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Fertility and Mortality in a Neoclassical Growth Model*

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Abstract

We develop and study a neoclassical growth model with a realistic demographic structure. Our model complements recent developments in the overlapping generations (OLG) literature by highlighting the natural link that exists between different classes of macrodynamic models. Within the model we show that there exists an overlooked equilibrium in which demographic factors play a minor role. In numerical simulations we show that the economic impact of a change in the population growth rate differs dramatically depending on whether the change was induced by a change in the birth rate, the mortality rate or a combination of the two. Finally, we show that the demographic transition experienced by the US over the last half century cannot explain the substantial aggregate economic changes that have occurred during this period.

Keywords: Demography, Growth, Overlapping Generations

JEL Codes: C62, D91, E13, J10

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1. Introduction

In general, the theory of economic growth has downplayed the significance of an economy's demographic structure as a determinant of its growth potential. Standard benchmark growth models continue to assume infinitely-lived representative agents, and therefore lack the structure necessary to address demographic issues.[1] The two primary approaches incorporating demographic features into growth theory are (i) the original overlapping generations model, pioneered by Samuelson (1958) and Diamond (1965) [SD], and (ii) the more recent "perpetual youth model", introduced by Blanchard (1985), and refined by Buiter (1988), and Weil (1989) [BBW].[2] Both approaches provide deep insights and have had profound impacts. But both are highly stylized, which limits their ability to incorporate demographic factors in a comprehensive way.

The canonical SD model usually adopts a two-period framework—a first period for working and a second period for retirement.[3] While the model can be used to analyze many inter-generational policy issues, the usual formulation is overly rigid with regard to its choice of time units for analyzing standard macroeconomic policy issues.[4] The BBW model is more flexible, but it assumes a mortality rate that is independent of the household's age. While this assumption has the advantage of analytical tractability, and captures the finite horizon aspect of life, it comes at the cost of being unable to incorporate changing behavior over the life cycle, a limitation that Blanchard himself originally acknowledged.

During the last decade substantial progress has been made in extending the BBW model to incorporate more realistic demographic structures. Bommier and Lee (2003), d'Albis (2007), Lau (2009), and Gan and Lau (2010) employ very general mortality structures to study the existence and

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[1] This can be readily confirmed by consulting any one of the leading textbooks on modern growth theory (or macrodynamics) where some version of the Ramsey model or the Romer (1986) model – depending upon the underlying production structure – is the overwhelmingly dominant paradigm.

[2] A key component of the BBW model to deal with uncertainty of lifespan, the existence of an actuarially fair insurance market, was originally introduced by Yaari (1965).

[3] Many variants of the model exist, including extensions to an initial third period, for education; see e.g. Docquier and Michel (1999).

[4] For this reason, Auerbach and Kotlikoff (1987), in their comprehensive study of fiscal policy, introduced 55 periods in order to accommodate multiple generations, while employing a time unit of the order of one year.

uniqueness of the steady-state equilibrium.[5] Complementing this approach, Boucekkine, de la Croix, and Licandro (2002), Faruqee (2003), Heijdra and Romp (2008), Heijdra and Mierau (2010), and Bruce and Turnovsky (2010) adopt empirically plausible mortality functions as the basis for their analysis of structural and demographic changes.[6] To a large extent, the issues addressed by these authors are motivated by the empirical findings of Modigliani and Brumberg (1954), who first argued that varying behavior over the life-cycle has important consequences for the evolution of the aggregate economy. In a similar vein, recent empirical studies have acknowledged the existence of an intricate relationship between the demographic structure and the economic outcomes of developing and developed countries; see, e.g., Kelley (1988), Kelley and Schmidt (1995), Bloom, Canning, and Graham (2002), and Erlandsen and Nymoen (2008).

