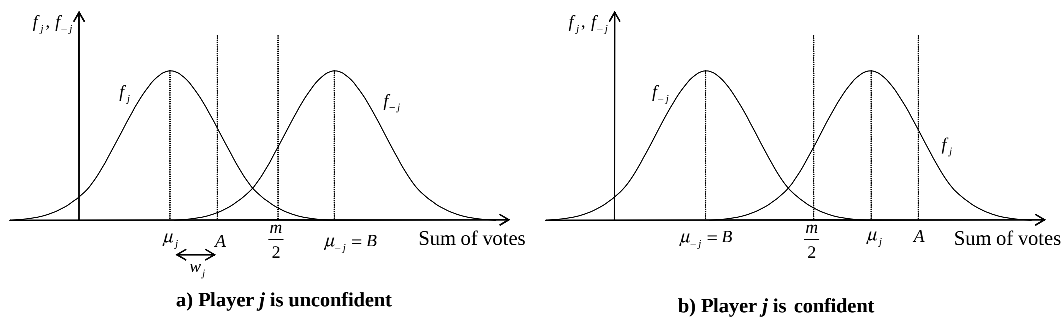

Figure 2: Distributions of other players' votes

a) **Non-confident** if $B > A$**.** In this case, any coalition that gets more than 50% of the votes is always more likely to vote for $\sim j$ than for j; or equivalently, any sum of the votes above $\frac{m}{2}$ is more likely to be reached by an adverse coalition than by a favorable one. This case is illustrated by graph (a) in figure 2.

b) **Confident** if $A > B$. This is the case where Player j is confident that the probability of a favorable majority is always higher than an unfavorable one (graph (b) in figure 2).

Because of the central role played in this model, it is necessary to clarify our idea of confidence. A player is confident when she thinks that winning is more likely than losing; or equivalently, when any sum of votes higher than $\frac{m}{2}$ is more likely to be reached by a type-j coalition than by a type-$\sim j$ coalition. Optimism ($p > \frac{1}{2}$) about how the other players will vote is only a sufficient condition for confidence. In fact, a sufficiently powerful player ends up being confident even with a certain degree of pessimism, since she knows she will be able to affect the outcome with her vote. In other words, high voting power plays the same role as high p in voting situations (see also section 2.3 below).

Lemma 1 below states that if the player is confident then q_0 always maximizes the expected utility of the voting lottery. On the contrary, if she is non-confident, then q_0 is the minimum.

Lemma 1 *q_0 in (7) is the maximum for (4) if and only if $A > B$.*

Proof. Rearranging (5) yields

$$\frac{f^j(q_0 - w_j)}{f^{\sim j}(q_0)} = RASQ_j \tag{11}$$

Substituting (11) for $RASQ_j$ into (10) yields

$$f_q^{\sim j\prime}(q_0) \cdot \frac{f^j(q_0 - w_j)}{f^{\sim j}(q_0)} < f_q^{j\prime}(q_0 - w_j)$$

from which

$$\frac{f_q^{\sim j\prime}(q_0)}{f^{\sim j}(q_0)} < \frac{f_q^{j\prime}(q_0 - w_j)}{f^j(q_0 - w_j)}$$

Recall that the $f(.)$ are normal distributions. Applying the formulas, and simplifying, yields

$$-\frac{1}{\sigma^2}(q_0 - \mu_{\sim j}) < -\frac{1}{\sigma^2}(q_0 - w_j - \mu_j)$$

thus,

$$w_j + \mu_j - \mu_{\sim j} > 0$$

from which the Lemma follows. ■

We use this Lemma in sections 2.1.1 and 2.1.2 to see where a non-confident player and a confident one set their preferred thresholds.

2.1.1 Non-confident players, $B > A$

When we have a non-confident player, q_0 is the minimum, thus we can only have a corner solution: either unanimity or the simple majority, as shown by figures 1 (b) and (d).[7] In particular, when q_0 is a minimum below q_s (graph (d)) we are sure that the player prefers unanimity. Proposition 1 shows that such a case occurs when $RASQ \geq 1$; i.e. when avoiding loss is preferred to winning.

