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A Model of Two-Stage All-Pay Auction and the Role of Commitment in Political Lobbying

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Abstract

The present paper investigates the role of commitment in political lobbying using a two-bidder two-period all-pay auction with complete information and asymmetric valuations. It is shown that, when the bidders (i.e., lobbyists) have a chance to commit their bids, the seller (i.e., politician) can simply and robustly resolve the "under-dissipation" problem. That is, the seller can raise a revenue arbitrarily close to that in the first- and second-price auctions, by having an arbitrarily small bias towards the bidders' early payments. This mechanism is robust in the sense that it does not depend on the seller's knowledge of the bidders' valuations. The results can be extended to general numbers of bidders and periods.

JEL Classification: D44, D72

Keywords: all-pay auction, all-pay contest, commitment, political lobbying, rent seeking, under-dissipation

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1 Introduction

The all-pay auction is often used as a model of political lobbying or fund-raising, because its "all-pay feature" captures the point that political donation or campaign cost is irreversible even if a lobbyist cannot enjoy a rent after all. However, by using the all-pay auction, we neglect another aspect: lobbying in the real world is not a static competition as is a sealed-bid auction but a dynamic activity. As long as a politician or a party raises the political fund through time, a lobbyist chooses not only the amount of his contribution but also its timing, taking the strategic effect of commitment into account. That is, a lobbyist can commit his contribution at an earlier point of time, or it is also possible that he waits and observes the rival's behavior and makes a decision later. Then, the timing itself can have a strategic effect. For example, an early contribution might be beneficial because it would work as a kind of preemption and lead the lobbyist to victory. On the contrary, it is also possible that an early commitment is harmful because, after observing it, the rival would commit more money and thereby intensify the competition.

To examine such strategic interactions through time and the roles of commitment in political lobbying, the present paper investigates a two-person two-period all-pay auction with complete information. In the first period, two bidders (i.e., lobbyists) simultaneously offer non-negative bids (i.e., contributions). In the second period, after the bids in the first period are publicly observed, each bidder can increase but not decrease his bid. Finally, a good (i.e., political reward or rent) is allocated to the bidder with the higher total bids and each bidder pays his total bid even if he does not win the good. It is assumed that the valuations for the reward are common knowledge between the bidders.

The primary finding is that in this dynamic setting, the seller (i.e., politician) can raise the same revenue as in the standard winner-pay auctions. This contrasts with the key property of the static model, which is so-called "under-dissipation." Namely, the static all-pay auction with complete information possesses an equilibrium only in non-degenerate mixed strategies and the expected revenue is smaller than in the standard first- and second-price auctions. In the dynamic model, this problem is fully reconciled in two steps. First, the present dynamic game possesses a subgame perfect equilibrium (henceforth, SPE) that yields the same revenue as in the standard auctions, although there also exists a continuum of SPEs that yield smaller revenue. Second, we perturb the original game and show that the maximal revenue equilibrium is the only SPE in the perturbed game. Therefore, the present study suggests that the seller can raise (almost) the same revenue in the dynamic all-pay auction as in the standard auctions. The perturbation can be interpreted as the seller's bias towards early contributions.[1] The (perturbed) mechanism demonstrated here is robust in the sense that it works even if the seller does not know the bidders' values, the rank order between the values, or even the distribution of the values.

The logic behind the under-dissipation is as follows. If a pure strategy equilibrium exists in the static all-pay auction between two bidders, the agent who cannot obtain the good must bid zero in the equilibrium. Then, the other agent has no incentive to offer any bid bounded away from zero. This yields a clear contradiction and thus, the static all-pay auction possesses an equilibrium only in mixed strategies. In the mixed strategy equilibrium, there is social inefficiency that the bidder with the lower value obtains the good with a positive probability, and this results in smaller revenue.

On the contrary, the present dynamic setting possesses an efficient SPE in which the higher value bidder obtains the good with probability one. On the equilibrium-path of this SPE, the bidders do not take probabilistic actions; in the first period, the higher value bidder submits the opponent's value while the lower value bidder bids zero, and in the second period, both bid zero. This SPE induces the socially optimal allocation with proba-

[1] In the literature on procurement auctions, the buyer's bias toward a particular bidder is frequently discussed. See, for example, Hubbard and Paarsch (2009) and Rezende (2009). In the present paper, on the contrary, the seller treats all the bidders equally.

bility one and yields the same revenue as of the standard winner-pay auctions. However, this is not a unique SPE. It can be shown that the lower value bidder bids zero in the first period of any SPE, but the higher value bidder's best response cannot be uniquely determined. Specifically, it is a best response of the higher value bidder (in the reduced game) to bid any amount between zero and the opponent's value in the first period, given that the opponent bids zero in the first period. As a consequence, there exists a continuum of equilibria on which the bidders' payoffs are equivalent. It is also shown that the seller's revenue is *not* equivalent on the continuum of SPE: it is increasing in the higher value bidder's commitment. This is because, a large commitment by the higher value bidder induces a larger probability of his winning and thus, larger social welfare.

Although multiple equilibria exist, the seller can implement the efficient equilibrium that yields the highest revenue at an arbitrarily small cost. Since the bidders' expected payoffs are the same among the equilibria, giving an arbitrarily small but positive incentive suffices to implement the efficient equilibrium. In this paper, the seller's bias towards the bids in the first period is considered; i.e., the seller evaluates a payment of $1 - \varepsilon$ dollars in the first period and that of 1 dollar in the second period as equal. It is shown that under a sufficiently small degree of bias ε, the efficient equilibrium that yields the highest revenue of $(1 - \varepsilon)$ times the lower value becomes the unique SPE. Thus, taking $\varepsilon \to 0$, the seller can raise almost the same revenue as in the standard auctions. It should be noted that this mechanism works as well as the standard auctions, even if the seller does not know the bidders' values. Namely, given a distribution of the bidders' values, as the rate of bias approaches zero, (i) the probability that the "best" equilibrium is implemented converges to one and (ii) the revenue converges to that in the standard auctions at any realization. Therefore, the seller can earn an expected revenue arbitrarily close to that in the standard auctions, even though the bidders' values are unknown to her. This feature distinguishes the present model from existing all-pay mechanisms, as discussed in Section 3.3.

The paper is organized as follows. Section 2 describes the model. Section 3 presents the main results and discussions. Section 4 contains an extension of the model to a general cost environment. Section 5 concludes. All proofs are presented in the Appendix.

1.1 Related Literature

The present study relates to a wide variety of existing research on the all-pay auction and all-pay contests.[2] Hillman and Samet (1987) first apply the all-pay auction to political lobbying. Hillman and Riley (1989) and Baye et al. (1993, 1996) provide the full characterization of the mixed strategy equilibria in the all-pay auction under complete information. Siegel (2009) investigates the mixed strategy equilibria of the all-pay contests with complete information in a general environment. Konrad and Leininger (2007) characterize the equilibria under complete information in the Stackelberg-type sequential move contests, where some contestants move first, and then the remainder decide their own bids.

Che and Gale (1998) shows a related result within the static framework. They investigate the mixed strategy equilibria in the all-pay auction with complete information and asymmetric value when a bid cap is imposed. They show that the seller can make her expected revenue arbitrarily close to that in the first- and second-price sealed bid auctions. This seems similar to the present study, but two distinctions are worth mentioning. First, the social welfare is higher in the present mechanism than in the static model with the bid cap. Second, the seller needs some information about the bidders' values in order to increase her revenue imposing the bid cap.

A number of papers further investigate the effect of bid cap in the all-pay auctions and contests. Kaplan and Wettstein (2006) and Che and Gale (2006) consider the situation in

[2] The term "all-pay auction" in this paper refers to the first-price all-pay auction with linear bidding costs. Non-linear cost models are referred to as (all-pay) contests.

which a bid cap cannot be rigidly enforced; i.e., bidding costs are continuous even though they are kinked at the cap. Kaplan and Wettstein (2006) show that any bid cap (i.e., any upward shift of cost functions) reduces the expected revenue as long as costs and shifts are identical between the bidders. On the contrary, Che and Gale (2006) consider asymmetric bidding costs and present a sufficient condition that asymmetric shifts in costs raise the revenue. Gavious et al. (2002) examine bid caps in the standard contests with incomplete information in which the valuation of the reward is private information and contestants are ex ante symmetric.

