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1 Introduction

We propose a revised version of the perfect Bayesian equilibrium in general multi-period games with observed actions. Fudenberg and Tirole (1991) formulated the perfect Bayesian equilibrium in the setting of finite games that allow only a finite number of types and strategies. In finite games, this perfect Bayesian equilibrium satisfies criteria of rational solution concepts such as weak consistency and the subgame perfect Nash equilibrium condition. However, it might not satisfy these criteria in general games that allow a continuum of types and strategies. To solve this problem with the perfect Bayesian equilibrium, we revise its definition by replacing Bayes' rule with a regular conditional probability. We refer to this revised version of the perfect Bayesian equilibrium as the perfect regular equilibrium. We show that it satisfies these criteria of rational solution concepts in general multi-period games with observed actions. In addition, this perfect regular equilibrium is equivalent to a simple version of the perfect Bayesian equilibrium in finite games. Therefore, we conclude that the perfect regular equilibrium extends the perfect Bayesian equilibrium into general games as a simple version of it.

In game theory, most of the solution concepts were developed as refinements of the Nash equilibrium introduced by Nash (1951). The Nash equilibrium, as the most popular solution concept, embodies the behavior of rational players. So, it consists of a set of strategies for each player such that each strategy is the best response to the other strategies. This Nash equilibrium became known as a compelling condition for rational strategies, and thus it became a necessary condition for rational solution concepts. However, the Nash equilibrium

was defined in strategic form games in which all players chose their strategies once and simultaneously. Hence, this solution concept might not properly predict players' behavior in multi-period games where players choose their actions in each period after observing actions taken before. In multi-period games, players could also have different incentives in different periods. Since the Nash equilibrium requires all players to decide what actions they should take once and simultaneously, it might not reflect these changes in incentives in multi-period games. As a result, a Nash equilibrium could include incredible threats.

The subgame perfect Nash equilibrium by Selten (1975) improved the Nash equilibrium. The basic idea behind this solution concept was to break a whole game into subgames and to find Nash equilibria in every subgame. When we analyze each of the subgames separately, we are able to consider players' incentives within those subgames. Thus, if situations in different periods lead to the formation of different subgames, then the subgame perfect Nash equilibrium could reflect different incentives in different periods in multi-period games. As a result, it could exclude incredible threats¹. Here, subgames can be regarded as complete units in the analysis of games in that we can find Nash equilibria, which reflect players' rational behavior, within subgames without referring to any information outside those subgames. In games with incomplete information, however, these complete units of analysis are too large to catch each of the players' incentives separately. So, the subgame perfect Nash equilibrium might fail to reflect players' incentives in different periods.

¹ To find a subgame perfect Nash equilibrium in practice, it is convenient to analyze subgames from back to front. This is because, by analyzing backward, we can naturally consider players' future incentives in any period. In this sense, we may think the subgame perfect Nash equilibrium is a combination of the Nash equilibrium and backward induction.

Perfect Bayesian equilibrium and sequential equilibrium introduced by Kreps and Wilson (1982) improved the subgame perfect Nash equilibrium. These solution concepts break a whole game into information sets and search strategies that satisfy sequential rationality at each information set. The sequential rationality is a condition for the strategies of rational players and requires that each strategy be the best response to the other strategies at each of the information sets. This sequential rationality, therefore, inherits the spirit of the Nash equilibrium condition. As units of analysis in multi-period games, information sets are small enough to catch each of the players' incentives separately. Consequently, these solution concepts could reflect different incentives in different periods in multi-period games, and thus they could exclude incredible threats. Particularly in finite games, these solution concepts can exclude all of the incredible threats. In general games that allow a continuum of types and strategies, however, these solution concepts might cause more serious problems than including incredible threats because of their new approaches through information sets.

Information sets can be regarded as the smallest units of analysis. In games, players cannot distinguish decision points in a common information set. So, whatever action they choose, the same action must be applied to all decision points in a common information set. That is, players can choose only one action at each of their information sets. Hence, information sets would be the smallest units used to analyze players' rational behavior.

These smallest units of analysis, however, might be smaller than complete units of analysis. For this reason, we might need more information to find rational strategies at each information set. In games, sufficient information to find rational strategies at each of the

information sets encompasses players' beliefs regarding probability distributions over information sets. Accordingly, the perfect Bayesian equilibrium and the sequential equilibrium require the players' beliefs to be part of these solution concepts themselves. Note that these solution concepts embody players' rationalities in games. Therefore, the perfect Bayesian equilibrium and the sequential equilibrium propose conditions for rational beliefs as they propose sequential rationality, which is the condition for rational strategies.

In general games, however, those conditions for rational beliefs might cause problems with these solution concepts. The perfect Bayesian equilibrium and the sequential equilibrium propose reasonability and consistency, respectively, as their conditions for rational beliefs. These conditions are defined based on Bayes' rule. However, Bayes' rule has limited application in practice and this limited application could result in these solution concepts being incapable of satisfying the criteria of rational solution concepts in general games. In this paper, we propose two criteria of rational solution concepts in general games, namely, weak consistency and the subgame perfect Nash equilibrium condition. Therefore, the limited application of Bayes' rule might mean that these solution concepts are unable to satisfy the weak consistency and the subgame perfect Nash equilibrium condition in general games.

The weak consistency is a criterion of the rational beliefs that places restrictions only on the beliefs on the equilibrium path. This condition for weak consistency is a requirement for all criteria related to rational beliefs. Thus, it is a necessary condition for rational beliefs. However, it is weak in that it does not locate any restriction on the beliefs off the equilibrium path. The subgame perfect Nash equilibrium condition, on the other hand, is a criterion

of the rational strategies. It places restrictions on all actions on the equilibrium path. Moreover, it sets restrictions on some of the actions off the equilibrium path, and in this way it can indirectly inspect some of the beliefs off the equilibrium path. Consequently, it can compensate for the weakness of the weak consistency, and therefore these two conditions can serve as the criteria of the rational solution concepts. In fact, the sequential rationality is also known as an important criterion of the rational solution concepts in multi-period games. This condition, however, is a requirement for the perfect Bayesian equilibrium and the sequential equilibrium. Furthermore, it is a requirement for our solution concept, namely, the perfect regular equilibrium. So these solution concepts always satisfy the sequential rationality, and thus we do not use this criterion to evaluate the rationality of these solution concepts.