Proposition 1 *If $B > A$ and $RASQ_j \geq 1$ then player j prefers unanimity.*

Proof. From $B > A$ and Lemma 1, we know that q_0 is the minimum. From (7) we can see that $q_0 < q_s$ only if $\frac{\sigma_j^2 \ln RASQ_j}{w_j + \mu_j - \mu_{\sim j}} \leq 0$. By $B > A$ the denominator in this disequality is negative, and by $RASQ_j \geq 1$ the numerator is weakly positive. ■

In the case, q_0 is an internal minimum, as illustrated in figure 1 (b), the player has to choose between unanimity and the simple majority. Consider that player j is non-confident about the outcome of voting: she thinks that losing is more likely than winning. In general, we expect this player to

[7] In the case that both the simple majority and unanimity yield the same expected payoffs, the player is indifferent.

protect herself from a (likely) tyranny by choosing a high threshold. However, if her level of non-confidence is low and the advantage of winning is considerably higher than the disadvantage of losing, we could even imagine that she would prefer a voting lottery in which a majority is easy to form. Proposition 2 below sets the conditions by which a non-confident player selects a low majority threshold. This Proposition also shows that in this case the player also wants to make majority formation as easy as possible, by choosing the simple majority.

Proposition 2 *If $B > A$ and $RASQ_j < 1$, then player j prefers unanimity to simple majority if*

$$RASQ_j > \frac{x_j}{1 - x_j} \tag{12}$$

with $x_j = \Pr\left\{ j|_{q=q_s} \right\}$, the probability of winning when the simple majority threshold q_s is set up.

Proof. By Lemma 1 and Proposition 1, we know that q_0 is an internal minimum. Player j has to decide if she is better off under the simple majority or under unanimity. Let us disregard the trivial situation in which j has all the votes ($w_j = m$), and she is indifferent. Recall that under unanimity the status quo is "almost" certain; with the simple majority, the status quo is impossible, and outcomes j and $\sim j$ occur with probability x_j and $1 - x_j$, respectively. Thus the expected utilities of voting under unanimity and under the simple majority are $u_j(\varsigma)$ and $u_j(j) \cdot x_j + u_j(\sim j) \cdot (1 - x_j)$, respectively. Player j prefers unanimity to the simple majority if

$$u_j(\varsigma) > u_j(j) \cdot x_j + u_j(\sim j) \cdot (1 - x_j)$$

rearranging,

$$[u_j(j) - u_j\varsigma)] \cdot x_j + [u_j(\varsigma) - u_j(\sim j)] \cdot (x_j - 1) < 0$$

then,

$$x_j + RASQ_j \cdot (x_j - 1) < 0$$

from which the lemma follows. ■

Thanks to Proposition 2,we also conclude that, all other things being equal, a player with higher $RASQ$ cannot prefer the simple majority while at the same time another player with lower $RASQ$ chooses unanimity. In section 2.2 we will use this argument to analyze how players with different degrees of risk aversion select their optimal thresholds.

2.1.2 Confident players, $A > B$

The two situations represented in figure 1 (a) and (c) occur when a player is confident. In this case, the preferred threshold is either the simple majority (q_0 is external) or a qualified majority (q_0 is internal). The Proposition below shows how these two cases are related to $RASQ$.

Proposition 3 *If $A > B$ and:*
a) if $RASQ_j > 1$, then player j prefers a qualified majority;
b) if $RASQ_j \leq 1$, then player j prefers the simple majority.