Yildirim (2005) also provides a relevant result to the present study. He investigates the two-stage version of Tullock's (1980) auction and shows that there exists a continuum of SPEs.[3] In Tullock's auction, each bidder's winning probability is continuous and is usually assumed to be concave to ensure the existence of pure strategy equilibrium. Thus, the underlying game is different between Yildirim (2005) and the present paper.[4] Moreover, the present study puts emphasis on equilibrium selection and/or mechanism design perspective, whereas Yildirim (2005) considers endogenous information disclosure.

The dollar auction (Shubik, 1971; Leininger, 1989, 1991) is a model of dynamic all-pay auction. The major distinction between the present model and dollar auction can be summarized into two points. First, the players bid alternately in the dollar auction, whereas the bidders move simlutaneously at each stage in the present setting. Second, the dollar auction has no exgoneous final stage, while the horizon is fixed and known in the present model. The latter distinction is technically important because each theory exploits its ending rule. That is, the theory of dollar auction depends on its recursive structure, where the present paper employs backward induction.

[3]Tullock's model also possesses the all-pay feature and is intensively discussed in the literature. Recent research includes Baharad and Nitzan (2008), Fey (2008), Fu and Lu (2008), and Lim and Matros (2009).

[4]The intermediate cases where the winning probability is continuous but convex are not fully investigated probably because of technical difficulty. For research on such cases, see Baye et al. (1994) and Che and Gale (2000).

Finally, there also exist various papers on elimination tournaments (Amegashie and Runkel, 2007; Gradstein and Konrad, 1999; Groh et al., 2008; Moldovanu and Sela, 2006; Zhan and Wang, 2009) and on repeated competitions (Konrad and Kovenock, 2009; Sela 2008) in all-pay environments. These models also analyze some dynamic aspects of all-pay auctions and contests. However, in these models, each stage of the multi-stage game is a static competition.

2 The Model

Consider the following auction G of a single indivisible good. The set of bidders is $N = \{A, B\}$.[5] For each bidder $i \in N$, his valuation for the good is denoted by $v_i > 0$ and is common knowledge between the bidders. We focus on the asymmetric valuation case where $v_A \neq v_B$, and assume without loss of generality that $v_A > v_B$.[6] The time line is as follows. In the first period, each bidder i independently and simultaneously offers his first bid $b_i^1 \geq 0$. In the second period, after both bidders observe (b_A^1, b_B^1), they simultaneously submit the second bids denoted by $b_i^2 \geq 0$. Finally, the good is allocated to the bidder with the higher total bid $b_i^1 + b_i^2$. In case of a tie, each bidder obtains the good with an equal probability. Each bidder has to pay his total bid $b_i^1 + b_i^2$ even when he does not obtain the good. In summary, the payoff function for bidder i is given by

$$U_i\left((b_i^1, b_i^2), (b_j^1, b_j^2)\right) = \mathrm{Pr}_i \cdot v_i - (b_i^1 + b_i^2), \tag{1}$$

[5] The results in this paper does not depend on this two-bidder assumption, because (generically) only two bidders are active in an equilibrium. For the proof in the static model, see Baye et al. (1996). An analogue also holds in the present setting.

[6] The symmetric case in which $v_A = v_B$ is relatively uninteresting since herein, there is no allocational inefficiency. The results in the asymmetric case can be easily extended to the symmetric case.

where $j \in N \setminus \{i\}$ and $\Pr_i$ is the probability of i's winning, which is equal to 1 if $b_i^1 + b_i^2 > b_j^1 + b_j^2$, to $1/2$ if $b_i^1 + b_i^2 = b_j^1 + b_j^2$, and to 0 otherwise.

As in the usual static all-pay auction, the second stage subgames do not possess a pure strategy Nash equilibrium in general and thus, we have to consider mixed strategies. Throughout the paper, let F_i be a cumulative distribution function on $\mathbb{R}_+$ denoting i's mixed strategy in the second period.[7] Then, the payoff functions can be naturally extended to the domain of the mixed strategies as

$$U_i\left((b_i^1, F_i), (b_j^1, F_j)\right) := \iint U_i\left((b_i^1, b_i^2), (b_j^1, b_j^2)\right) dF_i(b_i^2)\, dF_j(b_j^2). \tag{2}$$

To simplify the description of the second stage subgames, we define

$$u_i(b_i^2, b_j^2; b_i^1, b_j^1) := U_i((b_i^1, b_i^2), (b_j^1, b_j^2)) + b_i^1. \tag{3}$$

Here, u_i represents i's payoff in the subgame after (b_i^1, b_j^1) exclusive of his sunk cost b_i^1. It can also be naturally extended to mixed strategies. In what follows, U_i and u_i are referred to as i's *net payoff* and *gross payoff*, respectively. Furthermore, b_i^1 and b_j^1 in u_i are omitted as long as they are given and fixed.

3 Results

3.1 Equilibrium Characterization

In this subsection, we investigate the SPEs of the two-stage all-pay auction G. To begin with, let us consider the Nash equilibria in the second stage subgames. Notice that in the second stage, bidder i still potentially has an incentive to bid $b_i^2 = v_i$, because

[7] We do not consider mixed strategies in the first period. This is just for simplicity and does not restrict the results at all.

b_i^1 is already sunk. Following Siegel's (2009) terminology, let us refer to $v_i + b_i^1$ as i's *reach*. Given (b_A^1, b_B^1), let $(H, L) = (A, B)$ or (B, A) so that H has a longer reach than L, i.e., $v_H + b_H^1 \geq v_L + b_L^1$. Siegel (2009) shows that in a general environment that includes the subgames of G as special cases, the equilibrium payoffs in the all-pay contests are determined by each player's reach. That is because the reach represents the range of undominated strategies, and the max-min payoff in undominated strategies (i.e., the payoff that one can reserve as long as the rivals take undominated strategies) determines the lower bound of the equilibrium payoff.[8] In fact, Siegel (2009, Theorem 1) proves that the equilibrium payoffs must be equal to these lower bounds. Although Siegel (2009) does not provide a characterization of equilibrium strategies that is directly applicable to the present setting, we can fully characterize the equilibria as follows.

If the difference of the bids in the first stage (i.e., $b_H^1 - b_L^1$) is so large that L does not have an incentive to submit a positive bid in order to catch up with H, both must bid zero in the second stage. The first proposition states this transparent fact.

Proposition 1. *Consider the second stage subgame after (b_H^1, b_L^1). If $b_H^1 - b_L^1 \geq v_L$, then the unique Nash equilibrium of the subgame is $(b_H^{2*}, b_L^{2*}) = (0, 0)$. The equilibrium gross payoffs are $(u_H^*, u_L^*) = (v_H, 0)$.*

Proof. See the Appendix. ■

In the other cases, the second stage subgame possesses no pure strategy Nash equilibrium. However, we can fully characterize the unique mixed strategy equilibrium along almost the same lines as the static all-pay auction.[9]

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[8]Namely, H and L can reserve the payoffs of $v_H + b_H^1 - v_L - b_L^1$ and 0 by bidding $v_L + b_L^1 - b_H^1$ and 0, respectively. (Recall that L does not have an incentive to bid $b_L^2 > v_L$.)

[9]Recall that the domain of F_i is restricted to $\mathbb{R}_+$. Thus, $F_i(0) > 0$ implies that a positive probability is assigned to $b_i^2 = 0$.

Proposition 2. *Consider the second stage subgame after* (b_H^1, b_L^1)*. If* $b_H^1 - b_L^1 < v_L$*, the unique mixed strategy equilibrium* (F_H^*, F_L^*) *of the subgame is characterized by*

$$F_H^*(b) = \min\left\{1, \frac{\max\{b + b_H^1 - b_L^1, 0\}}{v_L}\right\}, \text{ and} \tag{4}$$

$$F_L^*(b) = \min\left\{1, \frac{\Delta + \max\{b + b_L^1 - b_H^1, 0\}}{v_H}\right\}, \tag{5}$$

where $\Delta := v_H + b_H^1 - v_L - b_L^1$*. The equilibrium gross payoffs are* $(u_H^*, u_L^*) = (\Delta, 0)$*.*