Perfect regular equilibrium satisfies the weak consistency and the subgame perfect Nash equilibrium condition in general games, and thus it solves the incapability problem with the perfect Bayesian equilibrium and the sequential equilibrium. The perfect regular equilibrium is defined as a pair of beliefs and strategies such that the beliefs are updated from period to period according to the regular conditional probability and taking the beliefs as given, no player prefers to change its strategy at any of its information sets. So, the perfect regular equilibrium still breaks a whole game into information sets and searches strategies that satisfy sequential rationality at each of the information sets just as those two solution concepts do. However, this solution concept defines its condition for rational beliefs as not being based on Bayes' rule, but rather on a regular version of the conditional probability. This regular conditional probability does not have a limited application. Hence, the perfect regular

equilibrium based on the regular conditional probability can always satisfy the two criteria of the rational solution concepts in general games. Moreover, in finite games, the perfect regular equilibrium is equivalent to a simple version of the perfect Bayesian equilibrium. Therefore, this perfect regular equilibrium extends the perfect Bayesian equilibrium into general games as a simple version of it.

The rest of the paper is organized as follows. Section 2 formulates a general multi-period game with observed actions. Section 3 provides a simple extension of the perfect Bayesian equilibrium in general games and then illustrates its incapability to satisfy the two criteria of the rational solution concepts, namely, the weak consistency and the subgame perfect Nash equilibrium condition. Section 4 formally defines the perfect regular equilibrium. Finally, Section 5 shows that every perfect regular equilibrium satisfies these two criteria of the rational solution concepts and concludes that a perfect regular equilibrium is an extended and simple version of the perfect Bayesian equilibrium in general multi-period games with observed actions.

2 General multi-period game with observed actions

We adopt the “multi-period games with observed actions” from Fudenberg and Tirole (1991) and adapt it to general games that allow infinite actions and types, but only finite players. Hence, like the game from Fudenberg and Tirole (1991), a general multi-period game with observed actions is represented by five items: players, a type and state space, a probability measure on the type and state space, strategies, and utility functions. Based on these items,

we define two more items, namely, a system of beliefs and expected utility functionals. We use all seven items to define the solution concept, namely, the perfect regular equilibrium. Finally, based on this setting of the general game, we extend the definitions of the Nash equilibrium and the subgame perfect Nash equilibrium. Consequently, this section is devoted to defining the setting of the general multi-period game with observed actions and the basic solution concepts.

In a general multi-period game with observed actions, there are a finite number of *players* denoted by $i = 1, 2, \dots, I$. Each player i has its *type* $\theta_i \in \Theta_i$ and this type is its private information as in Harsanyi (1967–68). In addition, there exists a *state* $\theta_0 \in \Theta_0$ and the players do not have information about the actual state. Thus, each player has information about its type θ_i , but no information about the other players' types and the state $\theta_{-i} \in \Theta_{-i} = \Theta_0 \times (\times_{i' \neq i} \Theta_{i'})$. We assume that $\Theta = \times_{i=0}^I \Theta_i$ is a non-empty metric space. Realizations $\theta \in \Theta$ are governed by a *probability measure* η on the class of the Borel subsets² $\times_{i=0}^I \beta(\Theta_i)$ of Θ . Given players' types θ_i , a *conditional probability measure* of η exists and is denoted by $\eta_{-i} : \Theta_i \times (\times_{j \neq i} \beta(\Theta_j)) \longrightarrow [0, 1]$ so that for each $\theta_i \in \Theta_i$ and $B \in \times_{j \neq i} \beta(\Theta_j)$, $\eta_{-i}(\theta_i; B)$ represents a probability of B given θ_i .

The players play the game in *periods* $t = 1, 2, \dots, T$ where $T \in \mathbb{N} \cup \{\infty\}$. In each period t , all players simultaneously choose actions, and then their actions are revealed at the end of the period. We assume, for simplicity, that each player's available actions are independent of its type so that each player i 's *action space* in period t is A_i^t regardless of its type. In

² Given a metric space X , the class of the Borel sets $\beta(X)$ is the smallest class of subsets of X such that *i)* $\beta(X)$ contains all open subsets of X and *ii)* $\beta(X)$ is closed under countable unions and complements.

addition, we assume that $A^t = \times_{i=1}^I A_i^t$ is a non-empty metric space³ for each t . Finally, we consider only the perfect recall games introduced by Kuhn (1950).

A strategy is defined as follows. For each $i = 1, \dots, I$ and $t = 1, \dots, T$, let δ_i^t be a measure from $\Theta_i \times A^1 \times \dots \times A^{t-1} \times \beta(A_i^t)$ to $[0, 1]$. Then, a behavioral *strategy* δ_i is an ordered list of measures $\delta_i = (\delta_i^1, \dots, \delta_i^T)$ such that 1) for each $(\theta_i, a^1, \dots, a^{t-1}) \in \Theta_i \times A^1 \times \dots \times A^{t-1}$, $\delta_i^t(\theta_i, a^1, \dots, a^{t-1}; \cdot)$ is a probability measure on $\beta(A_i^t)$ and 2) for every $B \in \beta(A_i^t)$, $\delta_i^t(\cdot; B)$ is $\beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i'=1}^I \beta(A_{i'}^{t'}))$ measurable. The condition 1) requires that each $\delta_i^t(\theta_i, a^1, \dots, a^{t-1}; \cdot)$ specify what to play at each information set $\Theta_{-i} \times \{(\theta_i, a^1, \dots, a^{t-1})\}$. The condition 2) requires that δ_i^t allow a well-defined expected utility functional, which is defined later. Hereafter, we simply call a behavioral strategy a strategy. Let Π_i be the *set of strategies* for player i and let Π be the *set of strategy profiles*, that is, $\Pi = \times_{i=1}^I \Pi_i$. Note that these definitions originated from Milgrom and Weber (1985) and Balder (1988) and are adapted to the general multi-period games with observed actions.

A system of beliefs is a set of probabilistic assessments about other players' types conditional on reaching each of the information sets. It therefore consists of conditional probability measures over each of the information sets and each measure denotes players' beliefs about the others' types given actions taken before and private information on their own types. Its formal definition is similar to that for the strategy. For each i and t , let μ_i^t be a measure on $\Theta_i \times A^1 \times \dots \times A^{t-1} \times (\times_{j \neq i} \beta(\Theta_j))$ into $[0, 1]$. In addition, for each t , let μ^t denote $(\mu_1^t, \dots, \mu_I^t)$. Then, a *system of beliefs* μ is an ordered list of measures $\mu = (\mu^1, \dots, \mu^T)$ such that 1) for

³ Therefore, the space $\Theta \times A^1 \times \dots \times A^T$ is a non-empty metric space. On this space, expected utility functionals are well-defined according to Ash (1972, 2.6).

each $(\theta_i, a^1, \dots, a^{t-1}) \in \Theta_i \times A^1 \times \dots \times A^{t-1}$, $\mu_i^t(\theta_i, a^1, \dots, a^{t-1}; \cdot)$ is a probability measure on $\times_{j \neq i} \beta(\Theta_j)$ and 2) for every $B \in \times_{j \neq i} \beta(\Theta_j)$, $\mu_i^t(\cdot; B)$ is $\beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i'=1}^I \beta(A_{i'}^{t'}))$ measurable. Here, the condition 1) requires that each $\mu_i^t(\theta_i, a^1, \dots, a^{t-1}; \cdot)$ specify a probability distribution of other players' types over the information set $\Theta_{-i} \times \{(\theta_i, a^1, \dots, a^{t-1})\}$. The condition 2) requires that μ_i^t allow a well-defined expected utility functional. Let Ψ be the set of all systems of beliefs. Then, an element⁴ (μ, δ) in $\Psi \times \Pi$ is called an *assessment*.

