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Abstract

In this paper we will prove that if a submodular aggregative game satisfies the generalized Fisher-Hahn condition, then the unique Nash equilibrium in the game is globally stable under two alternative continuous adjustment processes with non-negativity constraints. The general result is applied to Cournot oligopoly without product differentiation and to the pure public good model. The first application gives the complete proof to the global stability result of Hahn (1962) and Okuguchi (1964), whose proofs contain a defect pointed out by Al-Nowaihi and Levine (1985), taking into account the non-negativity of firms' outputs during the adjustment periods. We next analyze the global stability of Nash equilibrium in a model of a pure public good, another example of aggregative games.

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1. Introduction

Many economists have analyzed Cournot oligopoly from static and dynamic points of view. The first dynamic analysis is due to Theocharis (1960), which has led to the most influential paper of Hahn (1962) on the global stability of the Cournot equilibrium for a continuous output adjustment system (hereafter, Hahn's best reply dynamics) in which a firm's rate of change of actual output is proportional to the difference between its profit-maximizing and actual outputs. Okuguchi (1964) has extended his result using a more general adjustment system.[1] However, Al-Nowaihi and Levine (1985) have presented a counter-example to a certain assertion used in the proof of the Hahn-Okuguchi result. According to Al-Nowaihi and Levine the equilibrium is globally stable if the number of firms is less than or equal to 5.

It is well known that the Cournot oligopoly without product differentiation (hereafter, Cournot game) is a submodular aggregative game under usual or traditional assumptions.[2] All works mentioned above have studied the global stability of the Cournot-Nash equilibrium in the submodular Cournot game. However, the Cournot game can be supermodular.[3] Recently some economists, notably Vives (1990) and Amir (1996), have analyzed the supermodular Cournot game. Vives (1999, Theorem 2.11) proves that the unique Nash equilibrium in the supermodular Cournot game is globally stable under Hahn's best reply dynamics. Okuguchi and Yamazaki (2008) study the global stability in a

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1 See Okuguchi (1976).

2 In other words, under usual assumptions, any firm's output is strategic substitute to any other firm's output in the sense of Bulow et al. (1985). If the inverse demand function is linear and each firm's cost function is convex, the Cournot game is submodular.

3 In other words, the Cournot game can have strategic complementarities in the terminology of Bulow et al. (1985).

general Cournot game, which can be neither supermodular nor submodular, to prove that the unique Cournot-Nash equilibrium is globally stable if the general Cournot game satisfies a set of reasonable assumptions and the number of firms is less than or equal to 3.

This paper revisits the global stability of the unique Nash equilibrium in the traditional submodular Cournot game. To do so, we first examine the global stability of a unique Nash equilibrium in a submodular aggregative game. The previous works on the global stability of the Cournot-Nash equilibrium do not explicitly describe the dynamics at the point where the trajectory moves out of the non-negative domain. However, the adjustment processes used in the previous works do not necessarily ensure the non-negativity of all strategic variables over time independently of the initial condition.[4] Hence, we will formulate the dynamics at the boundary and proves that if a submodular aggregative game satisfies the generalized Fisher-Hahn condition, a unique Nash equilibrium in the game is globally stable under two alternative continuous adjustment processes with the non-negativity constraints. We then apply this general stability result to the traditional submodular Cournot game to complete the proof of the Hahn-Okuguchi result. We will also analyze the global stability of Nash equilibrium in a public good model, another example of submodular aggregative games.

2. Submodular Aggregative Games

Let $u^i(\mathbf{x})$ be player i's payoff function in a general n-person game, where $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in R_+^n \equiv \{\mathbf{x} \in R^n : x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0\}$ and x_i is player i's choice of

4 If the Nash equilibrium is not interior, the dynamics at the boundary do matter even for local stability.

strategy. The game is (smooth) *submodular* (respectively, *supermodular*) if $u^i(\mathbf{x})$ is twice continuously differentiable and $\partial^2 u^i / \partial x_j \partial x_k \le 0$ (respectively, $\partial^2 u^i / \partial x_j \partial x_k \ge 0$) for all $j \ne k$. The game is *aggregative* if player *i*'s payoff function $u^i(\mathbf{x})$ can be written as $U_i(x_i, X)$, where $X = \sum_{j \in N} x_j$.[5]

Define

$$h^i(x_i, X) \equiv \frac{\partial}{\partial x_i} U_i(x_i, X) + \frac{\partial}{\partial X} U_i(x_i, X).$$

Let us assume that the partial derivatives of h^i satisfy the following assumptions.