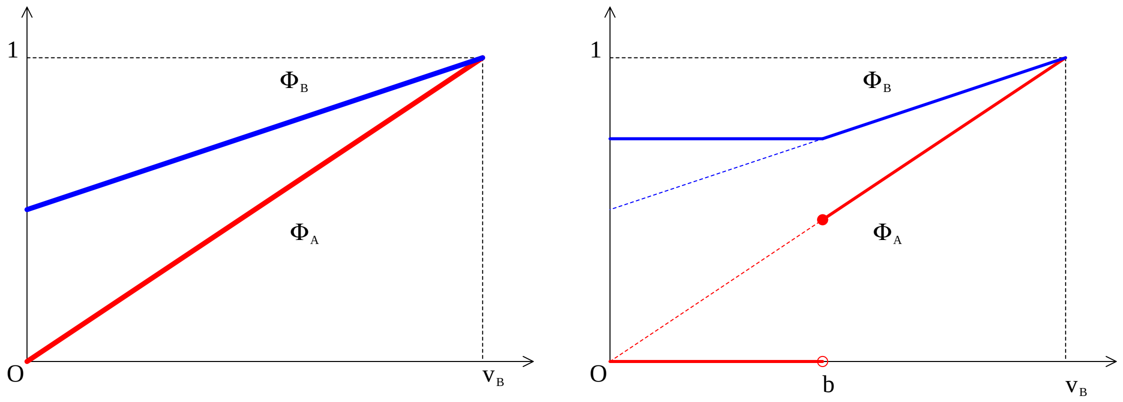
Proof. See the Appendix. ■

Let us briefly see how the equilibrium gross payoffs are determined. Let α_i and β_i denote i's lowest and highest bid in the support of his mixed strategy in an equilibrium, respectively; i.e., $\alpha_i := \min(\text{supp } F_i^*)$ and $\beta_i := \max(\text{supp } F_i^*)$. To begin with, recall that in a mixed strategy equilibrium, (almost) all bidding amounts in the support of his mixed strategy must be indifferent to each bidder. To sustain i's incentive to offer α_i, therefore, either (i) he can win with a positive probability even when he submits α_i or (ii) he earns the equilibrium gross payoff $u_i^* = 0$ so that he has an incentive to bid $\alpha_i = 0$ even though bidding zero yields the winning probability of zero. As a consequence, it is shown that at least one of the bidders earns the equilibrium gross payoff of zero. That is, both bidders can earn positive equilibrium payoffs only if both A and B win the reward with a positive probability by bidding α_A and α_B, respectively. This condition implies that $b_A^1 + \alpha_A = b_B^1 + \alpha_B$, and that F_A^* and F_B^* assign a positive mass at α_A and α_B, respectively. However, this is followed by a contradiction that each bidder i strictly prefers bidding slightly more than α_i to bidding α_i so as to increase the probability of winning discontinuously.

Hence, it follows that at least one of the bidders earns the equilibrium gross payoff zero. Given that one bidder i earns $u_i^* = 0$, the other bidder j must take a strategy so that i cannot earn a positive gross payoff by any bid. Then, $b_j^1 + \beta_j$ must be at least as large as i's reach $b_i^1 + v_i$, because otherwise i can profitably deviate by bidding $b_i^2 \in (b_j^1 + \beta_j - b_i^1, v_i)$. Since i has no incentive to offer any bid $b_i^2 > v_i$, it follows that $b_j^1 + \beta_j = b_i^1 + v_i$. Moreover, we can also conclude that j can win with probability one by bidding $b_j^2 = \beta_j$, because otherwise j prefers bidding slightly more than β_j to bidding β_j. These conditions pin down j's equilibrium gross payoff. Moreover, since j's equilibrium gross payoffs cannot be negative, it follows that $(i, j) = (L, H)$.

Using these results on the subgame equilibria, the SPEs of the whole game G can also be characterized. Notice that each bidder can reserve the net payoff $U_i = 0$ by bidding zero in both periods and thus, U_i^* must be non-negative in any equilibrium. Note also that by Propositions 1 and 2, at least one of the bidders must earn $u_i^* = 0$ in the subgame on the equilibrium-path. Recalling that $U_i \equiv u_i - b_i^1$, it follows that in any SPE, $U_i^* = 0$ and $b_i^{1*} = 0$ for (at least) one of the bidders. These arguments imply that $b_B^{1*} = 0$ in any SPE of G as follows. If $b_B^{1*} > 0$ in an SPE, it follows that $u_B^* > 0$ on the equilibrium-path and thus that $U_A^* = u_A^* = 0$ and $b_A^{1*} = 0$. Then, to sustain such an equilibrium, B's maximal total payment $b_B^{1*} + \beta_B$ must be equal to v_A, because otherwise A can profitably deviate by bidding (b_A^1, b_A^2) such that $b_A^1 + b_A^2 \in (b_B^{1*} + \beta_B, v_A)$. However, a clear contradiction occurs that B must earn a negative net payoff by offering a total payment of v_A even though he can obtain the reward.

Therefore, it is shown that $b_B^{1*} = 0$ in any SPE. On the contrary, b_A^{1*} cannot be uniquely determined. Unless $b_A^{1*} > v_B$, in the second stage, A must take a mixed strategy such that $b_A^{1*} + \beta_A = v_B$ in order to control B's incentive. In other words, no matter how much A commits in the first period, he is provided in the second stage an incentive to obtain the good by making a total payment of v_B. This implies that $U_A = v_A - v_B$ for any $b_A^1 \in [0, v_B]$, given that $b_B^1 = 0$. Since it is clearly suboptimal for A to bid $b_A^1 > v_B$, any $b_A^1 \in [0, v_B]$ is a best response to $b_B^1 = 0$ in the reduced game. The next proposition summarizes this equilibrium indeterminacy.



(a) SPE in which $b_A^{1*} = 0$.

(b) SPE in which $b_A^{1*} = b > 0$.

Figure 1: Distributions of total bids. The horizontal and vertical axes represent the total bid $b_i^1 + b_i^2$ and its cumulative distribution, respectively.

Proposition 3. *The two-stage all-pay auction G has a continuum of SPEs corresponding to each* $(b_A^1, b_B^1) \in [0, v_B] \times \{0\}$. *However, the equilibrium net payoffs are equivalent on the continuum:* $(U_A^*, U_B^*) = (v_A - v_B, 0)$.

Proof. See the Appendix. ∎

To observe how the equilibrium distributions of the bidders' total payments vary with b_A^{1*}, see Figure 1. Given (b_i^1, F_i), let Φ_i be the cumulative distribution of i's total bid $b_i^1 + b_i^2$; i.e., $\Phi_i(b_i^1 + b_i^2) = F_i(b_i^2)$. Figure 1 (a) shows the cumulative distributions Φ_A and Φ_B in the SPE where $b_A^{1*} = b_B^{1*} = 0$. Note that these distributions are equivalent to those of the equilibrium bids in the static all-pay auction. Figure 1 (b) illustrates Φ_A and Φ_B in the SPE where $(b_A^{1*}, b_B^{1*}) = (b, 0)$ with $b \in (0, v_B)$. As compared to Figure 1 (a), we can (roughly) check that these actually constitute an equilibrium as follows. Notice that Φ_i is identical between Figure 1 (a) and (b) on the interval $[b, v_B]$. Thus, the equilibrium conditions (in the second period) are satisfied on this interval. On the other hand, since A commits b in the first period, it is also obvious that B does not have an incentive to bid $b_B^2 < b$ and that A cannot decrease his total payment (in the second stage) so that $b_A^1 + b_A^2 < b$. Finally, both bidders clearly earn the same payoffs in the two SPEs.

Using these graphs, the next proposition that states the revenue non-equivalence on the continuum of SPEs is easily derived. Let Φ be a distribution function on $[0, T]$ given in the form of $\Phi(t) = \Phi(0) + \phi \cdot t$ for all $t \in [0, T]$. Such Φ is drawn in Figure 2 (a). Then, the expectation of t with respect to Φ is $\int_0^T t\, d\Phi(t) = \int_0^T t \cdot \phi\, dt = \int_0^T (\Phi(t) - \Phi(0))\, dt$, which is the shaded triangular region in Figure 2 (a). Applying this to Φ_i in Figure 1 (a) and (b), we can compare $\mathrm{E}[b_i^1 + b_i^2]$ in the two SPEs where $(b_A^{1*}, b_B^{1*}) = (0, 0)$ and $(b, 0)$. The differences in i's expected payment are illustrated in Figure 2 (b). The upper (resp. lower) shaded triangular region represents the decrease (increase) in B's (A's) expected payment. Since the lower triangular region is larger than the upper one, the sum of payments (i.e., the seller's revenue) is larger in the SPE with $(b_A^{1*}, b_B^{1*}) = (b, 0)$ than in the SPE with $(b_A^{1*}, b_B^{1*}) = (0, 0)$. It is also obvious from Figure 2 (b) that the revenue is monotonically increasing in b_A^{1*}. More precisely, the seller's expected revenue is given as follows.