A *Von Neumann-Morgenstern utility function* for player i is defined as $U_i : \Theta \times A^1 \times \dots \times A^T \longrightarrow \mathbb{R}$. We assume that each U_i is bounded above or bounded below and $\times_{i=0}^I \beta(\Theta_i) \times (\times_{t=1}^T \times_{i=1}^I \beta(A_i^t))$ measurable, which guarantees that U_i is integrable. In addition, we assume that each U_i can be expressed as a sum of finite-period utility functions. Formally, for each U_i , we assume that there exist both a partition $\{K\} \equiv \Gamma$ of $\{1, 2, \dots, T\}$ and its associated finite-period utility functions $U^K : \Theta \times (\times_{k \in K} A^k) \longrightarrow \mathbb{R}$ such that $U_i(\theta, a) = \sum_{K \in \Gamma} U^K(\theta, a^K)$ for every $(\theta, a) \in \Theta \times A^1 \times \dots \times A^T$ where $a^K = (a^k)_{k \in K} \in \times_{k \in K} A^k$. Here, the partition Γ is a disjoint collection of non-empty subsets K of $\{1, 2, \dots, T\}$ such that $\cup_{K \in \Gamma} K = \{1, 2, \dots, T\}$. These two assumptions ensure the existence of a well-defined expected utility functional.

An *expected utility functional* for player i is implicitly defined as a unique function $E_i : \Pi \longrightarrow \bar{\mathbb{R}} (= \mathbb{R} \cup \{-\infty, \infty\})$ satisfying the following two conditions given any arbitrary strategy profile δ . First, if $E_i(\delta)$ is finite, then for any $\varepsilon > 0$, there exist both a period $t' \leq T$

⁴ For each i and t , the measures $\mu_i^t(\cdot; \cdot)$ and $\delta_i^t(\cdot; \cdot)$ are known as transition probabilities. For more information on the transition probability, please refer to Neveu (1965, III), Ash (1972, 2.6), and Uglanov (1997).

and a sequence of actions $(\check{a}^{t'+1}, \dots, \check{a}^T) \in A^{t'+1} \times \dots \times A^T$ such that for any $t \geq t'$,

$$| E_i(\delta) - \int_{\Theta} \int_{A^1} \dots \int_{A^t} U_i(\theta, a^1, \dots, a^{t-1}, a^t, \check{a}^{t+1}, \dots, \check{a}^T) \delta^t(\theta, a^1, \dots, a^{t-1}; da^t) \dots \delta^1(\theta; da^1) \eta(d\theta) | < \varepsilon$$

where for each t , δ^t denotes the product measure of $\{\delta_1^t, \dots, \delta_I^t\}$ on $\times_{i=1}^I \beta(A_i^t)$, that is, $\delta^t = \delta_1^t \times \dots \times \delta_I^t$. Second, if $E_i(\delta)$ is infinite, then for any $M \in \mathbb{N}$, there exist both a period $t' \leq T$ and a sequence of actions $(\check{a}^{t'+1}, \dots, \check{a}^T) \in A^{t'+1} \times \dots \times A^T$ such that for any $t \geq t'$,

$$\begin{aligned} & \int_{\Theta} \int_{A^1} \dots \int_{A^t} U_i(\theta, a^1, \dots, a^{t-1}, a^t, \check{a}^{t+1}, \dots, \check{a}^T) \delta^t(\theta, a^1, \dots, a^{t-1}; da^t) \dots \delta^1(\theta; da^1) \eta(d\theta) \\ & > M \text{ when } E_i(\delta) = \infty \text{ and } < -M \text{ when } E_i(\delta) = -\infty. \end{aligned}$$

This definition of the expected utility functional makes sense according to Ash (1972, 2.6)⁵.

In this definition of the expected utility functional, the necessity of the second assumption on the utility function, which is that the utility functions U_i can be expressed as sums of finite-period utility functions U^K , that is, $U_i = \sum_{K \in \Gamma} U^K$, might not be clearly seen. This assumption is necessary to well-define an expected utility functional because the definition uses finitely iterated integrals. The following example shows that without this assumption, we might not be able to define an expected utility functional. Consider a game with just one player. Let a function $U : \{\alpha, \beta\}^\infty \longrightarrow \{0, 1\}$ be a utility function for the player such that for any $a \in \{\alpha, \beta\}^\infty$, $U(a) = 0$ if a contains infinitely many α , otherwise $U(a) = 1$.

⁵ Let F_j be a σ -field of subsets of Ω_j for each $j = 1, \dots, n$. Let μ_1 be a probability measure on F_1 , and for each $(\omega_1, \dots, \omega_j) \in \Omega_1 \times \dots \times \Omega_j$, let $\mu(\omega_1, \dots, \omega_j; B)$, $B \in F_{j+1}$, be a probability measure on F_{j+1} ($j = 1, 2, \dots, n-1$). Assume that $\mu(\omega_1, \dots, \omega_j; C)$ is measurable for each fixed $C \in F_{j+1}$. Let $\Omega = \Omega_1 \times \dots \times \Omega_n$ and $F = F_1 \times \dots \times F_n$.

(1) There is a unique probability measure μ on F such that for each measurable rectangle $A_1 \times \dots \times A_n \in F$, $\mu(A_1 \times \dots \times A_n) = \int_{A_1} \int_{A_2} \dots \int_{A_n} \mu(\omega_1, \dots, \omega_{n-1}; d\omega_n) \dots \mu(\omega_2; d\omega_1) \mu_1(d\omega_1)$.

(2) Let $f : (\Omega, F) \longrightarrow (\mathbb{R}, \beta(\mathbb{R}))$ and $f \geq 0$. Then, $\int_{\Omega} f d\mu = \int_{\Omega_1} \dots \int_{\Omega_n} f(\omega_1, \dots, \omega_n) \mu(\omega_1, \dots, \omega_{n-1}; d\omega_n) \dots \mu_1(d\omega_1)$.