Assumption 1: $h_2^i \equiv \frac{\partial}{\partial X} h^i(x_i, X) \le 0$ *for all feasible strategies and for all i.*

Assumption 2: $h_1^i \equiv \frac{\partial}{\partial x_i} h^i(x_i, X) < 0$ *for all feasible strategies and for all i.*

By Assumption 1, the aggregative game becomes submodular. Since Fisher (1961) and Hahn (1962) adopt Assumption 2 in the context of Cournot game, call it *generalized Fisher-Hahn condition*. Folmer and von Mouche (2004, Proposition 2) prove that the submodular aggregative game possesses a unique Nash equilibrium under Assumptions 1 and 2.[6] For $X_{-i} = \sum_{j \ne i} x_j \ge 0$, construct a new function from $h^i(x_i, X_{-i})$.

$$\tilde{h}^i(x_i, X) \equiv \begin{cases} h^i(x_i, X) & \textit{for} \quad x_i > 0 \\ \max\{h^i(0, X_{-i}), 0\} & \textit{for} \quad x_i = 0. \end{cases} \tag{1}$$

[5] See Okuguchi (1993) or Corchón (2001) for examples of aggregative games.
[6] As for the Cournot game, Gaudet and Salant (1991) prove that there exists a unique Cournot-Nash equilibrium under Assumptions 1 and 2.

Note that, at the unique Nash equilibrium $\mathbf{x}^* = \left(x_1^*, x_2^*, \cdots, x_n^*\right) \in R_+^n$, $\tilde{h}^i\left(x_i^*, X^*\right) = 0$ for all $i \in N$, where $X^* = \sum_{j \in N} x_j^*$.[7] Equivalently, at the unique Nash equilibrium $\mathbf{x}^*$, $x_i^* = R_i\left(X_{-i}^*\right)$, where $X_{-i}^* = \sum_{j \neq i} x_j^*$ and $R_i\left(X_{-i}\right)$ is player *i*'s best reply function, that is, $R_i\left(X_{-i}\right)$ is a unique non-negative solution to the problem of maximization problem of $U_i\left(x_i, X\right)$ with respect to x_i, given X_{-i}.[8]

There can be many ways to model how x_i is adjusted as a continuous function of time *t*. The following two dynamics are often assumed in the literature.

Assumption 3: x_i *as a non-negative function of continuous time t is adjusted according to*

$$\frac{d}{dt} x_i = \alpha_i \tilde{h}^i\left(x_i, X\right), \tag{2}$$

where a positive number α_i *denotes speed of adjustment.*

Assumption 4: x_i *as a non-negative function of continuous time t is adjusted according to*

$$\frac{d}{dt} x_i = \beta_i\left(R_i\left(X_{-i}\right) - x_i\right), \tag{3}$$

where a positive number β_i *denotes speed of adjustment.*

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[7] Gaudet and Salant (1991) describe the Cournot-Nash equilibrium in a similar but slightly different way.

[8] Assumptions 1 and 2 ensure that $R_i\left(X_{-i}\right)$ is a unique positive solution to the first order condition $h^i\left(x_i, x_i + X_{-i}\right) = 0$ for any X_{-i} such that $h^i\left(0, X_{-i}\right) > 0$ and that $R_i\left(X_{-i}\right) = 0$ for any X_{-i} such that $h^i\left(0, X_{-i}\right) \leq 0$.