Proof. From $A > B$ and Lemma 1, we know that q_0 is a maximum.
a) From (7) we can see that $q_0 \in [q_s, m]$ only if $\frac{\sigma_j^2 \ln RASQ_j}{w_j + \mu_j - \mu_{\sim j}} > 0$. Observe that the denominator in this disequality is $A - B$, and it is positive. This implies that $RASQ_j$ must be higher than one.
b) Following the same argument, we conclude that $q_0 < q_s$ if $RASQ_j \leq 1$.
■

This result suggests that in order for the simple majority to emerge as the preferred threshold, we only need the player to be confident and her $RASQ$ to be just under one. Voting weight, w_j, and the degree of optimism, p do not play any additional role, aside from ensuring that $A > B$.

2.2 Risk Aversion

In this section, we analyze the relationship between a player's preferred threshold and her attitude toward risk. As stated above, $RASQ_j$ in (8) represents the advantage of not being tyrannized relative to the advantage of being part of a majority. Observe that $RASQ_j$ depends on the concavity of player j's utility function and on the monetary values of j, $\sim j$ and ς. Given these three monetary values, a player with a more concave utility function gets high utility from avoiding tyranny (a higher numerator in $RASQ$) and lower utility from winning (a lower denominator). Thus we expect that $RASQ$ is positively related to the concavity of the utility function, and ultimately to the player's risk aversion. We prove this relationship in Lemma 2 below.

Lemma 2 *Given the monetary values of j, $\sim j$ and ς, $RASQ_j$ is positively related to player j's risk aversion.*

Proof. Let r and s be two type-j players on whom the three outcomes of the voting lottery have the same monetary consequences. Let

$u_r(.)$ and $u_s(.)$ be their monotone utility functions. Recall that the function $g(.)$ identifies the monetary consequence of any outcome. We know that $u_r(g(j)) > u_r(g(\varsigma)) > u_r(g(\sim j))$ and $u_s(g(j)) > u_s(g(\varsigma)) > u_s(g(\sim j))$. Suppose that r is more risk averse than s. Thus $u_r(.)$ is more concave than $u_s(.)$, or equivalently $u_r(.)$ can be represented as a monotonic and concave transformation t of $u_s(.)$. Therefore,

$$RASQ_r = \frac{u_r(\varsigma) - u_r(\sim j)}{u_r(j) - u_r(\varsigma)} = \frac{t\left(u_s(\varsigma)\right) - t\left(u_s(\sim j)\right)}{t\left(u_s(j)\right) - t\left(u_s(\varsigma)\right)}$$

We want to prove that

$$RASQ_r > RASQ_s.$$

Since $u_s(\varsigma)$ is between $u_s(j)$ and $u_s(\sim j)$, we can find in $[0,1]$ a number a such that $u_s(\varsigma) = a \cdot u_s(j) + (1-a) \cdot u_s(\sim j)$. Thus we can write

$$RASQ_s = \frac{[a \cdot u_s(j) + (1-a) \cdot u_s(\sim j)] - u_s(\sim j)}{u_s(j) - [a \cdot u_s(j) + (1-a) \cdot u_s(\sim j)]} = \frac{a}{1-a}$$

and

$$RASQ_r = \frac{t\left([a \cdot u_s(j) + (1-a) \cdot u_s(\sim j)]\right) - t\left(u_s(\sim j)\right)}{t\left(u_s(j)\right) - t\left([a \cdot u_s(j) + (1-a) \cdot u_s(\sim j)]\right)}$$

By the concavity of t, we know that $t\left[a \cdot u_s(j) + (1-a) \cdot u_s(\sim j)\right] > a \cdot t(u_s(j)) + (1-a) \cdot t(u_s(\sim j))$. Thus we can write

$$RASQ_r > \frac{a \cdot t(u_s(j)) + (1-a) \cdot t(u_s(\sim j)) - t\left(u_s(\sim j)\right)}{t\left(u_s(j)\right) - [a \cdot t\left(u_s(j)\right) + (1-a) \cdot t\left(u_s(\sim j)\right)]} = \frac{a}{1-a}$$

or

$$RASQ_r > RASQ_s$$

■

By Lemma 2, once we know the monetary consequences of voting, we can use $RASQ$ to measure the risk attitudes of the players. Lemma 3 shows that $RASQ$ and q_0 that satisfies the FOC move in the same direction when a player is confident. They go in the opposite directions if the player is non-confident. We use these results in Proposition 4 to show that risk averse players unambiguously prefer higher or equal majority thresholds.