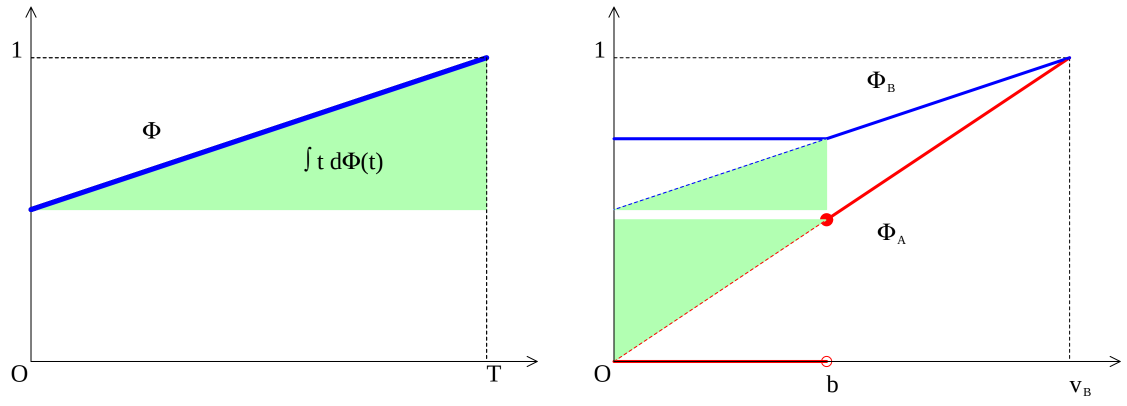
Proposition 4. *Let $R(b)$ denote the expected revenue of the seller in the SPE with $b_A^{1*} = b$. Then, $R(b)$ is given by*

$$R(b) = \frac{(v_A + v_B)v_B}{2v_A} + b^2 \cdot \frac{v_A - v_B}{2v_A v_B}, \tag{6}$$

which is increasing in b.

Proof. See the Appendix. ∎

The intuition behind Proposition 4 is as follows. Although A's large commitment does not benefit A himself, it has an externality. Recall that it is socially optimal that A obtains the good because $v_A > v_B$. Then, since a larger b_A^1 induces a longer reach of A and thus a larger probability of A's winning, an increase in b_A^1 enlarges social welfare. Since the



(a) Example of linear distribution.

(b) Differences in $\mathrm{E}[b_i^1 + b_i^2]$.

Figure 2: Expectation with respect to a linear distribution and the differences in the expected total bids between the two SPEs.

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payoffs are the same across all SPEs, the increment in social welfare must result in a rise in the seller's revenue. That is, the seller's expected revenue is increasing in b_A^{1*}.

Let us compare $R(b)$ with the revenues in other auctions. It is obvious that $R(0) = (v_A + v_B)v_B/2v_A$ is equal to the expected revenue in the static all-pay auction. Thus, any SPE in the present setting yields a (weakly) higher revenue than in the static all-pay auction. Notice also that $R(v_B) = v_B$ is the revenue of the first- and second-price auctions. Hence, if the SPE in which $b_A^{1*} = v_B$ is realized, the present auction performs as well as the standard auctions from the seller's point of view. In the next subsection, a perturbed mechanism that implements this efficient SPE is considered.

3.2 Bias towards Commitments

In this subsection, we consider a class of perturbed mechanisms by which the seller can implement the efficient equilibrium eliminating all the other inefficient equilibria. Given a parameter $\varepsilon \in (0, 1/2)$, the corresponding perturbed game G^ε is formulated by replacing the payoff function U_i in the original game G with

$$U_i^{\varepsilon}\left((b_i^1, F_i), (b_j^1, F_j)\right) := U_i\left((b_i^1, F_i), (b_j^1, F_j)\right) + \varepsilon \cdot b_i^1. \tag{7}$$

The restriction $\varepsilon < 1/2$ is to guarantee the existence of a pure strategy equilibrium in the reduced game.

The intended interpretation of G^{ε} is as follows. The seller is biased towards the payments of b_i^1 committed in the first period, in the sense that a payment of $1 - \varepsilon$ dollars in the first period has the same effect as that of 1 dollar in the second period. In this sense, ε represents the degree of the seller's bias towards commitments. When bidder i offers (b_i^1, b_i^2), he only needs to pay $(1 - \varepsilon)b_i^1 + b_i^2$, and the seller's *net* revenue is given by

$$(b_A^1 + b_A^2 + b_B^1 + b_B^2) - \varepsilon(b_A^1 + b_B^1). \tag{8}$$

However, bidder i's winning probability $\Pr_i$ still depends on $b_i^1 + b_i^2$. For example, if $\varepsilon = .1$ and $((b_A^1, b_A^2), (b_B^1, b_B^2)) = ((100, 10), (0, 105))$, A and B pay 100 and 105, respectively, although A wins the reward.

In the original auction G, a bid b_i^t has two aspects. On the one hand, it represents i's payment and thus his cost. On the other hand, it also refers to the score on which the allocation of the reward depends. Technically, the only important point in the definition of G^{ε} is that b_i^1 and b_i^2 are not perfectly substitutable from the viewpoint of i's cost, but are so from the viewpoint of the score. The gap between the two aspects represents the seller's bias.[10]

[10] This gap should not be regarded as a consequence of interest. If interest rate r is not negligible, 1 dollar in the first period is worth $1 + r$ dollars in the second period. Thus, one might feel natural that the seller assigns a larger score to 1 dollar in the first stage rather than that in the second stage. However, in such a case, the bidder's cost should be measured by the present value rather than the face value and thus, 1 dollar in the first period and $1 + r$ dollars in the second period are identical also from the viewpoint of bidder's cost. What is assumed here is that in such a case, bidding 1 dollar in the first period is better than bidding $1 + r$ dollars in the second period from the viewpoint of score.

Recall that the bidders earn the same payoffs across the continuum of SPEs in the original game G. The multiple equilibria arise due to the multi-valuedness of A's best response. Hence, small incentives provided by a small ε suffice to induce the maximal commitment. More specifically, under a sufficiently and arbitrarily small ε, the revenue maximising SPE becomes the unique equilibrium. The seller can make her expected net revenue arbitrarily close to v_B by imposing a sufficiently small ε.

Proposition 5. *For any $\varepsilon \in (0, 1/2)$, the biased auction G^ε has an SPE of which outcome is $((b_A^{1*}, b_A^{2*}), (b_B^{1*}, b_B^{2*})) = ((v_B, 0), (0, 0))$. This SPE is unique if $\varepsilon \notin [\frac{v_A - v_B}{v_A}, \frac{v_B}{v_A + v_B}]$; otherwise, there is another SPE whose outcome is $((b_A^{1*}, b_A^{2*}), (b_B^{1*}, b_B^{2*})) = ((0, 0), (v_A, 0))$.*

Proof. See the Appendix. ■

Proposition 5 implies that the seller can raise almost the same revenue as in the standard auctions, maintaining the all-pay feature of the mechanism. As discussed in the next subsection, there exist other all-pay mechanisms that yield (almost) the same revenue. Before comparing these mechanisms, one property of the present mechanism should be noted: it is robust in the sense that it is independent of the seller's knowledge of the bidders' values. For a moment, ignore the assumption that $v_A > v_B$ and suppose that (v_A, v_B) is randomly drawn from a probability distribution. Suppose also that at each realization, (v_A, v_B) is known to both bidders but *not* to the sellers. Even then, as ε approaches zero, at any realization of (v_A, v_B), the seller's expected revenue converges to $\min\{v_A, v_B\}$, which is equivalent to that in the standard auctions.[11] This property distinguishes the present mechanism from other all-pay mechanisms, as discussed in the next subsection.

[11] In the symmetric case of $v_A = v_B$, two SPEs exist under any $\varepsilon > 0$. However, the two SPEs yield the same revenue. Therefore, the probability of an SPE being unique does not converge to one if $v_A = v_B$ with a positive probability, but this does not matter from the viewpoint of the revenue.

3.3 Discussion

In this subsection, the results in the previous subsections are compared with other all-pay mechanisms that yield the same revenue as the standard auctions. In the context of political lobbying or rent seeking, the seller (politician) has to design an all-pay mechanism. This is so because the politician cannot explicitly "sell" a rent in the literal sense, and the fund must be raised as if it is irrelevant to the rent.