First consider the gradient dynamics in Assumption 3, which is assumed by Dixit (1986), Furth (1986), Dastidar (2000) among others. Define

$$I \equiv \left\{ i \in N : x_i > x_i^* \right\}, \quad J \equiv \left\{ j \in N : x_j < x_j^* \right\} \text{ and } K \equiv \left\{ k \in N : x_k = x_k^* \right\}. \tag{4}$$

For the sake of notational simplicity, define

$$X^A \equiv \sum\nolimits_{i \in A} x_i \text{ and } X^{A*} \equiv \sum\nolimits_{i \in A} x_i^* , \tag{5}$$

where A is a subset of *N*. Since an increase in *t* may change the index sets *I* and *J*, we occasionally write $I(t)$ and $J(t)$ instead of *I* and *J*, respectively. By the same reason, we may write X^A as $X^A(t)$. Now we are ready to prove our two main theorems.[9]

Theorem 1: *Under Assumptions 1, 2 and 3, the unique Nash equilibrium in the aggregative game is globally stable.*

Proof: For $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in R_+^n$, define a Lyapunov function as follows.

$$V(\mathbf{x}) \equiv \begin{cases} \dfrac{1}{2}\left(X^I - X^{I*}\right)^2 & if \quad X \geq X^*, \\ \dfrac{1}{2}\left(X^J - X^{J*}\right)^2 & if \quad X < X^*, \end{cases} \tag{6}$$

It is clear that $V(\mathbf{x})$ in (6) is zero for $\mathbf{x} = \mathbf{x}^*$ and positive for $\mathbf{x} \neq \mathbf{x}^*$.[10] It is also clear that $V(\mathbf{x})$ is continuous in $\mathbf{x}$ for $X < X^*$ and $X > X^*$. If $X = X^*$,

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9 Corchón (2001) has proved the global stability of the Nash equilibrium adopting Hahn's method of proof as well as taking into account the property of aggregative game. His proof, however, is not free from the same defect as in Hahn (1962) and Okuguchi (1964).

10 If $\mathbf{x} \neq \mathbf{x}^*$ and $X > X^*$, the set *I* is not empty. If $\mathbf{x} \neq \mathbf{x}^*$ and $X = X^*$, the sets *I* and *J* are not empty. Hence, if $\mathbf{x} \neq \mathbf{x}^*$ and $X \geq X^*$, $V(\mathbf{x})$ in (6) is positive. Similarly, if $\mathbf{x} \neq \mathbf{x}^*$ and $X < X^*$, $V(\mathbf{x})$ in (6) is positive.

$\left(X^I - X^{I*}\right) = -\left(X^J - X^{J*}\right)$. Since $V(\mathbf{x})$ is continuous in $\mathbf{x}$ for $X < X^*$ and $X > X^*$, the fact $\left(X^I - X^{I*}\right) = -\left(X^J - X^{J*}\right)$ at $X = X^*$ implies that $V(\mathbf{x})$ is continuous in $\mathbf{x}$. Since x_i is non-negative for all *i*, $\|\mathbf{x}\| \equiv \sum_{i=1}^{n} x_i^2$ is infinite if and only if x_i is infinite for some *i*. Hence, $\|\mathbf{x}\|$ is infinite if and only if $\left(X^I - X^{I*}\right)$ and/or $\left(X^J - X^{J*}\right)$ is infinite. Therefore, $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$.

$V(\mathbf{x})$ may not be differentiable with respect to *t*.[11] Following Brock and Malliaris (1989, p.95) or Hale (1969, p.293), define

$$\dot{V}(\mathbf{x}) \equiv \limsup_{\Delta t \to 0} \frac{1}{\Delta t}\left[V\left(\mathbf{x}(t+\Delta t)\right) - V\left(\mathbf{x}(t)\right)\right].$$

If $X(t) > X^*$, as we prove in the Appendix, $\dot{V}(\mathbf{x})$ satisfies

$$\dot{V}(\mathbf{x}) \le \left(X^I(t) - X^{I*}(t)\right) \sum_{i \in I(t)} \alpha_i \tilde{h}^i . \tag{7}$$

Since $x_i > x_i^*$ for all $i \in I$, $\left(X^I(t) - X^{I*}(t)\right) > 0$. Since $x_i > x_i^*$ for all $i \in I$ and $X > X^*$, Assumptions 1 and 2 ensure $\tilde{h}^i(x_i, X) < \tilde{h}^i\left(x_i^*, X^*\right) = 0$ for all $i \in I$.[12] Hence,

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[11] If $X(t) \ne X^*$ and the set $K(t)$ in (4) is empty, $V(\mathbf{x})$ is differentiable with respect to *t* and its time derivative can be calculated as $\dot{V}(\mathbf{x}) = \left(X^I(t) - X^{I*}(t)\right) \sum_{i \in I(t)} \alpha_i \tilde{h}^i$ for $X(t) > X^*$ and $\dot{V}(\mathbf{x}) = \left(X^J(t) - X^{J*}(t)\right) \sum_{j \in J(t)} \alpha_j \tilde{h}^j$ for $X(t) < X^*$.