The objective of the current paper is to study the theoretical and quantitative aspects of a neoclassical growth model having a realistic demographic structure. For the theoretical part we build on the contribution of d'Albis (2007), by highlighting the mechanisms whereby the demographic structure impinges on the macrodynamic equilibrium. This is through what we call the "generational turnover term". This refers to the reduction in aggregate consumption due to the addition of newborn agents having no accumulated assets, together with the departure of agents with their accumulated lifetime assets. All demographic structures share the feature that they impact on the aggregate macrodynamic equilibrium through their effect on the aggregate consumption growth rate, so that differences among them reduce to differences in the specification of the generational turnover term. By explicitly setting up the underlying dynamic system, we are able to establish that there are in fact two viable steady-state equilibria, rather than just the one identified by d'Albis.

The two equilibria contrast sharply in how they are influenced by the demographic structure. In the first equilibrium (the one indentified in the literature) demographic factors play an important

[5] Somewhat analogously, Edmond (2008) introduces overlapping generations into a continuous-time, "endowment economy" version of the SD model, in which agents live for a given finite period. He shows how the equilibrium can be represented as the solution to a rather abstract nonlinear integral equation involving intertemporal prices.

[6] Boucekkine, de la Croix, and Licandro (2002) adopt a generalization of the Blanchard mortality function, thereby embedding the latter as a special case. This formulation is also adopted by Heijdra and Mierau (2010). Heijdra and Romp (2008) use the Gompertz (1825) exponential mortality hazard function in a small open-economy overlapping generations model. Faruqee (2003) approximates the Gompertz function with an estimated hyperbolic function, which he introduces into the Blanchard (1985) model. Finally, Bruce and Turnovsky (2010) represent survival by a de Moivre (1725) function.

role. They impinge on equilibrium per capita consumption directly, through the impact of the mortality function on the discounting of future consumption. In contrast, in the second equilibrium we identify, demographic factors play no direct role, except insofar as they determine the overall population growth rate. The key feature of this equilibrium is that the equilibrium growth rate of consumption just equals the growth rate of population. As a result, the amount of consumption given up by the dying just equals that required to sustain the consumption of the growing population. Accordingly, steady-state consumption is sustainable, independent of the time profile of the mortality function.

In order to get a better understanding of the dynamics of the model and to prepare for the numerical analysis, we must add more demographic structure, and we do so by adopting the Boucekkine, de la Croix, and Licandro (2002) (BCL) mortality function. This functional form is not only fairly tractable but also tracks the observed mortality data very well.[7] Using the BCL function we provide an explicit representation of the aggregate macrodynamic system. This turns out to be a highly nonlinear fifth order system involving not only capital and consumption, as in the standard representative agent economy, but also the dynamics of the various elements of the intergenerational turnover term. This model embeds the BBW model, the dynamics of which simplifies dramatically due to the constant mortality assumption, which carries the implication that both human wealth and the marginal propensity to consume are independent of age.

In our numerical simulations we study the long-run behavior of the model in response to both structural and demographic changes, illustrating their effects on aggregate quantities, as well as on the distributions of consumption and wealth across cohorts. Our numerical results show how the effects of a given increase in the population growth rate contrast sharply – both qualitatively and quantitatively – depending upon whether it occurs through an increase in the birth rate or a decrease in mortality. Whereas in the former case an increase in the population growth rate is associated with a mild decline in the capital stock, in the latter case it leads to a substantial increase in the per capita stock of capital. These differences in turn carry over to other aspects of the aggregate economy.

[7] We employ US mortality data, but Heijdra and Mierau (2010) successfully apply the BCL function to Dutch data. However, like other functions, such as the de Moivre function, the BCL function fails in the extreme old age tail of the mortality distribution; see Fig. 2.

This contrast is consistent with empirical evidence obtained by Blanchet (1988) and by Kelley and Schmidt (1995). The latter summarize the difference in terms of children, having little accumulated wealth, being "resource users" and working adults with their accumulated capital being "resource creators".[8] Our numerical results also confirm the empirical findings of Bloom, Canning, and Graham (2003) who find that increases in life expectancy leads to higher savings, as well as the life-cycle consumption patterns obtained by Fair and Dominguez (1991), Attfield and Cannon (2003), and Erlandsen and Nymoen (2008).