Lemma 3 *q_0 in (7) is positively related to $RASQ_j$ if and only if $A > B$.*

Proof. We want to prove that

$$\frac{\partial\left[RASQ_j\right]}{\partial q_0} > 0 \tag{13}$$

Recall that rearranging (5) yields

$$\frac{f^j(q_0 - w_j)}{f^{\sim j}(q_0)} = RASQ_j \tag{14}$$

where

$$\frac{f^j(q - w_j)}{f^{\sim j}(q)} = e^{\frac{\left(q-\mu_{\sim j}\right)^2 - \left(q - w_j - \mu_j\right)^2}{2\sigma_j^2}}$$

then satisfying (13) implies that

$$\frac{\partial\left[\frac{f^j(q-w_j)}{f^{\sim j}(q)}\right]}{\partial q} > 0$$

which in turn is satisfied if and only if

$$w_j + \mu_j > \mu_{\sim j}$$

where $w_j + \mu_j = A$ and $\mu_{\sim j} = B$. ■

Recall that when $A > B$ then q_0 is a maximum. Lemma 3 shows that q_0 increases in $RASQ$ only when the player is confident. We can now see how risk aversion affects the preferred threshold.

Proposition 4 *All the other things being equal, player j's risk aversion is (weakly) positively related to the preferred threshold.*

Proof. Let us distinguish two sub-cases: a) player j is confident ($A > B$); b) player j is non-confident ($B > A$).
Let us consider the sub-case a) first. From Lemmas 2 and 3 we know that when player j is confident her q_0 increases in her risk aversion. Moreover, from Proposition 3 we know that if $RASQ_j$ is larger than one then she prefers a qualified majority. Therefore, after the increases in risk aversion we can have three situations: 1. $RASQ_j$ increases, but it is still lower than one; 2. $RASQ_j$ increases and becomes larger than one; 3. $RASQ_j$ is already larger than one, and it increases. In Case 1, player j continues to prefer simple majority. In Case 2, she stops preferring a simple majority and she switches in favor of a qualified majority. In Case 3, she prefers a

higher qualified majority. Thus, the preferred majority never decreases.
The proof of sub-case b) works in the opposite way. From Lemma 2 we know that if player j is non-confident then higher risk aversion negatively affects $RASQ_j$. From Lemma 1 we know that she can only prefer either the simple majority or unanimity, and Proposition 1 states that she prefers the simple majority only if $RASQ_j$ is lower than one. Therefore if risk aversion increases we can have three cases: 1. she continues to prefer the simple majority; 2. she continues to prefer unanimity; 3. she shifts from the simple majority to unanimity. Then, also in sub-case b), the preferred threshold cannot decrease in risk aversion. ■

A more risk averse player undergoes a higher risk of utility drop in the event that she falls into a minority. Thus she prefers a system were an undesirable majority formation can be blocked more easily.

2.3 Voting weight

We can now explore the impact of w_j on the optimal threshold. More voting power gives better control over the final decision so that the final outcome is less uncertain. All other things being equal, we expect that players use higher voting power to increase their chance to win, via a lower threshold. Proposition 5 presents this negative correlation between weight and the preferred threshold.