Suppose the player chooses its strategy δ such that 1) $\delta^1(\alpha) = \delta^1(\beta) = \frac{1}{2}$ and 2) for any $t \geq 2$, $\delta^t(a^1, \dots, a^{t-1}; \alpha) = 1$ if $a^{t-1} = \alpha$ and $\delta^t(a^1, \dots, a^{t-1}; \beta) = 1$ if $a^{t-1} = \beta$. Then, the expected utility value with respect to δ is obviously $\frac{1}{2}$. However, according to our definition of the expected utility functional E , $E(\delta)$ cannot be $\frac{1}{2}$ since $E(\delta') = 0$ or 1 for any arbitrary strategy δ' . Accordingly, we cannot define an expected utility functional for this game. In fact, this assumption regarding the utility function is a weak requirement in that it is always satisfied in finite-period games and also satisfied in repeated games that consist of infinitely repeated finite-period games. Nevertheless, this assumption is so potent that we can define an expected utility functional by using only finitely iterated integrals.

Based on this expected utility functional, the *Nash equilibrium* by Nash (1951) and the *subgame perfect Nash equilibrium* by Selten (1975) are extended in the general multi-period games with observed actions. In this paper, we suggest two conditions for rational solution concepts in the general games. One is the subgame perfect Nash equilibrium condition. The other is weak consistency introduced by Myerson (1991, 4.3). This weak consistency is a criterion of a consistent relation between players' beliefs and players' actual strategies. A formal definition of the weak consistency is presented in Section 5.

Definition 1 A strategy profile $\delta = (\delta_1, \dots, \delta_I)$ is a **Nash equilibrium** if δ satisfies $E_i(\delta) = \max_{\delta'_i \in \Pi_i} E_i(\delta'_i, \delta_{-i})$ for each $i \leq I$. A Nash equilibrium is **subgame perfect** if it induces a Nash equilibrium in every subgame⁶.

⁶ For a formal definition of the subgame, please refer to Selten (1975, Section 5).

3 Example: Incapable simple perfect Bayesian equilibrium in a general game

This section shows that an extension of the perfect Bayesian equilibrium in general multi-period games with observed actions might be incapable of satisfying the weak consistency and the subgame perfect Nash equilibrium condition. We first provide a simple and formal extension of the perfect Bayesian equilibrium in general games. Originally, the perfect Bayesian equilibrium was defined in finite games. However, it has been extended and applied to general games on various economic issues, such as the Auction, Bargaining game, and Signaling game. Here, we try to present a universal definition of the perfect Bayesian equilibrium that can be commonly applied to such general games. Next, we describe the setting of the example which is the famous signaling game by Crawford and Sobel (1982). Then, based on this setting, we show that the simple extension of the perfect Bayesian equilibrium might be incapable of satisfying the weak consistency and the subgame perfect Nash equilibrium condition.

3.1 Simple perfect Bayesian equilibrium in a general game

According to Fudenberg and Tirole (1991), a perfect Bayesian equilibrium in a finite game is defined as an assessment (μ, δ) , which is a pair consisting of a system of beliefs μ and a strategy profile δ , such that (μ, δ) is both 1) reasonable and 2) sequentially rational. Here, an assessment (μ, δ) is said to be reasonable *i*) if μ is updated from period to period with respect to δ and μ itself according to Bayes' rule whenever possible and *ii*) if it satisfies the “no-signaling-what-you-don’t-know” condition that constrains μ off the equilibrium path which

players would not reach if they would play according to δ . In addition, δ is sequentially rational with respect to μ if, taking μ as given, no player prefers to change its strategy δ_i at any of its information sets.

Of these two conditions for a perfect Bayesian equilibrium, the first condition, reasonability, might lead it to being incapable of satisfying the weak consistency and the subgame perfect Nash equilibrium condition in general multi-period games with observed actions. To be precise, the incapability of a perfect Bayesian equilibrium is caused by the weakness of Bayes' rule. Bayes' rule is a way of formulating a conditional probability or a conditional probability density function and defines them as a fraction between two probabilities or a fraction between two probability density functions. So, Bayes' rule can be employed only when the probability of a given event, which becomes a denominator in the fraction, is positive or when the probability density functions are well-defined. This limited application of Bayes' rule consequently gives rise to the incapability of a perfect Bayesian equilibrium in general games.

To clearly see this incapability of a perfect Bayesian equilibrium, we formally extend the definition of the reasonability into general multi-period games with observed actions. Notice that, in this extension, we omit the “no-signaling-what-you-don’t-know” condition for simplicity’s sake. This condition was designed to improve the reasonability condition so that this reasonability condition might become as plausible as the consistency condition introduced by Kreps and Wilson (1982). As shown by Osborne and Rubinstein (1994, 234.3), however, the reasonability condition including the no-signaling-what-you-don’t-know

condition is fundamentally different from the consistency condition. Hence, we conclude that its contribution to the reasonability condition is not sufficient compared with the complexity caused by this condition⁷. As a result, we simplify the definition of the reasonability in general games by excluding this no-signaling-what-you-don't-know condition. We call this simple extension of the reasonability *reasonable consistency*.

Definition 2 An assessment (μ, δ) is **reasonably consistent** if given each i , 1) μ_i^1 is the same as η_{-i} and 2) for each $(\theta_i, a^1, \dots, a^{t-1}) \in \Theta_i \times A^1 \times \dots \times A^{t-1}$ and each $B \in \times_{j \neq i} \beta(\Theta_j)$, $\mu_i^t(\theta_i, a^1, \dots, a^{t-1}; B)$ indicates the same probability as

$$\frac{\int_{\Theta_{-i}} \int_{\{a^{t-1}\}} I_B(\theta_{-i}) \delta^{t-1}(\theta, a^1, \dots, a^{t-2}; da^{t-1}) \mu_i^{t-1}(\theta_i, a^1, \dots, a^{t-2}; d\theta_{-i})}{\int_{\Theta_{-i}} \int_{\{a^{t-1}\}} \delta^{t-1}(\theta, a^1, \dots, a^{t-2}; da^{t-1}) \mu_i^{t-1}(\theta_i, a^1, \dots, a^{t-2}; d\theta_{-i})}$$

whenever $t \geq 2$ and $\int_{\Theta_{-i}} \int_{\{a^{t-1}\}} \delta^{t-1}(\theta, a^1, \dots, a^{t-2}; da^{t-1}) \mu_i^{t-1}(\theta_i, a^1, \dots, a^{t-2}; d\theta_{-i}) > 0$ where $I_B(\cdot)$ is an indicator function, i.e. $I_B(\theta_{-i}) = 1$ if $\theta_{-i} \in B$ and $I_B(\theta_{-i}) = 0$ if $\theta_{-i} \notin B$.