The remainder of the paper is structured as follows. Section 2 lays out the components of the underlying analytical framework, while Section 3 describes the corresponding macrodynamic equilibrium and steady state. Section 4 focuses on specific demographic structures and section 5 performs the numerical simulations. The final section concludes and provides some suggestions for directions in which this research might be extended.

2. The Analytical Framework

2.1. Individual household behavior

Consider an individual born at time v. The probability that this agent lives to become $t-v$ years old is governed by the survival function $S(t-v)$, where $S'(s)\equiv dS(s)/ds<0$, decreases with age. Given this function, the hazard rate or instantaneous probability of death is given by

$$\mu(t-v)=-\frac{S'(t-v)}{S(t-v)}, \tag{1}$$

which is positive. The probability that an individual dies before reaching age $t-v$ is described by the cumulative mortality rate:

$$M(t-v)=\int_0^{t-v}\mu(\tau)d\tau. \tag{2}$$

Combining (1) and (2) the survival function can be conveniently related to the mortality function by:

[8] It is also consistent with the related evidence from cross-country studies of fertility and growth. These have typically found the correlations between economic growth and population growth to be negative for less developed economies, having higher birth rates, and positive for developed economies, with their lower mortality rates (Kelley, 1988).

$$S(t-v)=e^{-M(t-v)}. \tag{3a}$$

where

$$S(0)=e^{-M(0)}=1,\ \ S(D)=e^{-M(D)}=0 \tag{3b}$$

so that D defines the maximum age that individuals can attain.[9]

Given this specification of the mortality function, the discounted expected life time utility of an individual newborn at time v is:

$$E\Lambda(v)=\int_{v}^{v+D}U\left(C(v,t)\right)\cdot e^{-\rho(t-v)-M(t-v)}dt, \tag{4a}$$

where $C(v,t)$ denotes the consumption at time t of an individual born at time v, and ρ is the pure rate of time preference of a newborn. Written in this way, the agent's discount rate, $\rho+\mu(t-v)$, varies with age.[10] The agent supplies a unit of labor inelastically and is assumed to make his consumption and asset accumulation decisions to maximize his expected utility (4a) subject to his budget constraint:

$$A_t(v,t)\equiv\frac{\partial A(v,t)}{\partial t}=\left(r(t)+\mu(t-v)\right)A(v,t)+w(t)-C(v,t), \tag{4b}$$

where $A(v,t)$ are assets held at time t of an individual born at time v, $w(t)$ is the wage rate, and $r(t)$ is the interest rate (see below).

Individuals are born without assets, have no bequest motive, and are not allowed to die indebted. Therefore, $A(v,v)=0$ and individuals fully annuitize all their assets. Annuities are life-insured financial assets that pay, conditional on the survival of the individual. Individuals receive a premium on these annuities equal to their instantaneous probability of death[11], $\mu(t-v)$, and, in return, if an individual dies his assets flow to the insurance company. Thus the overall rate of return received by an agent on his assets is $r(t)+\mu(t-v)$. Alternatively, an individual may engage in

[9] Depending on the specific mortality structure one assumes, D may be taken to be either finite or infinite.

[10] From (1)-(3) we see that the discount increases with age if and only if $SS''<(S')^2$, which is certainly met if the mortality function is concave.

[11] This result follows from perfect competition between annuity firms. If competition between annuity firms is less-than-perfect there is a load factor, $0\leq\lambda<1$, on the annuity premium and individuals receive only $\lambda\mu(t-v)$ on their annuities. This is studied in Heijdra and Mierau (2009).

borrowing. In that case he pays a premium of $\mu(t-v)$ and if he dies his debts are cancelled.