[12] Since $\tilde{h}^i\left(x_i^*, X^*\right) = h^i\left(x_i^*, X^*\right) = 0$ for $x_i^* > 0$, Assumptions 1 and 2 directly imply that $\tilde{h}^i(x_i, X) < \tilde{h}^i\left(x_i^*, X^*\right) = 0$ for $x_i^* > 0$. If $x_i^* = 0$, $\tilde{h}^i\left(x_i^*, X^*\right) = 0$ and $h^i\left(x_i^*, X^*\right) \le 0$.

if $X(t) > X^*$,

$$\dot{V}(\mathbf{x}) \le \left(X^I(t) - X^{I*}(t)\right) \sum_{i \in I(t)} \alpha_i \tilde{h}^i < 0 . \qquad (7')$$

If $X(t) < X^*$, the arguments similar to the ones in the Appendix lead to

$$\dot{V}(\mathbf{x}) \le \left(X^J(t) - X^{J*}(t)\right) \sum_{j \in J(t)} \alpha_j \tilde{h}^j . \qquad (8)$$

Since $x_j < x_j^*$ for all $j \in J$, $\left(X^J(t) - X^{J*}(t)\right) < 0$. Since $x_j < x_j^*$ for all $j \in J$ and $X < X^*$, Assumptions 1 and 2 ensure $\tilde{h}^j\left(x_j, X\right) > \tilde{h}^j\left(x_j^*, X^*\right) = 0$ for all $j \in J$.[13] Hence, if $X(t) < X^*$,

$$\dot{V}(\mathbf{x}) \le \left(X^J(t) - X^{J*}(t)\right) \sum_{j \in J(t)} \alpha_j \tilde{h}^j < 0 . \qquad (8')$$

Consider the case of $X(t) = X^*$. If there exists $\varepsilon > 0$ such that $X(t') = X^*$ for any $t' \in (t - \varepsilon, t + \varepsilon)$,

$$\dot{V}(\mathbf{x}) \le \left(X^I(t) - X^{I*}(t)\right) \sum_{i \in I(t)} \alpha_i \tilde{h}^i < 0 .$$

If there is no $\varepsilon > 0$ such that $X(t') = X^*$ for any $t' \in (t - \varepsilon, t + \varepsilon)$, then, without loss of

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Assumptions 1 and 2 imply that $h^i\left(x_i, X\right) < h^i\left(x_i^*, X^*\right) \le 0$ for $x_i^* = 0$. If $x_i^* = 0$, the fact $i \in I$ implies $x_i > 0$. Hence, by (1), $\tilde{h}^i\left(x_i, X\right) = h^i\left(x_i, X\right) < 0$ for $x_i > x_i^* = 0$.

[13] The fact $j \in J$ implies that x_j^* is positive for all $j \in J$, since $x_j^* > x_j \ge 0$. Assumptions 1 and 2 imply that $h^j\left(x_j, X\right) > h^j\left(x_j^*, X^*\right) = 0$. Hence, by (1), $\tilde{h}^j\left(x_j, X\right) = h^j\left(x_j, X\right) > 0$ for $x_j < x_j^*$, even if $x_j = 0$.

generality, $X(t+\Delta t) > X^*$ and $X(t-\Delta t) < X^*$ for $\Delta t > 0$ small enough. If we define $\dot{V}_+(\mathbf{x}) \equiv \lim_{\Delta t \to 0} \dot{V}(x(t+\Delta t))$ and $\dot{V}_-(\mathbf{x}) \equiv \lim_{\Delta t \to 0} \dot{V}(x(t-\Delta t))$, $\dot{V}(\mathbf{x}) = \max\left\{\dot{V}_+(\mathbf{x}), \dot{V}_-(\mathbf{x})\right\}$.