In other words, an assessment (μ, δ) is reasonably consistent if 1) in the first period, each player correctly forms its beliefs μ_i^1 based on the type and state probability measure η , and 2) from the second period, each player employs Bayes' rule to update its beliefs μ_i^t with respect to the previous action plans δ^{t-1} and the previous beliefs μ_i^{t-1} whenever possible. Here, “whenever possible” means whenever an information set $\Theta_{-i} \times \{(\theta_i, a^1, \dots, a^{t-1})\}$ is reached with positive probability with respect to δ^{t-1} and μ_i^{t-1} , that is, $\int_{\Theta_{-i}} \int_{\{a^{t-1}\}} \delta^{t-1}(\theta, a^1, \dots, a^{t-2}; da^{t-1}) \mu_i^{t-1}(\theta_i, a^1, \dots, a^{t-2}; d\theta_{-i}) > 0$. Note that, in finite games, this definition of the reasonable consistency represents the same condition as the definition of the reasonability in

⁷ According to Fudenberg and Tirole (1991), this condition requires that “no player i 's deviation be treated as containing information about things that player i does not know.” Here, “player i 's deviation” is its behavior off the equilibrium path, so this condition places restrictions on the beliefs off the equilibrium path. However, the incapability problem with a perfect Bayesian equilibrium occurs more significantly on the equilibrium path than off the equilibrium path. Therefore, this condition cannot solve the problem with the solution concept of the perfect Bayesian equilibrium. This is another reason why we conclude that the contribution of this condition is not sufficient.

Fudenberg and Tirole (1991, Definition 3.1) except for the no-signaling-what-you-don't-know condition.

There is another version of Bayes' rule, a continuous version of Bayes' rule, but we cannot use this version to extend the definition of the reasonability into general multi-period games with observed actions. A continuous version of Bayes' rule defines a conditional probability density function as a fraction between two probability density functions. Accordingly, this version requires well-defined probability density functions. In general games with a continuum of actions, however, only mixed strategies that assign zero probability to every single action can be represented as probability density functions. As a result, this version of Bayes' rule is not well-defined for any strategies that assign positive probability to a single action. In particular, this version is not well-defined for any of the pure strategies under which players would play a single action at each information set. Therefore, we cannot extend the definition of the reasonability into general games by using this continuous version of Bayes' rule⁸.

Based on the reasonable consistency condition, a perfect Bayesian equilibrium is extended in general multi-period games with observed actions. We call this simple extension of the perfect Bayesian equilibrium a *simple perfect Bayesian equilibrium*.

Definition 3 *An assessment (μ, δ) is a **simple perfect Bayesian equilibrium** if (μ, δ)*

⁸ There is a way to combine these two versions of Bayes' rule. This way does not solve the limited application problem with Bayes' rule in general games, either. This is because it requires well-defined probability or probability density functions. In general games, however, players' strategies might induce neither probability nor a probability density function. For example, the sender's strategy introduced in the next subsection induces neither probability nor a probability density function. As a result, this combined version of Bayes' rule still has limited application, and therefore it could result in the incapability of a perfect Bayesian equilibrium in general games.

is both 1) *reasonably consistent* and 2) *sequentially rational*⁹ .

3.2 Example

Now, we are ready to exemplify the incapability of a simple perfect Bayesian equilibrium in a general multi-period game with observed actions. Consider the information transmission game introduced by Crawford and Sobel (1982). There are two players, a sender and a receiver. The sender is assigned a type θ that is a random variable from a uniform distribution on $[0, 1]$ and she makes a signal $s \in [0, 1]$ to the receiver. Then, after observing the signal s , the receiver chooses his action $a \in [0, 1]$. The sender has a von Neumann-Morgenstern utility function $U^S(\theta, a, b) = -(\theta - (a + b))^2$ where $b > 0$ and the receiver has another von Neumann-Morgenstern utility function $U^R(\theta, a) = -(\theta - a)^2$.

In this game, the sender's strategy $s(\theta) = \theta$ and the receiver's strategy $a(s) = \max\{s - b, 0\}$ are a simple perfect Bayesian equilibrium together with the receiver's system of beliefs $\mu(\max\{s - b, 0\}; s) = 1$ which denotes that given a signal s , the type $\max\{s - b, 0\}$ would be assigned to the sender with probability one. First, the system of beliefs $\mu(\max\{s - b, 0\}; s) = 1$ is reasonably consistent with the sender's strategy $s(\theta) = \theta$ because it does not violate the conditions for the reasonable consistency in Definition 2. Under the strategy $s(\theta) = \theta$, each signal θ occurs with probability zero, and thus we cannot employ Bayes' rule. In this case, no system of beliefs is considered to violate Bayes' rule formulated in Definition 2. Consequently, $\mu(\max\{s - b, 0\}; s) = 1$ is reasonably consistent with $s(\theta) = \theta$. Second, the sender's strategy $s(\theta) = \theta$ is the best response to the receiver's strategy $a(s) = \max\{s - b, 0\}$,

⁹ For a formal definition of the sequential rationality, please refer to Definition 5 in Section 4.

and $a(s) = \max\{s - b, 0\}$ is the best response to his system of beliefs $\mu(\max\{s - b, 0\}; s) = 1$. This proves that they satisfy the sequential rationality. Therefore, these strategies and the system of beliefs are a simple perfect Bayesian equilibrium.

This simple perfect Bayesian equilibrium, however, is incapable of satisfying the weak consistency and the subgame perfect Nash equilibrium condition. In the scenario of this equilibrium, the receiver constantly mistakes a true type θ for a wrong type $\max\{\theta - b, 0\}$. As a result, the sender's strategy $s(\theta) = \theta$ and the receiver's system of beliefs $\mu(\max\{s - b, 0\}; s) = 1$ do not induce the same probability distribution on the equilibrium path which the players would actually reach if they were to play according to their strategies $s(\cdot)$ and $a(\cdot)$. Since the weak consistency¹⁰ requires them both to induce the same probability distribution on the equilibrium path, this simple perfect Bayesian equilibrium does not satisfy the weak consistency. Moreover, the receiver's strategy $a(s) = \max\{s - b, 0\}$ is not the best response to the sender's strategy $s(\theta) = \theta$. So this simple perfect Bayesian equilibrium does not satisfy the Nash equilibrium condition, and thus it does not satisfy the subgame perfect Nash equilibrium condition¹¹.

This incapability of the simple perfect Bayesian equilibrium is caused mainly by the

¹⁰ Definition 7 in Section 5 formally defines this weak consistency in general multi-period games with observed actions.