3.3.1 Comparison with the optimal bid cap

Che and Gale (1998) show that in the static all-pay auction with complete information, the seller can increase the revenue by imposing a bid cap. Let m denote the level of bid cap, i.e., the maximum permissible bidding amount. They demonstrate that the revenue converges to v_B as $m \nearrow v_B/2$, while there exists a continuum of equilibria under $m = v_B/2$. This might seem similar to Propositions 3–5 in the present study. However, comparing the two mechanisms, two points should be noted.

First, the social welfare is higher in the present dynamic mechanism than in the static all-pay auction with the optimal bid cap. Note that the socially optimal allocation (i.e., A wins the reward) realizes with probability one in the present mechanism under sufficiently small ε. Thus, the social surplus is always maximized. In contrast, as Che and Gale (1998) emphasize, a bid cap increases the revenue only by decreasing the welfare. Under a bid cap $m < v_B/2$, lower value bidder B has a larger chance to win than when without the cap.[12] This increases B's expected bid, and hence the revenue. However, since $v_A > v_B$, a higher probability of B winning the reward implies a larger social inefficiency. Therefore, the bid cap decreases the welfare although it increases the revenue.

Second, the performances of the two mechanisms are different even for the seller, if

[12]Note that when $m < v_B/2$, B can reserve a positive payoff because he can win at least with probability $1/2$ by bidding m.

she does not know v_A and v_B. As discussed above, the present mechanism works even in such cases. On the contrary, the bid cap does not do the same. Since the optimal level of bid cap depends on (v_A, v_B), any fixed m cannot raise the revenue of $\min\{v_A, v_B\}$ with probability one when v_A and v_B are uncertain for the seller.

3.3.2 Comparison with the Stackelberg auction

There is another all-pay mechanism that yields the highest expected revenue v_B: the Stackelberg auction studied by Konrad and Leininger (2007). Consider a sequential move game in which A only bids in the first period and B only bids in the second period. In this setting, there is a unique equilibrium where A and B bid v_B and 0 on the equilibrium-path, respectively. This outcome in the Steckelberg auction might seem similar to that of the efficient SPE in the present model.

However, this mechanism also has a problem when (v_A, v_B) is unknown to the seller. It requires the seller to know which bidder has the larger (or the largest if there are more than two bidders) value. If the seller let the lower value bidder move first, the unique equilibrium revenue becomes zero.[13] Thus, when the seller does not know the rank order between the bidders' valuations, she is unable to determine which bidder should move first.

4 Extension

In this section, an extension of the results in the previous section to a generalized bidding cost environment is considered. Although the linear cost is frequently assumed in the literature and may seem plausible as a model of lobbying, such a generalization could

[13] As Konrad and Leininger (2007) show, this unpleasant timing is likely to realize when the timing is endogenized by the observable-delay game. On the contrary, the present model corresponds to the action-commitment game. See Hamilton and Slutsky (1990) for the original formulation of the two timing games.

be of interest possibly because of, for example, non-risk-neutral lobbyists, incomplete financial markets, or non-linear transaction costs.[14] Furthermore, it would be important to consider non-linear costs when we apply the all-pay auction to other competitions such as promotion tournaments where the bids are not interpreted as monetary payments.

The generalized contest Γ is the game defined by replacing U_i in the original model G with

$$U_i((b_i^1, b_i^2), (b_j^1, b_j^2)) = \Pr_i \cdot v_i - C(b_i^1 + b_i^2), \tag{9}$$

where $\Pr_i$ is defined the same as in the previous sections and $C : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing and twice continuously differentiable function with $C(0) = 0$. As in the previous section, the gross payoffs in the second stage are defined by $u_i(b_i^2, b_j^2) := U_i + C(b_i^1)$, given b_i^1 and b_j^1.

Notice that the definition of Γ does not exclude a certain class of cost asymmetries. Suppose that i's bidding cost is $C_i = \theta_i \cdot C$, where $\theta_i \in \mathbb{R}_+$ is a parameter representing i's efficiency. Even if $\theta_A \neq \theta_B$, the present model can incorporate the cost asymmetry by replacing v_i and C_i with v_i/θ_i and C, respectively. Note also that this transformation does not affect the scale of equilibrium bids; i.e., the implications on the seller's revenue do not change.

The assumption that the bidding cost only depends on the total bid $b_i^1 + b_i^2$ might be more restrictive. This means that, given an amount of total bid $b_i^1 + b_i^2$, the division into b_i^1 and b_i^2 does not matter. This might seem non-plausible in some situations other than lobbying, in particular when the bids represent non-monetary effort. However, as a model of political lobbying and fund-raising in which b_i^t is a monetary transfer, there is a possible justification. When the bids are monetary, b_i^t should be regarded as the present value of the transfer in period t, as discussed in footnote 10. Then, it would not be peculiar to assume that the cost depends only on the total present value of the transfers $b_i^1 + b_i^2$,

[14] Fibich et al. (2006) study the all-pay auction among risk-averse bidders under incomplete information.

even though C is non-linear.

As in the previous section, given (b_A^1, b_B^1), let $(H, L) = (A, B)$ or (B, A) so that $v_H + C(b_H^1) \geq v_L + C(b_L^1)$.[15] Then, Propositions 1 and 2 are naturally extended as follows.

Proposition 6. *Consider the second stage subgame after (b_H^1, b_L^1). Then, the unique equilibrium is characterized as follows. (A) When $C(b_H^1) - C(b_L^1) \geq v_L$, then the equilibrium is given by $b_H^{2*} = b_L^{2*} = 0$. The equilibrium gross payoffs are $(u_H^*, u_L^*) = (v_H, 0)$. (B) Otherwise, the unique equilibrium mixed strategies are given by*

$$F_H^*(b) = \min\left\{1, \frac{C\left(M(b) + b_L^1\right) - C(b_L^1)}{v_L}\right\}, \text{ and} \tag{10}$$

$$F_L^*(b) = \min\left\{1, \frac{\Delta + C\left(m(b) + b_L^1\right) - C(b_L^1)}{v_H}\right\}, \tag{11}$$

where $M(b) := \max\{b + b_H^1 - b_L^1, 0\}$, $m(b) := \max\{b + b_L^1 - b_H^1, 0\}$, and $\Delta := v_H + C(b_H^1) - v_L - C(b_L^1) \geq 0$. The equilibrium gross payoffs are $(u_H^*, u_L^*) = (\Delta, 0)$.

Proof. See the Appendix. ■

Further, an analogue of Proposition 3 also holds in this environment.

Proposition 7. *The generalized contest Γ has a continuum of SPEs corresponding to each $(b_A^1, b_B^1) \in [0, C^{-1}(v_B)] \times \{0\}$. However, the equilibrium net payoffs are equivalent on the continuum: $(U_A^*, U_B^*) = (v_A - v_B, 0)$.*

Proof. See the Appendix. ■

It should be noted that even in this generalized setting, the SPEs with $(b_A^1, b_B^1) = (0, 0)$ and $(C^{-1}(v_B), 0)$ correspond to the equilibrium outcomes in the static all-pay contest and the standard winner-pay auctions under bidding cost C, respectively. However, in this

[15] Note that $v_i + C(b_i^1)$ is the maximal cost that i would pay given that $C(b_i^1)$ is sunk. Notice also that a larger total cost is equivalent to a larger total bid because C is increasing.

generalized environment, the SPE with $(b_A^1, b_B^1) = (C^{-1}(v_B), 0)$ may not be the best for the seller. In other words, Proposition 4 can only be partially extended. The seller's revenue may not be monotonically increasing in b_A^{1*}, although the sum of the bidders' costs is.

Proposition 8. *In the SPE with $b_A^{1*} = b$, the expected revenue is given by*

$$R(b) = \int_0^{C^{-1}(v_B)} \frac{v_A + v_B}{v_A v_B}\, t \cdot C'(t)\, dt + \int_0^b \left(\frac{b}{v_B} - \frac{(v_A + v_B)t}{v_A v_B} \right) C'(t)\, dt, \tag{12}$$

which may or may not be increasing in b. On the contrary, the sum of the expected costs, $\mathrm{E}[C(b_A^1 + b_A^2)] + \mathrm{E}[C(b_B^1 + b_B^2)]$, is given by

$$X(b) = \frac{v_B(v_A + v_B)}{2v_A} + \frac{v_A - v_B}{2v_A v_B}[C(b)]^2, \tag{13}$$

which is increasing in b.