Defining the present value Hamiltonian for an agent born at time v:

$$H \equiv e^{-\rho(t-v)-M(t-v)} \left\{ U\left(C(v,t)\right) + \lambda(v,t)\left[\left(r(t)+\mu(t-v)\right)A(v,t)+w(t)-C(v,t)\right]\right\}$$

and optimizing with respect to $C(v,t)$ and $A(v,t)$, we obtain:

$$U'\left(C(v,t)\right) = \lambda(v,t), \tag{5a}$$

$$\rho - \frac{\lambda_t(v,t)}{\lambda(v,t)} - \frac{S'(t-v)}{S(t-v)} = r(t) + \mu(t-v) \tag{5b}$$

Equation (5a) equates the marginal utility of consumption to the shadow value of financial wealth, while (5b) equates the rate of return on consumption, adjusted by the mortality hazard rate, to the rate of return on financial assets. In addition, the agent must satisfy the transversality condition:

$$A(v, v+D) = 0 \tag{5c}$$

That is, in the absence of a bequest motive, individuals want to make sure that $A(v,v+D) \le 0$ and annuity firms want to ensure that[12] $A(v,v+D) \ge 0$, so that the only feasible solution is $A(v,v+D) = 0$.

For analytical convenience we follow the contemporary growth literature and assume an iso-elastic utility function:

$$U\left(C(v,t)\right) = \frac{C(v,t)^{1-1/\sigma}-1}{1-1/\sigma},$$

where σ is the inter-temporal elasticity of substitution. Combining (5a), (5b) and noting (1), enables us to write the Euler equation as:

$$\frac{\partial C(v,t)/\partial t}{C(v,t)} = \sigma\left(r(t)-\rho\right), \tag{6}$$

which expresses how the agent's consumption changes with age. In particular, equation (6) implies

[12] Although annuity firms cancel debts of individuals they will not take up debts of individuals who die indebted for sure. That is, at some time $\varepsilon - D$ annuity firms will refuse to issue life insurance and recall all debts. Letting ε go into zero gives the constraint implied by the annuity firms.

that consumption of all agents grows at a common (time-varying) rate, independent of age or their level of wealth.

Solving (6) forward from time *t*, the agent's consumption at an arbitrary time $\tau > t$ is:

$$C(v,\tau) = C(v,t)e^{\sigma\left(R(t,\tau)-\rho(\tau-t)\right)}, \tag{7}$$

where $R(t,\tau) \equiv \int_t^\tau r(s)ds$ is the cumulative interest rate over the period (t,τ). To express the agent's consumption in terms of his financial resources, we integrate the budget constraint (4b) forward from time *t* and impose the transversality condition, (5c), to yield the agent's intertemporal budget constraint operative from time *t*:

$$A(v,t) + e^{R(v,t)+M(t-v)}\int_t^{v+D} w(\tau)e^{-R(v,\tau)-M(\tau-v)}d\tau = e^{R(v,t)+M(t-v)}\int_t^{v+D} C(v,\tau)e^{-R(v,\tau)-M(\tau-v)}d\tau . \tag{8}$$

Substituting (7) into (8) we obtain the following expression for $C(v,t)$:

$$C(v,t) = \frac{A(v,t) + \int_t^{v+D} w(\tau)e^{-R(t,\tau)-(M(\tau-v)-M(t-v))}d\tau}{\int_t^{v+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-(M(\tau-v)-M(t-v))}d\tau} = \frac{A(v,t)+H(v,t)}{\Delta(v,t)}, \tag{9a}$$

where:

$$H(v,t) \equiv \int_t^{v+D} w(\tau)e^{-R(t,\tau)-\left(M(\tau-v)-M(t-v)\right)}d\tau \tag{9b}$$

is discounted future labor income (human wealth) at time t of an individual born at time v, and:

$$\Delta(v,t) \equiv \int_t^{v+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-\left(M(\tau-v)-M(t-v)\right)}d\tau \tag{9c}$$

is the inverse of the marginal propensity to consume out of total wealth (i.e. financial wealth, $A(v,t)$, plus human wealth, $H(v,t)$) at age $t-v$. Expressions (9b) and (9c) show that an increase in mortality leads to a decline in human wealth and an increase in the marginal propensity to consume, as agents will have a shorter expected lifespan over which to accumulate assets and to consume the income they generate. Setting $t = v$ yields the corresponding quantities at birth.