¹¹ Crawford and Sobel (1982) tried to solve this problem with a simple perfect Bayesian equilibrium by adopting a continuous version of Bayes' rule. Their approach to the problem naturally led them to only consider the probability density strategies of the sender. That is, they did not consider the overall strategies of the sender. Their partial consideration of the sender's strategies might be justified by Lemma 1 in their paper which guaranteed that, in equilibrium, any sender's strategy can be replaced with her probability density strategies while preserving the strategies of the receiver. Lemma 1, however, was not proven correctly, and thus it cannot justify their partial consideration of the sender's strategies or any other results. For more information, please refer to Jung (2009).

setting¹² that the sender has a continuum of types and signals. Accordingly, most games having similar settings can testify that there exist simple perfect Bayesian equilibria that break the weak consistency and the subgame perfect Nash equilibrium condition. Since this setting represents a usual situation, there is a large class of games including similar settings. Therefore, we conclude that this incapability is a ubiquitous problem with the solution concept of the simple perfect Bayesian equilibrium. In the next section, we revise this simple perfect Bayesian equilibrium to develop a solution concept that is capable of satisfying both the weak consistency and the subgame perfect Nash equilibrium condition.

4 Perfect Regular Equilibrium

The incapability of a simple perfect Bayesian equilibrium is due to the limited application of Bayes' rule in general multi-period games with observed actions. Bayes' rule cannot be employed if a conditional event, whose probability becomes a denominator in a conditional probability formula according to Bayes' rule, takes place with probability zero. In general games, however, it is possible for every conditional event to take place with probability zero. In this case, we cannot employ Bayes' rule at all either on the equilibrium path or off the equilibrium path. Hence, no system of beliefs is considered to violate Bayes' rule, which means that every system of beliefs satisfies the reasonable consistency for a simple perfect Bayesian equilibrium. As a result, some intuitively inconsistent system of beliefs could be part of a simple perfect Bayesian equilibrium, and this system of beliefs could lead the simple

¹² Jung (2010) showed that, under this setting, an extension of the sequential equilibrium in general games can cause the same problem, namely, the incapability to satisfy both the weak consistency and the Nash equilibrium condition.

perfect Bayesian equilibrium to the incapability as shown in the example.

To solve this incapability problem with the simple perfect Bayesian equilibrium, we revise it by replacing Bayes' rule with a regular version of the conditional probability and refer to the revised solution concept as a perfect regular equilibrium. The regular version of the conditional probability is another way of formulating a conditional probability and is especially designed to well-define the probability given probability zero events. It therefore defines a conditional probability implicitly through a functional equation without referring to a fraction between probabilities of events. Accordingly, it does not show the limited application problem as Bayes' rule does, and it can well-define the conditional probabilities given 'almost every' probability zero event. Therefore, the perfect regular equilibrium equipped with this regular conditional probability¹³ can solve the incapability problem with the simple perfect Bayesian equilibrium.

The definition of the perfect regular equilibrium is the same as that of the simple perfect Bayesian equilibrium except for its approach to the conditional probabilities. Thus, the perfect regular equilibrium is defined as an assessment (μ, δ) such that 1) μ is updated from period to period with respect to δ and μ itself according to the regular conditional probability, and 2) taking μ as given, no player prefers to change its strategy δ_i at any of its information sets. The first condition for the perfect regular equilibrium is referred to as regular consistency and the second condition is referred to as the sequential rationality. In this section, we formally define these conditions and the perfect regular equilibrium.

¹³ For more information regarding the regular version of the conditional probability, please refer to Ash (1972, 6.6).

We first provide a formal definition of the *regular consistency* in general multi-period games with observed actions.

Definition 4 An assessment (μ, δ) is **regularly consistent** if given each i , 1) μ_i^1 is the same as η_{-i} and 2) for each $t \geq 2$ and each $(\theta_i, a^1, \dots, a^{t-2}) \in \Theta_i \times A^1 \times \dots \times A^{t-2}$, μ_i^t satisfies the following functional equation: $\int_B \int_A \delta^{t-1}(\theta, a^1, \dots, a^{t-2}; da^{t-1}) \mu_i^{t-1}(\theta_i, a^1, \dots, a^{t-2}; d\theta_{-i}) = \int_{\Theta_{-i}} \int_A \mu_i^t(\theta_i, a^1, \dots, a^{t-1}; B) \delta^{t-1}(\theta, a^1, \dots, a^{t-2}; da^{t-1}) \mu_i^{t-1}(\theta_i, a^1, \dots, a^{t-2}; d\theta_{-i})$ for every $B \in \times_{j \neq i} \beta(\Theta_j)$ and $A \in \times_{i=1}^I \beta(A_i^{t-1})$.

That is, an assessment (μ, δ) is regularly consistent if 1) in the first period, each player correctly forms its beliefs μ_i^1 based on the type and state probability measure η , and 2) from the second period, each player employs the regular conditional probability to update its beliefs μ_i^t with respect to the previous action plans δ^{t-1} and the previous beliefs μ_i^{t-1} given the information about its type and the previous actions $(\theta_i, a^1, \dots, a^{t-2})$. Here, Definition 4 implicitly defines μ_i^t as a regular conditional probability measure through the functional equation governed by δ^{t-1} and μ_i^{t-1} . In this way, Definition 4 can avoid the limited application problem since the functional equation is well-defined for any arbitrary set $B \times \{(\theta_i, a^1, \dots, a^{t-2})\} \times A$ where $B \in \times_{j \neq i} \beta(\Theta_j)$ and $A \in \times_{i=1}^I \beta(A_i^{t-1})$. As a result, beliefs μ_i^t can be properly updated with respect to δ^{t-1} and μ_i^{t-1} . Note that the functional equation can determine a conditional probability of μ_i^t only within the support¹⁴ of the product measure of δ^{t-1} and μ_i^{t-1} . This is because if a set $A \in \times_{i=1}^I \beta(A_i^{t-1})$ is outside the support of the product measure $\delta^{t-1} \mu_i^{t-1}$, then both sides in the functional equation become zero, and so the conditional probability of μ_i^t given A can be arbitrary. Consequently, a conditional probability of μ_i^t is only meaningful given a set of actions within the support of the product measure

¹⁴ In a metric space, a support of a measure is defined as the smallest closed set within which the measure lives.

$$\delta^{t-1} \mu_i^{t-1}.$$

Next, we formally define the *sequential rationality* in general games. For notational simplicity, given a player i and a period t , let a functional $E_i^t : \Theta_i \times A^1 \times \dots \times A^{t-1} \times \Psi \times \Pi \longrightarrow \bar{\mathbb{R}} (= \mathbb{R} \cup \{-\infty, \infty\})$ be a *conditional expected utility functional*¹⁵ where $\Psi \times \Pi$ is the set of all assessments. That is, for each $(\theta_i, a^1, \dots, a^{t-1}) \in \Theta_i \times A^1 \times \dots \times A^{t-1}$, $E_i^t(\theta_i, a^1, \dots, a^{t-1}; \mu, \delta)$ denotes an expected utility value with respect to the system of beliefs μ and the strategy profile δ conditional on reaching the information set $\Theta_{-i} \times \{(\theta_i, a^1, \dots, a^{t-1})\}$.