Proof. See the Appendix. ■

We have to distinguish the seller's revenue and the bidders' costs in the generalized model Γ that are equivalent in the linear model G. If we replace the revenue with cost in Proposition 4, it is fully extended by the second half of Proposition 8. Even with a general cost function, a larger commitment of the high value bidder yields a higher probability of optimal allocation. Since the equilibrium payoffs are the same across SPEs, the benefit from optimal allocation must be offset by the increment in the expected costs. However, when C is non-linear, the increment in the expected costs does not necessarily mean that the expected revenue increases.

That is, the reason why Proposition 4 cannot be fully extended is that there is another source of social inefficiency in the generalized cost environment. Even when taking the revenue $(b_A^1 + b_A^2) + (b_B^1 + b_B^2)$ as fixed, its allocation between $(b_A^1 + b_A^2)$ and $(b_B^1 + b_B^2)$

matters under a non-linear C. For example, given that the revenue is R, it is socially optimal that either bidder pays all if C is concave and that A and B equally pay $R/2$ if C is convex.

The seller's bias still works in the sense that she can raise the revenue arbitrarily close to that in the standard auctions. It is easily seen that the seller can implement the SPE with $((b_A^{1*}, b_A^{2*}), (b_B^{1*}, b_B^{2*})) = ((v_B, 0), (0, 0))$ by a sufficiently small rate of bias towards commitments even in Γ. However, when C is non-linear, Proposition 8 states that the revenue under the maximal commitment SPE may not be the highest. In other words, the simple bias for commitments may not implement the best equilibrium for the seller.

The last proposition provides a sufficient condition that the revenue is monotonically increasing in b_A^{1*} for any (v_A, v_B) and thus, the maximal commitment SPE is the best for the seller. The property that the revenue is increasing in b_A^{1*} regardless of (v_A, v_B) is of importance from the viewpoint of the robustness of the mechanism. Under this property, using the simple bias, the seller can raise the revenue as if she knows the bidders' values and can implement the best equilibrium, even though she does not have such information. Note also that the concavity of C is sufficient for the condition in Proposition 9.[16]

Proposition 9. *The expected revenue $R(b)$ is increasing in b for any (v_A, v_B), if the elasticity of C is globally less than one, i.e., $(xC'(x))/C(x) \leq 1$ for all $x \in \mathbb{R}_{++}$.*

Proof. See the Appendix. ■

To conclude this section, two situations in which the conditions of Proposition 9 would be plausible are mentioned. First, there might exist a fixed cost to offer a positive bid, or a concave transaction cost. Suppose that $C(b) = I(b > 0) \cdot c + b$, where $c > 0$ and I is an index function that takes one if $b > 0$ and zero, otherwise. This function is discontinuous

[16] Since $C(0) = 0$, if C is concave, $x \cdot C'(x) \leq C(x)$ for all $x \in \mathbb{R}_{++}$.

and thus does not satisfy the assumptions made above. However, we can approximate it arbitrarily well by a continuous concave function. Therefore, Proposition 9 provides an insight on such a situation with a fixed or concave transaction cost. Moreover, it is easily checked that the revenue is also increasing in the limit where C is discontinuous. Second, in other applications of the all-pay contest such as R&D races, there could be increasing returns to scale, which can be represented by the concavity of C.

5 Concluding Remarks

This paper examines the role of commitment in political lobbying using a simple model of two-stage all-pay auction. When the bidders have a chance of commitment, the politician can overcome the under-dissipation problem, which is a central property of the static all-pay auction. Specifically, the seller can raise almost the same revenue as in the standard winner-pay auctions by having an arbitrarily small degree of bias. Moreover, this dynamic mechanism is robust in the sense that it does not require the seller to have any information about the bidders' values. These results provide a new insight to the literature, which mainly investigates static models, elimination tournaments, or repeated contests.

In the literature, the campaign expenditure of lobbyists is often regarded as a social waste rather than the politician's revenue. From such a perspective, the equilibria in the dynamic all-pay auction is indifferent for the society. A larger commitment of the high value bidder implies a larger social waste, but it is offset by the allocational efficiency. However, as long as the campaign expenditure has *some* benefit to the society, the maximal commitment equilibrium is the most preferable.

To conclude, let us discuss the assumptions made in the present paper. First, it is worth mentioning that we can extend the present results to general numbers of bidders and periods. As noted in footnote 5, adding additional bidders does not change the results, because only two bidders make a positive bid in equilibria as in the static all-pay auctions. Moreover, the results do not depend on the two-period setting, either. Adding additional periods just increases the dimension of continuum of equilibria. For example, in a three-period model, for each (b^1, b^2) with $b^1 + b^2 \leq v_B$, there exists an SPE in which higher value bidder A bids b^1 in the first period and b^2 in the second period, on the equilibrium-path. Therefore, we can extend the present results to a general n-bidder T-period model. On the contrary, it might be crucial to assume that the horizon is fixed and certain, because the equilibrium characterization in the present paper depends on backward induction. It remains for future research to analyze models with uncertain horizons.

Second, although the present mechanism does not require the seller to have any information, it is assumed that the values are common knowledge among bidders. It is also of interest to investigate dynamic incomplete-information models in future research. Under incomplete information, early commitments can play a different role because bids in earlier stages can reveal bidders' private information.

Appendix: Proofs

Proof of Proposition 1.

The statement is obvious, because any $b_L^2 > 0$ is strongly dominated by bidding zero. ■

Proof of Proposition 2.

Let (F_H^*, F_L^*) and (u_H^*, u_L^*) be a Nash equilibrium and the pair of gross equilibrium payoffs, respectively. For each $i \in \{H, L\}$, define $S_i := \operatorname{supp} F_i^*$ and $(\alpha_i, \beta_i) := (\min S_i, \max S_i)$. Notice that,

$$u_i^* = \lim_{b \searrow \alpha_i} u_i(b; F_j^*) = \lim_{b \nearrow \beta_i} u_i(b; F_j^*), \tag{14}$$

must hold by definition.[17]

First of all, we show that $b_H^1 + \beta_H = b_L^1 + \beta_L$. By way of contradiction, suppose $b_i^1 + \beta_i > b_j^1 + \beta_j$. Then, since $b_j^1 + b_j^2 \leq b_j^1 + \beta_j$ with probability one, any $b_i^2 > b_j^1 - b_i^1 + \beta_j$ cannot be a best response to F_j^*, which is a contradiction to the supposition.

Then, it directly follows that β_i is not a mass point of F_i^* for all $i \in \{H, L\}$. This is because, if β_i is a mass point of F_i^*,

$$\lim_{b \nearrow \beta_j} u_j(b; F_i^*) < u_j(\beta_j; F_i^*) < \lim_{b \searrow \beta_j} u_j(b; F_i^*), \tag{15}$$

which is a contradiction to equation (14) and the equilibrium condition.

Second, it is demonstrated that $\beta_L = v_L$. On the contrary, suppose that $\beta_L < v_L$. (It is clear that $\beta_L \leq v_L$ because any $b_L^2 > v_L$ is strongly dominated by $b_L^2 = 0$.) Then, it follows that $\beta_H < v_H$, because $b_H^1 + v_H \geq b_L^1 + v_L$ by definition and $b_H^1 + \beta_H = b_L^1 + \beta_L$. Therefore, for each $(i, j) \in \{(H, L), (L, H)\}$, equation (14) implies

$$u_i^* = \lim_{b \nearrow \beta_i} u_i(b; F_j^*) = v_i - \beta_i > 0, \tag{16}$$

since β_j is not a mass point of F_j^*. That is, $u_H^*, u_L^* > 0$. Now, assume without loss of generality that $b_i^1 + \alpha_i \leq b_j^1 + \alpha_j$. If $b_i^1 + \alpha_i < b_j^1 + \alpha_j$ or α_j is not an atom of F_j^*, then,

$$\lim_{b \searrow \alpha_i} u(b; F_j^*) = u_i(\alpha_i; F_j^*) = -\alpha_i, \tag{17}$$

which is a contradiction to equations (14) and (16). Hence, the only possible case is that $b_i^1 + \alpha_i = b_j^1 + \alpha_j$ and α_j is an atom of F_i^*. In order that α_j is an atom of F_j^*, a similar discussion yields that α_i is an atom of F_i^*, which implies $u_i(\alpha_i; F_j^*) = u_i^*$. Then, however,

[17] This statement follows from an analogue of Pitchik's (1982) Lemma 1 and the transparent fact that $b_A^1 + b_A^2 = b_B^1 + b_B^2$ only with probability zero in an equilibrium.

a contradiction occurs that

$$u_i^* = u_i(\alpha_i; F_j^*) < \lim_{b \searrow \alpha_i} u(b; F_j^*). \tag{18}$$

Thus, we have obtained that $\beta_L = v_L$. Now, we can pin down the equilibrium gross payoffs. Recalling that β_i is not a mass point of F_i^* for any i, it follows that

$$u_H^* = \lim_{b \nearrow \beta_H} u_H(b; F_L^*) = \Delta \qquad \text{and} \qquad u_L^* = \lim_{b \nearrow \beta_L} u_L(b; F_H^*) = 0, \tag{19}$$

where $\Delta := (b_H^1 + v_H) - (b_L^1 + v_L)$.