2.2. Aggregate household behavior

Let $P(t)$ denote the size of the total population at time *t*. The birth rate is constant, β, so that at every instant *v*, a cohort of size $P(v,v) = \beta P(v)$ is born. Given the mortality function, the number of individuals of cohort *v* still alive at time *t* is $P(v,t) = \beta P(v) e^{-M(t-v)}$. Similarly, at every instant *v*, a mass of $\bar{\mu} P(v)$ individuals dies, where $\bar{\mu}$ is the average mortality rate across cohorts:

$$\bar{\mu} \equiv \int_{t-D}^{t} \mu(t-v) \frac{P(v,t)}{P(t)} dv. \tag{10}$$

In the absence of migration, the growth rate of the population is equal to $\beta - \bar{\mu} = n$. Hence, from the perspective of time v the population at time t is equal to

$$P(t) = P(v) e^{n(t-v)}. \tag{11}$$

The relative weight of a cohort is:

$$p(v,t) \equiv \frac{P(v,t)}{P(t)} = \beta e^{-n(t-v)-M(t-v)}, \tag{12}$$

the dynamics of which are

$$\frac{p_t(v,t)}{p(v,t)} \equiv \frac{\partial p(v,t)/\partial t}{p(v,t)} = -\left[n + \mu(t-v)\right] \tag{13}$$

Thus the decline in the relative size of each cohort reflects both its individual mortality rate and the overall population growth rate.

Aggregating over the surviving cohort members at each point of time, the total population at any time t is equal to:

$$P(t) = \beta \int_{t-D}^{t} P(v) e^{-M(t-v)} dv \tag{14}$$

Equating (11) and (14) yields the relationship:

$$\frac{1}{\beta} = \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} dv, \tag{15}$$

which defines the demographic steady-state; see d'Albis (2007, p.416) and Heijdra and Romp (2008, p.94). That is, (15) defines a constraint linking the birth rate, β, mortality structure,

$(M(t-v)$ and $D)$, and the overall population growth rate, n. For example, given the birth rate and mortality function, (15) implies the corresponding solution for the population growth rate. This relationship is an integral component of any consistently specified aggregate demographic structure.[13]

To obtain aggregate per capita quantities we sum across cohorts by employing the following generic aggregator function:

$$x(t) \equiv \int_{t-D}^{t} p(v,t)\, X(v,t)\, dv = \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} X(v,t)\, dv \tag{16}$$

Taking the time derivative of (16), the evolution of $x(t)$ is given by:

$$\dot{x}(t) = \beta X(t,t) - \int_{t-D}^{t} p(v,t)\, X_t(v,t)\, dv - nx(t) - \int_{t-D}^{t} \mu(t-v)\, p(v,t)\, X(v,t)\, dv \tag{17}$$

where we have used the fact (see (3b) and (12)) that $p(t,t) = \beta$, $p(t-D,t) = 0$, and (13).