By (7') and (8'), the following two inequalities must hold.[14]

$$\dot{V}_+(\mathbf{x}) \le \lim_{\Delta t \to 0}\left(X^I(t+\Delta t) - X^{I*}(t+\Delta t)\right) \sum_{i \in I(t+\Delta t)} \alpha_i \tilde{h}^i , \tag{9}$$

$$\dot{V}_-(\mathbf{x}) \le \lim_{\Delta t \to 0}\left(X^J(t-\Delta t) - X^{J*}(t-\Delta t)\right) \sum_{j \in J(t-\Delta t)} \alpha_j \tilde{h}^j . \tag{10}$$

Since the sets *I* and *J* are not empty for any $\mathbf{x} \neq \mathbf{x}^*$ such that $X = X^*$, as in (7'), the right hand side of (9) is negative. Similarly, the right hand side of (10) is negative. Hence, $\dot{V}(\mathbf{x}) = \max\left\{\dot{V}_+(\mathbf{x}), \dot{V}_-(\mathbf{x})\right\}$ is negative for any $\mathbf{x} \neq \mathbf{x}^*$ such that $X = X^*$. In any case, $\dot{V}(\mathbf{x})$ is negative for any $\mathbf{x} \neq \mathbf{x}^*$. ■

Now consider Hahn's best reply dynamics in Assumption 4, which is assumed by Hahn (1962), Okuguchi (1964), Seade (1980) among others.

Theorem 2: *Under Assumptions 1, 2 and 4, the unique Nash equilibrium in the aggregative game is globally stable.*

Proof: Consider the Lyapunov function in (6). Since for all $i \in N$, by Assumptions 1 and 2, h^i is strictly concave in x_i for $X_{-i}(t) = \sum_{j \neq i} x_j(t)$, the definition of $\tilde{h}^i$ in (1)

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[14] (A2) in Appendix is proved for $X(t) > X^*$. If $X(t) = X^*$, (A2) holds with equality. Hence, (9) must hold with equality. Similarly, (10) holds with equality.

ensures that $R_j\left(X_{-j}(t)\right)-x_j(t)=0$ if and only if $\tilde{h}^j\left(x_j(t),X(t)\right)=0$.[15] Similarly, $R_i\left(X_{-i}(t)\right)-x_i(t)>0$ (<0) if and only if $\tilde{h}^i\left(x_i(t),X(t)\right)=h^i\left(x_i(t),X(t)\right)>0$ (<0, respectively). Hence, if $X>X^*$,

$$\dot{V}(\mathbf{x})\le\left(X^I(t)-X^{I*}(t)\right)\sum_{i\in I(t)}\beta_i\left(R_i\left(X_{-i}(t)\right)-x_i(t)\right)<0.$$

Arguments are similar for other two cases. Hence, even under Assumption 4, $\dot{V}(\mathbf{x})$ is negative for any $\mathbf{x}\neq\mathbf{x}^*$. ■

3. Applications

3.1 Cournot Game

In Cournot oligopoly without product differentiation,

$$U_i\left(x_i,X\right)=x_iP\left(X\right)-C_i\left(x_i\right), \tag{11}$$

$$h^i\left(x_i,X\right)=P\left(X\right)+x_iP'\left(X\right)-C_i'\left(x_i\right), \tag{12}$$

where x_i is firm *i*'s output, $P\left(X\right)$ with $P'\left(X\right)<0$ is the inverse demand function and C_i is firm *i*'s cost function. If

$$h_1^i\left(x_i,X\right)=\frac{\partial}{\partial x_i}h^i\left(x_i,X\right)=P'\left(X\right)-C_i''\left(x_i\right)<0, \tag{13}$$

$$h_2^i\left(x_i,X\right)=\frac{\partial}{\partial X}h^i\left(x_i,X\right)=P'\left(X\right)+x_iP''\left(X\right)\le 0, \tag{14}$$

We can prove on the basis of Theorems 1 and 2 that the unique Cournot-Nash equilibrium

[15] See Footnote 7.

is globally stable under the gradient dynamics formulated as Assumption 3 and Hahn's best reply dynamics stated as Assumption 4.[16]