Third, it is shown that F_i^* must be continuous on $(0, \beta_i)$ for each $i \in \{H, L\}$. By way of contradiction, suppose that F_i^* has a mass point at $\bar{b}_i \in (0, \beta_i)$. It is clear that $\bar{b}_i \geq b_j^1 - b_i^1$, because otherwise $\bar{b}_i$ is a strongly dominated strategy. Moreover, $\bar{b}_i = b_j^1 - b_i^1$ also yields a contradiction, since it implies that $u_i(\bar{b}_i; F_j^*) < 0$ if $F_j^*(0) = 0$ and that $u_i(\bar{b}_i; F_j^*) < \lim_{b \searrow \bar{b}_i} u_i(b; F_j^*)$ otherwise. Hence, the only possible case is $\bar{b}_i > b_j^1 - b_i^1$. Let $\widetilde{b}_j = \bar{b}_i + b_i^1 - b_j^1 > 0$. Then, it follows that $\lim_{b \nearrow \widetilde{b}_j} u_j(b; F_i^*) < \lim_{b \searrow \widetilde{b}_j} u_j(b; F_i^*)$ and thus, F_j^* must be constant on a left neighborhood of $\widetilde{b}_j$. However, it implies that there exists $\bar{\bar{b}}_i < \bar{b}_i$ such that $u_i(\bar{\bar{b}}_i; F_j^*) > u_i(\bar{b}_i; F_j^*)$, which is a contradiction.

Finally, let us explicitly solve for (F_H^*, F_L^*). Notice that as long as $b + b_i^1 > b_j^1$,

$$u_i(b; F_j^*) = F_j^*(b + b_i^1 - b_j^1) v_i - b, \tag{20}$$

and hence, the equilibrium condition, $u_i(b; F_j^*) \leq u_i^*$, implies

$$F_j^*(b + b_i^1 - b_j^1) \leq \frac{u_i^* + b}{v_i}. \tag{21}$$

Therefore, for $b' \geq \max\{b_i^1 - b_j^1, 0\}$, it follows that

$$F_j^*(b') \leq \frac{u_i^* + (b' + b_j^1 - b_i^1)}{v_i}. \tag{22}$$

By way of contradiction, suppose that there exists $\widehat{b} \in (\max\{b_i^1 - b_j^1, 0\}, \beta_j)$ such that

$$F_j^*(\widehat{b}) < \frac{u_i^* + (\widehat{b} + b_j^1 - b_i^1)}{v_i}. \tag{23}$$

Notice that we can take $\widehat{b} \in S_j$ without any loss of generality, because of the continuity of F_j^*.[18] Moreover, the continuity also implies that there exists an open interval $I \ni \widehat{b}$ such that

$$F_j^*(b) < \frac{u_i^* + (b + b_j^1 - b_i^1)}{v_i} \text{ for all } b \in I. \tag{24}$$

Then, $u_i(b + b_j^1 - b_i^1; F_j^*) < u_i^*$ for any $b \in I$ and thus, F_i^* must be constant on the interval $(\inf I + b_j^1 - b_i^1, \sup I + b_j^1 - b_i^1)$. However, it follows that for bidder j, any strategy $b \in I$ is strongly dominated by $\inf I$, which is a contradiction to the assumption that $\widehat{b} \in S_j$. Hence, we can conclude that

$$F_j^*(b) = \frac{u_i^* + (b + b_j^1 - b_i^1)}{v_i}, \tag{25}$$

for all $b \in (\max\{b_i^1 - b_j^1, 0\}, \beta_i)$ and hence, for all $b \in [\max\{b_i^1 - b_j^1, 0\}, \beta_i]$ by continuity. When $b_i^1 - b_j^1 > 0$, any $b \in (0, b_i^1 - b_j^1)$ is a strongly dominated strategy for j, and thus F_j^* must be constant on $(0, b_i^1 - b_j^1)$.

[18]This can be formally shown as follows. Let $B := \{b : F_j^*(b) < [u_i^* + (b + b_j^1 - b_i^1)]/v_i\}$. By way of contradiction, suppose that $B \neq \varnothing$ and $B \cap S_j = \varnothing$. Since B is open and S_j is closed, for an arbitrary $b \in B$, there exists an open interval $I \ni b$ such that $I \subset B \cap S_j^c$. Let I^*, which is an open interval itself, be the union of all such intervals. Then, it is clear that $\sup I^* \in \partial(B \cap S_j^c) \subset \partial B \cup \partial S_j^c$. Moreover, by the supposition that $B \cap S_j = \varnothing$ and the fact that $\partial S_j^c = \partial S_j \subset S_j$, a contradiction occurs if $\sup I^* \in B^\circ \cap \partial S_j^c$. Therefore, $\sup I^* \in \partial B = \{b : F_j^*(b) = [u_i^* + (b + b_j^1 - b_i^1)]/v_i\}$. However, since F_j^* must be constant on I^*, this gives rise to a clear contradiction to the definition of I^*.

As a summary, if (F_H^*, F_L^*) is an equilibrium, it follows that

$$F_H^*(b) = \min\left\{1, \frac{\max\{b + b_H^1 - b_L^1, 0\}}{v_L}\right\}, \tag{26}$$

$$F_L^*(b) = \min\left\{1, \frac{\Delta + \max\{b + b_L^1 - b_H^1, 0\}}{v_H}\right\}. \tag{27}$$

It can be easily checked that these actually constitute an equilibrium. ■

Proof of Proposition 3.

To begin with, it is shown that $b_B^1 = 0$ is B's strongly dominant strategy in the reduced game. Fix an arbitrary $\bar{b}_B^1 > 0$. If $b_A^1 + v_A \geq \bar{b}_B^1 + v_B$, B's gross payoff in the second stage is $u_B^* = 0$ and thus, his net payoff is $U_B = -\bar{b}_B^1 < 0$. If $b_A^1 + v_A < \bar{b}_B^1 + v_B$, he can earn a positive gross payoff, $(\bar{b}_B^1 + v_B) - (b_A^1 + v_A)$, but his net payoff is still negative: $U_B = v_B - (v_A + b_A^1) < 0$. On the contrary, if $b_B^1 = 0$, B's gross payoff u_b^* in the second stage equilibrium is zero no matter what b_A^1 is, and thus so is his net payoff U_B. Therefore, $b_B^1 = 0$ dominates $\bar{b}_B^1$.

Then, what remains to examine is A's best response to $b_B^1 = 0$ in the reduced game. Given $b_B^1 = 0$, A's gross equilibrium payoff in the second stage subgame is v_A if $b_A^1 \geq v_B$ and $b_A^1 + v_A - v_B$ otherwise. Thus, his net payoff is $U_A = v_A - v_B$ as long as $b_A^1 \in [0, v_B]$ and $U_A = v_A - b_A^1 < v_A - v_B$ otherwise. This implies that any $b_A^1 \in [0, v_B]$ is a best response to $b_B^1 = 0$ and the proof is complete. ■

Proof of Proposition 4.