Definition 5 A strategy profile δ is **sequentially rational** with respect to a system of beliefs μ if, given each i and t , and given each $(\theta_i, a^1, \dots, a^{t-1}) \in \Theta_i \times A^1 \times \dots \times A^{t-1}$, we have $E_i^t(\theta_i, a^1, \dots, a^{t-1}; \mu, \delta) \geq E_i^t(\theta_i, a^1, \dots, a^{t-1}; \mu, (\delta'_i, \delta_{-i}))$ for every $\delta'_i \in \Pi_i$.

Here, a set $\Theta_{-i} \times \{(\theta_i, a^1, \dots, a^{t-1})\}$ denotes an information set of player i . Thus, the sequential rationality requires that, in responding to the other players' strategies δ_{-i} , each player i make its best response δ_i with respect to the system of beliefs μ , which would induce the greatest expected utility value given any of its information sets $\Theta_{-i} \times \{(\theta_i, a^1, \dots, a^{t-1})\}$.

¹⁵ Formally, the conditional expected utility functional is implicitly defined just like the expected utility functional. So, given i and t , the conditional expected utility functional E_i^t is a unique function satisfying the following conditions for any arbitrary strategy profile δ . First, if $E_i^t(\theta_i, \dot{a}^1, \dots, \dot{a}^{t-1}; \mu, \delta)$ is finite, then, for any $\varepsilon > 0$, there exist both a period $t' \in \{t, \dots, T\}$ and a sequence of actions $(\check{a}^{t'+1}, \dots, \check{a}^T) \in A^{t'+1} \times \dots \times A^T$ such that for any $t'' \geq t'$,

$$\begin{aligned} & | E_i^t(\dot{\theta}_i, \dot{a}^1, \dots, \dot{a}^{t-1}; \mu, \delta) - \int_{\Theta_{-i}} \int_{A^t} \dots \int_{A^{t''}} U_i(\dot{\theta}_i, \theta_{-i}, \dot{a}^1, \dots, \dot{a}^{t-1}, a^t, \dots, a^{t''}, \check{a}^{t''+1}, \dots, \\ & \check{a}^T) \delta^{t''}(\dot{\theta}_i, \theta_{-i}, \dot{a}^1, \dots, \dot{a}^{t''-1}; da^{t''}) \dots \delta^t(\dot{\theta}_i, \theta_{-i}, \dot{a}^1, \dots, \dot{a}^{t-1}; da^t) \mu_i^t(\dot{\theta}_i, \dot{a}^1, \dots, \dot{a}^{t-1}; d\theta_{-i}) | < \varepsilon. \end{aligned}$$

Second, if $E_i^t(\dot{\theta}_i, \dot{a}^1, \dots, \dot{a}^{t-1}; \mu, \delta)$ is infinite, then, for any $M \in \mathbb{N}$, there exist both a period $t' \in \{t, \dots, T\}$ and a sequence of actions $(\check{a}^{t'+1}, \dots, \check{a}^T) \in A^{t'+1} \times \dots \times A^T$ such that for any $t'' \geq t'$,

$$\begin{aligned} & \int_{\Theta_{-i}} \int_{A^t} \dots \int_{A^{t''}} U_i(\dot{\theta}_i, \theta_{-i}, \dot{a}^1, \dots, \dot{a}^{t-1}, a^t, \dots, a^{t''}, \check{a}^{t''+1}, \dots, \check{a}^T) \delta^{t''}(\dot{\theta}_i, \theta_{-i}, \dot{a}^1, \\ & \dots, \dot{a}^{t''-1}, da^{t''}) \dots \delta^t(\dot{\theta}_i, \theta_{-i}, \dot{a}^1, \dots, \dot{a}^{t-1}; da^t) \mu_i^t(\dot{\theta}_i, \dot{a}^1, \dots, \dot{a}^{t-1}; d\theta_{-i}) \\ & > M \text{ when } E_i^t(\dot{\theta}_i, \dot{a}^1, \dots, \dot{a}^{t-1}; \mu, \delta) = \infty \text{ and } < -M \text{ when } E_i^t(\dot{\theta}_i, \dot{a}^1, \dots, \dot{a}^{t-1}; \mu, \delta) = -\infty. \end{aligned}$$

Again, this definition of the conditional expected utility functional makes sense according to Ash (1972, 2.6).

As a result, no player prefers to change its strategy at any of its information sets. Originally, Kreps and Wilson (1982) defined the sequential rationality in finite games. We adapt their definition to general multi-period games with observed actions.

Finally, Definition 6 defines the *perfect regular equilibrium*.

Definition 6 *An assessment (μ, δ) is a **perfect regular equilibrium** if (μ, δ) is both 1) regularly consistent and 2) sequentially rational.*

5 Properties of the perfect regular equilibrium

The first property of the perfect regular equilibrium is that it always satisfies the weak consistency, which is a criterion of the rational beliefs. Since the weak consistency was originally defined in finite games, we start by extending its definition to general multi-period games with observed actions.

Definition 7 *An assessment (μ, δ) is **weakly consistent**¹⁶ if given each i , 1) μ_i^1 is the same as η_{-i} and 2) for each $t \geq 2$, μ_i^t satisfies the following functional equation: $\int_{\Theta} \int_{A^1} \cdots \int_{A^{t-1}} I_{B \times A}(\theta, a^1, \dots, a^{t-1}) \delta^{t-1}(\theta, a^1, \dots, a^{t-2}; da^{t-1}) \cdots \delta^1(\theta; da^1) \eta(d\theta) = \int_{\Theta} \int_{A^1} \cdots \int_{A^{t-1}} I_A(\theta_i, a^1, \dots, a^{t-1}) \mu_i^t(\theta_i, a^1, \dots, a^{t-1}; B) \delta^{t-1}(\theta, a^1, \dots, a^{t-2}; da^{t-1}) \cdots \delta^1(\theta; da^1) \eta(d\theta)$ for every $B \in \times_{j \neq i} \beta(\Theta_j)$ and $A \in \beta(\Theta_i) \times (\times_{t'=1}^{t-1} \times_{i'=1}^I \beta(A_{i'}^{t'}))$ where $I_{B \times A}(\cdot)$ and $I_A(\cdot)$ are indicator functions, i.e. $I_{B \times A}(\theta, a^1, \dots, a^{t-1}) = 1$ if $(\theta, a^1, \dots, a^{t-1}) \in B \times A$ and $I_{B \times A}(\theta, a^1, \dots, a^{t-1}) = 0$ otherwise.*

In plain words, an assessment (μ, δ) is weakly consistent if 1) in the first period, each player correctly forms its beliefs μ_i^1 based on the type and state probability measure η , and 2) from the second period, each player employs the regular conditional probability to update its beliefs μ_i^t with respect to all the previous action plans $\delta^1, \dots, \delta^{t-1}$ and the probability measure η given the information about its type and the previous actions $(\theta_i, a^1, \dots, a^{t-1})$.