Hahn (1962) and Okuguchi (1964) have proved that if assumptions (13) and (14) are satisfied, the Cournot-Nash equilibrium is globally stable under Hahn's best reply dynamics. However, Al-Nowaihi and Levine (1985) have presented a counter-example to a certain assertion used in the proof of the Hahn-Okuguchi result. Al-Nowaihi and Levine (1985, Theorem 6) prove that if the number of firms is less than or equal to 5 in the Cournot game with assumptions (13) and (14), the Cournot-Nash equilibrium is globally stable under Hahn's best reply dynamics. We have thus extended Al-Nowaihi and Levine (1985, Theorem 6) and Hahn-Okuguchi result for two alternative continuous adjustment processes, taking into account the non-negativity of firms' outputs during the adjustment periods .[17]

3.2 Pure Public Good Model

In this section we will analyze the global stability of the unique Nash equilibrium of the pure public good model to show how powerful our Theorems 1 and 2 are.The pure public good model discussed here is another example of aggregative games with strategic

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16 For example, if the inverse demand function $P(X)$ is linear and each firm *i*'s cost function C_i is convex, the Fisher-Hahn condition (13) and submodularity (14) are satisfied so that the unique Cournot-Nash equilibrium is globally stable under two continuous adjustment processes.

17 Many researchers have also studied local stability of the Cournot-Nash equilibrium. Al-Nowaihi and Levine (1985, Theorem 3) prove that if assumptions (13) and (14) are satisfied, the unique Cournot-Nash equilibrium is locally stable under Hahn's best reply dynamics. Dastidar (2000, Proposition 2) makes use of Kolstad and Mathiesen's (1987) necessary and sufficient condition for the existence of the unique Cournot-Nash equilibrium to prove that if assumptions (13) and (14) are satisfied, the unique Cournot-Nash equilibrium is locally stable under the gradient dynamics in Assumption 3.

substitutes.[18] In the pure public good model, agent i maximizes $u^i\left(y_i, x_i + X_{-i}\right)$ with respect to y_i, agent i's quantity consumed of a private good, and x_i, the contribution made by agent i to the pure public good, subject to agent i's budget constraint $y_i + px_i = m_i$, where p is the unit price of the public good and m_i is agent i's income. Agent i's utility function $u^i\left(y_i, X\right)$ is assumed to be twice continuously differentiable, strictly increasing and strictly quasi-concave in its two arguments,that is,

$$u_1^i\left(y_i, X\right) \equiv \partial u^i / \partial y_i > 0,\ u_2^i\left(y_i, X\right) \equiv \partial u^i / \partial X > 0, \text{ and } H^i \equiv \begin{vmatrix} u_{11}^i & u_{12}^i & u_1^i \\ u_{21}^i & u_{22}^i & u_2^i \\ u_1^i & u_2^i & 0 \end{vmatrix} > 0$$

for any $\left(y_i, X\right) \in R_+^2$, where $u_{11}^i\left(y_i, X\right) \equiv \partial u_1^i / \partial y_i$, $u_{12}^i\left(y_i, X\right) \equiv \partial u_1^i / \partial X$, and $u_{22}^i\left(y_i, X\right) \equiv \partial^2 u^i / \partial X^2$.[19] Let the the exogenously given price p to be one. Partial derivatives of agent i's demand functions of the private good and of public good, $y_i\left(m_i, X_{-i}\right)$ and $x_i\left(m_i, X_{-i}\right)$, with respect to agent i's income can be calculated as

$$\frac{\partial y_i}{\partial m_i} = \frac{1}{D^i}\left(u_{12}^i\left(y_i, X\right) - u_{22}^i\left(y_i, X\right)\right) \text{ and } \frac{\partial x_i}{\partial m_i} = \frac{1}{D^i}\left(u_{12}^i\left(y_i, X\right) - u_{11}^i\left(y_i, X\right)\right)$$

where

$$D^i \equiv \begin{vmatrix} u_{11}^i & u_{12}^i & -1 \\ u_{21}^i & u_{22}^i & -1 \\ -1 & -1 & 0 \end{vmatrix} = \left(u_{12}^i - u_{11}^i\right) + \left(u_{12}^i - u_{22}^i\right).$$

[18] Frasca (1980) and Okuguchi (1984) examine stability under a discrete adjustment process in McGuire's (1974) pure public good model. Cornes and Sandler (1996, Section 6.2) graphically argue that if there are multiple Nash equlibria, some of them are locally unstable.