Let (F_A^0, F_B^0) denote the equilibrium mixed strategies in the subgame after $(b_A^1, b_B^1) = (0, 0)$. That is, $F_A^0(b) = b/v_B$ and $F_B^0(b) = (b + v_A - v_B)/v_A$ on $[0, v_B]$. Fix an arbitrary $(b_A^1, b_B^1) \in [0, v_B] \times \{0\}$. Then, using the results in Proposition 2, the distributions of

$b_A^1 + b_A^2$ and $b_B^1 + b_B^2$ are given by

$$\Phi_A(b) = \begin{cases} 0 & \text{for } b \in [0, b_A^1] \\ b/v_B & \text{for } b \in [b_A^1, v_B] \end{cases} \tag{28}$$

and,

$$\Phi_B(b) = \begin{cases} (b_A^1 + v_A - v_B)/v_A & \text{for } b \in [0, b_A^1] \\ (b + v_A - v_B)/v_A & \text{for } b \in [b_A^1, v_B]. \end{cases} \tag{29}$$

Note that $F_i^0(b) = \Phi_i(b)$ for $b \in [b_A^1, v_B]$. Therefore, the difference in expected revenue is

$$\begin{aligned} R(b_A^1) - R(0) &= \int_0^{b_A^1} \left[\frac{b_A^1 - t}{v_B} - \frac{t}{v_A} \right] dt \\ &= \left(b_A^1\right)^2 \left(\frac{1}{v_B} - \frac{1}{2v_B} - \frac{1}{2v_A} \right) \\ &= \left(b_A^1\right)^2 \frac{v_A - v_B}{2v_A v_B}. \end{aligned} \tag{30}$$

Since

$$R(0) = \int_0^{v_B} t \, dF_A^0(t) + \int_0^{v_B} t \, dF_B^0(t) = \frac{v_B(v_A + v_B)}{2v_A}, \tag{31}$$

we obtain

$$R(b_A^1) = \frac{v_B(v_A + v_B)}{2v_A} + \left(b_A^1\right)^2 \frac{v_A - v_B}{2v_A v_B}, \tag{32}$$

and the proof is complete. ■

Proof of Proposition 5.

Notice that the bias does not affect the second stage gross payoff u_i. Thus, the equilibria of the subgames do not change and Propositions 1 and 2 also hold in G^ε. That is, we only need to investigate the reduced game.

Notice that there exists $i \in \{A, B\}$ such that $u_i^* = 0$ in any subgame. If $u_i^* = 0$ on an equilibrium-path, it clearly implies $b_i^{1*} = 0$. That is, in any (pure strategy) equilibrium of the reduced game, there exists i with $b_i^{1*} = 0$.

Therefore, it is easily seen that the only candidates of equilibria of the reduced game are $(b_A^{1*}, b_B^{1*}) = (v_B, 0)$ and $(0, v_A)$. In the former case, the equilibrium conditions are

$$(1-\varepsilon)v_B \leq v_A \qquad \text{and} \qquad (1-\varepsilon)(v_A + v_B) \geq v_B, \tag{33}$$

which are always satisfied as long as $\varepsilon \in (0, 1/2)$. In the latter, the equilibrium conditions,

$$(1-\varepsilon)v_A \leq v_B \qquad \text{and} \qquad (1-\varepsilon)(v_A + v_B) \geq v_A, \tag{34}$$

are satisfied if and only if $\varepsilon \in [\frac{v_A - v_B}{v_A}, \frac{v_B}{v_A + v_B}]$. ■

Proof of Proposition 6. Part (A) is transparent as well as Proposition 1. We can also prove part (B) along the same line as Proposition 2 and thus, we present only the sketch of the proof.

Let (F_H^*, F_L^*) be an equilibrium pair of mixed strategies and define S_i, α_i, and β_i the same as in the proof of Proposition 2. First, it is shown that $b_H^1 + \beta_H = b_L^1 + \beta_L$ and β_i is not a mass point of F_i^* for each $i \in \{H, L\}$. Second, it can be shown that $\beta_L = C^{-1}(v_L + C(b_L^1)) - b_L^1$ and it follows that $(u_H^*, u_L^*) = (\Delta, 0)$, where $\Delta := v_H + C(b_H^1) - v_L - C(b_L^1)$. Then, as long as $b + b_i^1 > b_j^1$, the equilibrium condition,

$$u_i(b; F_j^*) = F_j^*(b + b_i^1 - b_j^1)v_i - \Big(C(b + b_i^1) - C(b_i^1)\Big) \leq u_i^*, \tag{35}$$

implies that

$$F_j^*(b + b_i^1 - b_j^1) \leq \frac{u_i^* + \big(C(b + b_i^1) - C(b_i^1)\big)}{v_i}. \tag{36}$$

Letting $b' = b + b_i^1 - b_j^1 > 0$ yields

$$F_j^*(b') \leq \frac{u_i^* + \left(C(b' + b_j^1) - C(b_i^1)\right)}{v_i}. \tag{37}$$

By continuity, inequality (37) must hold with equality for all $b' \in (\max\{b_i^1 - b_j^1, 0\}, \beta_i)$. When $b_i^1 - b_j^1 > 0$, it is clear that F_j^* must be constant on $(0, b_i^1 - b_j^1)$. It follows that

$$F_j^*(b') = \frac{u_i^*}{v_i}, \tag{38}$$

for all $b' \in (0, b_i^1 - b_j^1)$. As a summary, we can obtain that

$$F_j^*(b) = \min\left\{1, \frac{u_i^* + C\left(\max\{b + b_j^1 - b_i^1, 0\} + b_i^1\right) - C(b_i^1)}{v_i}\right\}, \tag{39}$$

and the proof is complete. ■

Proof of Proposition 7. The proof is exactly the same as of Proposition 3 and thus omitted. ■

Proof of Proposition 8. Define (F_A^0, F_B^0) and (Φ_A, Φ_B) the same as in the proof of Proposition 4. Further, let $\lambda := C^{-1}(v_B)$ for expositional simplicity. Then, the revenue and the costs in the subgame after $(b_A^1, b_B^1) = (0, 0)$ are given by

$$\begin{aligned} R(0) &= \int_0^\lambda t\, dF_A^0(t) + \int_0^\lambda t\, dF_B^0(t) \\ &= \int_0^\lambda \frac{t \cdot C'(t)}{v_B}\, dt + \int_0^\lambda \frac{t \cdot C'(t)}{v_A} t\, dt \\ &= \int_0^\lambda \frac{v_A + v_B}{v_A v_B} t \cdot C'(t)\, dt, \end{aligned} \tag{40}$$

and,

$$
\begin{aligned}
X(0) &= \int_0^\lambda C(t)\, dF_A^0(t) + \int_0^\lambda C(t)\, dF_B^0(t) \\
&= \int_0^\lambda \frac{t \cdot C'(t)}{v_B}\, dt + \int_0^\lambda \frac{t \cdot C'(t)}{v_A} t\, dt \\
&= \int_0^\lambda \frac{v_A + v_B}{v_A v_B} C(t) C'(t)\, dt = \frac{v_B(v_A + v_B)}{2v_A},
\end{aligned} \tag{41}
$$

respectively. (Note that $\int C \cdot C' = C^2/2$ follows from integration by parts.) Regarding that $\Phi_i(b) = F_i^0(b)$ for $b \in [b_A^1, \lambda]$, the difference in R and X are given by

$$
\begin{aligned}
R(b) - R(0) &= \int_0^b (b-t) dF_A^0(t) - \int_0^b t\, dF_B^0(t) \\
&= \int_0^b \frac{b-t}{v_B} C'(t) dt - \int_0^b \frac{t}{v_A} C'(t) dt \\
&= \int_0^b \left(\frac{b}{v_B} - \frac{(v_A + v_B)t}{v_A v_B} \right) C'(t) dt,
\end{aligned} \tag{42}
$$

and,

$$
\begin{aligned}
X(b) - X(0) &= \int_0^b (C(b) - C(t)) dF_A^0(t) - \int_0^b C(t) dF_B^0(t) \\
&= \int_0^b \frac{C(b) - C(t)}{v_B} C'(t) dt - \int_0^b \frac{C(t)}{v_A} C'(t) dt \\
&= \int_0^b \left(\frac{C(b)}{v_B} - \frac{(v_A + v_B) C(t)}{v_A v_B} \right) C'(t) dt \\
&= \frac{\{C(b)\}^2}{v_B} - \frac{v_A + v_B}{v_A v_B} \frac{\{C(b)\}^2}{2} = \frac{v_A - v_B}{2 v_A v_B} [C(b)]^2,
\end{aligned} \tag{43}
$$

and the proof is complete. ■

Proof of Proposition 9. Suppose that $b \cdot C'(b)/C(b) \leq 1$ for all b. Then,

$$
\begin{aligned}
\frac{d}{db} \int_0^b \left(\frac{b}{v_B} - \frac{(v_A + v_B)t}{v_A v_B} \right) C'(t) dt &= \frac{C(b) + b \cdot C'(b)}{v_B} - \frac{(v_A + v_B) b \cdot C'(b)}{v_A v_B} \\
&> \frac{1}{v_B} \left(C(b) - b \cdot C'(b) \right) \geq 0,
\end{aligned} \tag{44}
$$

for all $b \in (0, v_B)$ and the proof is complete. ■

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