¹⁶ We adapt Myerson's (1991, 4.3) definition to general multi-period games with observed actions.

In this definition, the functional equation can determine a conditional probability of μ_i^t only on the equilibrium path. This is because if a set $A \in \times_{i=1}^I \beta(A_i^{t-1})$ is off the equilibrium path, then both sides in the functional equation become zero, so a conditional probability of μ_i^t given A can be arbitrary. Consequently, this definition indicates that the weak consistency imposes restrictions only on the beliefs on the equilibrium path, and thus it imposes no restriction on the beliefs off the equilibrium path. Note that the regular consistency places restrictions on the beliefs on the support of the product measure $\delta^{t-1} \mu_i^{t-1}$, which includes all the beliefs on the equilibrium path. As a result, in general multi-period games with observed actions, if an assessment satisfies the regular consistency, then it also satisfies the weak consistency. This statement is formulated in Proposition 1.

Proposition 1 *If an assessment is regularly consistent, then it is weakly consistent.*

Proof. The result directly follows from the definitions. ■

Kreps and Ramey (1987) introduced another criterion of the rational beliefs, *convex structural consistency*. According to them, the convex structural consistency is defined as a consistency criterion under which the beliefs of the players should reflect the informational structure of a game through a convex combination of players' strategies. Thus, under this consistency criterion, if players would be unexpectedly located, they should then form their beliefs such that a convex combination of strategies can induce the beliefs¹⁷. This criterion

¹⁷ In fact, this convex structural consistency is a weak version of the *structural consistency* of Kreps and Wilson (1982). Kreps and Wilson defined the structural consistency as a consistency criterion under which the beliefs of the players should reflect the informational structure of a game through a single strategy profile. Thus, this structural consistency requires players to use only one strategy profile to form one belief. Because of the strong requirement, however, most of the solution concepts including the sequential equilibrium and the perfect equilibrium do not satisfy this criterion even in finite games.

of the rational beliefs imposes restrictions on all the beliefs off the equilibrium path as well as on the equilibrium path. Note that the perfect regular equilibrium, just like the perfect Bayesian equilibrium in finite games, places restrictions only on the beliefs on the support of the product measure $\delta^{t-1}\mu_i^{t-1}$. Since the support of the product measure $\delta^{t-1}\mu_i^{t-1}$ might not cover all the beliefs off the equilibrium path, the perfect regular equilibrium might not put restrictions on all the beliefs off the equilibrium path. Accordingly, the perfect regular equilibrium might not satisfy the convex structural consistency¹⁸.

The second property of the perfect regular equilibrium is that it always satisfies the subgame perfect Nash equilibrium condition, which is a criterion of the rational strategies. This property is due to the sequential rationality, which is one of the two conditions for the perfect regular equilibrium. If an information set initiates a subgame, then the conditional probability on the information set given the information set itself is uniquely determined as one. Then, the sequential rationality condition, given the information set, becomes the same as the Nash equilibrium condition, which means that the perfect regular equilibrium induces a Nash equilibrium in the subgame. As a result, the perfect regular equilibrium satisfies the subgame perfect Nash equilibrium condition. Proposition 2 formally presents this second property of the perfect regular equilibrium in general multi-period games with observed actions.

Proposition 2 *Every perfect regular equilibrium is a subgame perfect Nash equilibrium.*

Proof. The result directly follows from the definitions. ■

¹⁸ Jung (2010) introduced a new solution concept, complete sequential equilibrium, in general finite-period games with observed actions and presented conditions under which the complete sequential equilibrium satisfies the convex structural consistency.

Next, Proposition 3 reveals the relationship between the regular consistency for the perfect regular equilibrium and the reasonable consistency for the simple perfect Bayesian equilibrium in finite games. In general games, Bayes' rule in the reasonable consistency might give rise to the incapability problem with a simple perfect Bayesian equilibrium. In finite games, however, Bayes' rule does not cause this problem, and it functions as well as the regular conditional probability does. As a result, the reasonable consistency based on Bayes' rule is equivalent to the regular consistency based on the regular conditional probability in finite games.

Proposition 3 *In finite games, an assessment is regularly consistent if and only if it is reasonably consistent.*

Proof. The result directly follows from the definitions. ■

Finally, Theorem 1 aggregates all the results. Propositions 1 and 2 together ensure that the perfect regular equilibrium satisfies both conditions, namely, the weak consistency and the subgame perfect Nash equilibrium condition, which we have suggested as criteria of rational solution concepts in general games. So, we conclude that the perfect regular equilibrium successfully extends the perfect Bayesian equilibrium to general games. In addition, Proposition 3 guarantees that the perfect regular equilibrium is equivalent to the simple perfect Bayesian equilibrium in finite games. Note that the simple perfect Bayesian equilibrium is defined as a simple version of the perfect Bayesian equilibrium. Therefore, as a corollary of these propositions, Theorem 1 brings all the properties of the perfect regular equilibrium together and provides evidence that it is indeed an extended and simple version of the perfect Bayesian equilibrium in general multi-period games with observed actions.

Theorem 1 *Every perfect regular equilibrium satisfies both the weak consistency and the subgame perfect Nash equilibrium condition. Furthermore, in finite games, an assessment (μ, δ) is a perfect regular equilibrium if and only if it is a simple perfect Bayesian equilibrium.*

There is another solution concept for general games. Jung (2010) developed complete sequential equilibria in general finite-period games with observed actions by improving sequential equilibria. In general games, the sequential equilibrium might give rise to the incapability problem as in the case of the perfect Bayesian equilibrium. The complete sequential equilibrium solves this incapability problem by replacing beliefs with *complete beliefs*. The complete beliefs are probability measures defined, not on each information set, but on the whole class of information sets in each period. Note that all strategy profiles lead to the whole class of information sets in each period with probability one and thus they can well-define probability distributions over the whole class of information sets. As a result, any arbitrary strategy profile can properly induce consistent complete beliefs, and therefore the complete sequential equilibrium can improve the sequential equilibrium in general games.

This complete sequential equilibrium, however, is not closely related to the perfect regular equilibrium in general games in that it might not be a perfect regular equilibrium and vice versa. This is because the consistency for the complete sequential equilibrium and the regular consistency for the perfect regular equilibrium place different restrictions on the beliefs off the equilibrium path. Consequently, a complete sequential equilibrium might not be a perfect regular equilibrium in general games and a perfect regular equilibrium might not be a complete sequential equilibrium either.

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