[19] For quasi-concavity, see Arrow and Enthoven (1961), Crouzeix and Ferland (1982) or De La Fuente (2000, especially Problem 3.12).

Note that $H^i > 0$ together with the first-order conditions implies $D^i > 0$. Bergstrom et al. (1992) prove that if $0 < \partial x_i / \partial m_i < 1$ or equivalently, if both the public and private goods are normal, there exists a unique Nash equilibrium in the pure public good model. Hence, if

$$u_{12}^i(y_i, X) - u_{11}^i(y_i, X) > 0 \text{ and } u_{12}^i(y_i, X) - u_{22}^i(y_i, X) > 0, \tag{15}$$

there exists a unique Nash equilibrium in the pure public good model.

Agent *i*'s original maximization problem is equivalent to the problem of maximizing

$$U^i(x_i, X) \equiv u^i(m_i - x_i, X) \tag{16}$$

with respect to x_i. Hence, in the pure public good model,

$$h^i(x_i, X) = -u_1^i(m_i - x_i, X) + u_2^i(m_i - x_i, X). \tag{17}$$

If

$$h_1^i(x_i, X) = \frac{\partial}{\partial x_i} h^i(x_i, X) = u_{11}^i(m_i - x_i, X) - u_{21}^i(m_i - x_i, X) < 0, \tag{18}$$

$$h_2^i(x_i, X) = \frac{\partial}{\partial X} h^i(x_i, X) = -u_{12}^i(m_i - x_i, X) + u_{22}^i(m_i - x_i, X) \leq 0, \tag{19}$$

Theorem 1 and 2 prove that the unique Nash equilibrium of the pure public good model is globally stable under the gradient dynamics formulated as Assumption 3 and Hahn's best reply dynamics given by Assumption 4. Hence, if the pure public good model satisfies normality condition (15) of Bergstrom et al. (1992) for the unique existence of Nash equilibrium, then the unique interior Nash equilibrium is globally stable under two continuous adjustment processes.

4. Conclusion

In this paper we have proved that if a submodular aggregative game satisfies the generalized Fisher-Hahn condition, then the unique Nash equilibrium in the game is globally stable under two alternative continuous adjustment processes with non-negativity constraints. The general result has been applied to Cournot oligopoly without product differentiation and the pure public good model. The first application, which is free from the defect found by Al-Nowaihi and Levine (1985), gives the complete proof of the Hahn-Okuguchi result, taking into account the non-negativity of firms' outputs during the adjustment periods. It would be interesting to apply our general result in Theorems 1 and 2 to other submodular aggregative games.

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Appendix

This appendix proves the inequality in (7), that is,

$$\dot{V}(\mathbf{x}) \le \left(X^{I}(t) - X^{I*}(t)\right)\sum_{i\in I(t)} \alpha_i \tilde{h}^i .$$

Let $K_-(t)$ and $K_+(t)$ be subsets of $K(t)$ in (4) such that $I(t-\Delta t) = I(t)\cup K_-(t)$ and $I(t+\Delta t) = I(t)\cup K_+(t)$ for any $\Delta t > 0$ small enough.

$$\frac{1}{\Delta t}\left[V\left(\mathbf{x}(t+\Delta t)\right) - V\left(\mathbf{x}(t)\right)\right]$$

$$= \frac{1}{\Delta t}\left[\frac{1}{2}\left\{\left(X^{I(t)}(t+\Delta t) - X^{I(t)*}\right) + \left(X^{K_+(t)}(t+\Delta t) - X^{K_+(t)*}\right)\right\}^2 - \frac{1}{2}\left(X^{I(t)}(t) - X^{I(t)*}\right)^2\right]$$

where $X^{A(t)}(t') \equiv \sum_{i\in A(t)} x_i(t')$ and $X^{A(t)*} \equiv \sum_{i\in A(t)} x_i^*$. A simple calculation shows