Proposition 5 *w_j is (weakly) negatively related to the preferred threshold.*

Proof. Let us consider three cases: a) $A > B$ before and after the increase in w_j; b) $A > B$ only after the increase in w_j; c) $A < B$ before and after the increase in w_j.
a.1) If $RASQ_j \leq 1$ then by Proposition 3, player j continues to prefer the simple majority.
a.2) If $RASQ_j > 1$ then by Proposition 3, player j prefers a lower qualified majority.
b.1) If $RASQ_j \leq 1$ then by Propositions 3 and 1, either player j continues to prefer the simple majority if, before the increase in w_j, disequality (12) was not satisfied, or she switches from unanimity to the simple majority if disequality (12) was satisfied before the increase in w_j.
b.2) If $RASQ_j > 1$ then by Proposition 1, player j switches from unanimity to a qualified majority.
c.1) If $RASQ_j \geq 1$ then by Proposition 1, player j continues to prefer unanimity.

c.2) If $RASQ_j < 1$ then by Proposition 1, either player j continues to prefer unanimity if, after the increase in w_j, disequality (12) is still satisfied, or she switches from unanimity to the simple majority if disequality (12) is no longer satisfied. ■

The intuition of this result is straightforward: a powerful player has more leverage on the voting outcome, therefore she is less subject to the expropriation risk. Thus she prefers to facilitate majority formation by lowering the threshold. The simple majority is more likely to be preferred by a powerful player.[8] Finally, observe that voting weight has the same impact on outcome probabilities as optimism. We can even say that power allows a player to afford a certain degree of pessimism about the other players' behavior.

[8]It is easy to verify that if w_j is high enough to give player j veto power, losing probability is zero. Thus she can only either win or stay in the status quo. In this extreme case, independently of both her gains and losses and her degree of optimism, she always prefers the lowest threshold: the simple majority.

3 Conclusion

In this paper we have described voting from the perspective of an individual who is uncertain about how other individuals will vote. The random outcome is a gain, if the majority makes decisions favorable to the individual, or a loss, if those decisions are harmful to her. Therefore, voting is a lottery. The probabilities of winning and losing depend on the voting rules, such as the majority threshold and the apportionment of voting weights. We compute the individual preferred majority threshold as a function of her risk attitudes, her priors about how other individuals will vote, and her voting power.

High thresholds reduce the chance of winning, but they also protect the individual from the risk of losing. We find that the optimal threshold is higher when an individual is more risk averse, less powerful and less optimistic about the likelihood that others will vote like her. De facto, raising the threshold is a form of protection against the higher risk of being tyrannized by an unfavorable majority.

Interestingly, in our setting, the simple majority and unanimity occur very frequently as the preferred thresholds. An individual who is confident about ending up in the majority prefers the simple majority when her gain is sufficiently higher than her loss. The simple majority also results when the individual is completely unaware of how the others will vote, when her gain equals her loss in absolute value, and when her voting power is negligible. One could imagine that this is situation occurs when writing a constitution that sets up the rules for the functioning of the legislature.

Unanimity is the preferred threshold when the individual is even slightly non-confident about the outcome and her gain is lesser than her loss. This possibly explains why it is so difficult to overcome the unanimity rule in international institutions.

We have not presented a way to determine which threshold is "socially" optimal. We have only looked at the individual perspective, without describing any mechanism to coordinate individual preferences regarding the voting rules during the "constitutional phase". Nevertheless our findings can contribute to a better understanding of important normative questions, such as: Should the voting rules reflect the risk attitudes of citizens where crucial policy issues are concerned? How should voting weights be apportioned across voters, as a function of the nature of the decisions? How many super-majority thresholds should a statute include, and for which issues? What degree of conflict on decisional rules should we expect within a constituency whose members have diversified preferences? How should voting rules take into account the presence of politically weak minorities that are greatly harmed by the majority decisions?

Our model is unique by describing voting as a lottery, in which the outcome probabilities depend on the voting rules. Our model runs without any restrictions on the voters' utility functions. Despite the fact that our setting is rather general, there are some caveats. We consider a simplified majority voting context. We assume that individuals have the same priors about how any other player will vote. Since we use the central limit theorem, our setting requires that the number of voters be sufficiently high (say more than thirty). Despite all this, our results are reasonable, thus we think that they can also provide a good description of more complex voting schemes or situations with fewer voters. This could be verified by future research.

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