$$\frac{1}{\Delta t}\left[V\left(\mathbf{x}(t+\Delta t)\right) - V\left(\mathbf{x}(t)\right)\right]$$

$$= \frac{1}{\Delta t}\left[\frac{1}{2}\left(X^{I(t)}(t+\Delta t) - X^{I(t)*}\right)^2 - \frac{1}{2}\left(X^{I(t)}(t) - X^{I(t)*}\right)^2\right]$$

$$+ \frac{1}{\Delta t}\left(X^{K_+(t)}(t+\Delta t) - X^{K_+(t)*}\right)\left(X^{I(t)}(t+\Delta t) - X^{I(t)*}\right) + \frac{1}{\Delta t}\frac{1}{2}\left(X^{K_+(t)}(t+\Delta t) - X^{K_+(t)*}\right)^2 .$$

It is clear that

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\frac{1}{2}\left(X^{I(t)}(t+\Delta t) - X^{I(t)*}\right)^2 - \frac{1}{2}\left(X^{I(t)}(t) - X^{I(t)*}\right)^2\right] = \left(X^{I}(t) - X^{I*}(t)\right)\sum_{i\in I(t)} \alpha_i \tilde{h}^i .$$

By L'Hospital's rule,

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\frac{1}{2}\left(X^{K_+(t)}(t+\Delta t) - X^{K_+(t)*}\right)^2$$

$$= \lim_{\Delta t \to 0} \left(X^{K_+(t)}(t+\Delta t) - X^{K_+(t)*} \right) \sum_{k \in K_+(t)} \alpha_k \tilde{h}^k \left(x_k(t+\Delta t), X(t+\Delta t) \right) = 0,$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(X^{K_+(t)}(t+\Delta t) - X^{K_+(t)*} \right) \left(X^{I(t)}(t+\Delta t) - X^{I(t)*} \right)$$

$$= \lim_{\Delta t \to 0} \left\{ \left(\sum_{k \in K_+(t)} \alpha_k \tilde{h}^k \right) \left(X^{I(t)}(t+\Delta t) - X^{I(t)*} \right) + \left(X^{K_+(t)}(t+\Delta t) - X^{K_+(t)*} \right) \left(\sum_{i \in I(t)} \alpha_i \tilde{h}^i \right) \right\}$$

$$= \left(\sum_{k \in K_+(t)} \alpha_k \tilde{h}^k \left(x_k(t), X(t) \right) \right) \left(X^{I(t)}(t) - X^{I(t)*} \right).$$

Hence,

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[V\left(\mathbf{x}(t+\Delta t) \right) - V\left(\mathbf{x}(t) \right) \right] = \left(X^I(t) - X^{I*}(t) \right) \sum_{i \in I(t) \cup K_+(t)} \alpha_i \tilde{h}^i . \qquad \text{(A1)}$$

Since $x_k(t) = x_k^*$ for any $k \in K_+(t)$ and $X(t) > X^*$, Assumption 1 implies $\tilde{h}^k\left(x_k(t), X(t) \right) \le 0$ for any $k \in K_+(t)$ so that

$$\left(X^I(t) - X^{I*}(t) \right) \sum_{i \in I(t) \cup K_+(t)} \alpha_i \tilde{h}^i \le \left(X^I(t) - X^{I*}(t) \right) \sum_{i \in I(t)} \alpha_i \tilde{h}^i . \qquad \text{(A2)}$$

Similarly, we can show

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[V\left(\mathbf{x}(t) \right) - V\left(\mathbf{x}(t-\Delta t) \right) \right] = \left(X^I(t) - X^{I*}(t) \right) \sum_{i \in I(t) \cup K_-(t)} \alpha_i \tilde{h}^i , \qquad \text{(A3)}$$

$$\left(X^I(t) - X^{I*}(t) \right) \sum_{i \in I(t) \cup K_-(t)} \alpha_i \tilde{h}^i \le \left(X^I(t) - X^{I*}(t) \right) \sum_{i \in I(t)} \alpha_i \tilde{h}^i . \qquad \text{(A4)}$$

Hence, by (A1)-(A4), the inequality in (7) holds.