Thus, aggregate per capita consumption is:

$$c(t) \equiv \int_{t-D}^{t} p(v,t)\, C(v,t)\, dv \tag{18}$$

Taking the time derivative of (18) and using (6) and (13), the dynamics of per capita consumption are described by:

$$\dot{c}(t) = \left(\sigma\left[r(t) - \rho\right] - n\right) c(t) + \beta C(t,t) - \int_{t-D}^{t} \mu(t-v)\, p(v,t)\, C(v,t)\, dv, \tag{19}$$

Combining (19) with (6) we see that:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\partial C(v,t)/\partial t}{C(v,t)} - \frac{\Phi(t)}{c(t)} \tag{20a}$$

where

$$\Phi(t) \equiv \int_{t-D}^{t} \mu(t-v)\, p(v,t)\, C(v,t)\, dv - \beta C(t,t) + nc(t) \tag{20b}$$

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13 It can also be written in the form $\beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} dv = 1$. In the case of a stationary population this constraint reduces to $\beta \int_{t-D}^{t} e^{-M(t-v)} dv = 1$ which describes the necessary offsetting relationship between births and mortality.

is the "generational turnover term". That is, the reduction in aggregate per capita consumption (below the common consumption growth rate of each cohort) due to the addition of newborn agents with no accumulated assets and the departure of agents with assets. It depends upon (i) total consumption given up by the dying relative to the average, and (ii) the difference between the consumption of a newborn and the overall average per capita consumption due to growth.

The expression in (20b) provides a very general specification that encompasses all of the standard demographic models. With zero population growth, the textbook infinitely-lived representative agent model is obtained by setting $\beta = \mu = 0$ (implying $D \to +\infty$). With a growing population, we need to take account of the fact that at each instant, each newborn is immediately endowed with the average capital stock, part of which he must immediately set aside for the individuals born at the next instant. With the intertemporal elasticity of substitution σ, this reduces the per capita consumption growth rate by $\Phi(t)/c(t) = \sigma n$, so that (19) reduces to the familiar Euler equation $\dot{c}(t) = \sigma\left(r(t) - \rho - n\right)c(t)$.

Substituting for equations (1), (3), and (12) yields

$$\Phi(t) = -\beta\int_{t-D}^{t} S'(t-v)e^{-n(t-v)}C\left(v,t\right)dv - \beta C\left(t,t\right) + nc(t)$$

Integrating by parts and simplifying, yields

$$\Phi(t) = -\beta\int_{t-D}^{t} e^{-n(t-v)-M(t-v)}[nC\left(v,t\right) + C_v\left(v,t\right)]dv + nc(t) = -\beta\int_{t-D}^{t} e^{-n(t-v)-M(t-v)}C_v\left(v,t\right)dv, \qquad \text{(20b')}$$

where $C_v(v,t)$ represents the change in consumption across cohorts at a given point in time. Hence, using (20b') in (19) the evolution of aggregate per capita consumption can be written as:

$$\dot{c}\left(t\right) = \sigma\left[r\left(t\right) - \rho\right]c\left(t\right) - \beta\int_{t-D}^{t} e^{-n(t-v)-M(t-v)}C_v\left(v,t\right)dv. \qquad (21)$$

To determine whether or not $\Phi(t) > 0$, we use the fact that at any instant of time, the rate of change of consumption of agents of age $(t-v)$ is $\dot{C}(v,t) = C_v(v,t) + C_t(v,t)$.[14] Recalling (6), and letting $\gamma(v,t) \equiv \dot{C}(v,t)/C(v,t)$ denote the growth rate of consumption this implies

[14] Formally it is $\lim_{h\to 0}\dfrac{C(v+h,t+h)-C(v,t)}{h}$.

$$C_v(v,t) = \left[\gamma(v,t) - \sigma[r(t) - \rho]\right] C(v,t)$$

Thus a sufficient condition to ensure that $\Phi(t) > 0$ is that the growth rate of consumption with age exceeds the overall growth rate of consumption. In steady state, $\gamma(v,t) = 0$ and we immediately derive $\Phi(t) = \sigma(r - \rho)c > 0$.

Employing (16) again, aggregate per capita assets are:

$$a(t) \equiv \int_{t-D}^{t} p(v,t) A(v,t) dv \tag{22}$$

Taking the time derivative of (22) and using (4b) and (13), per capita asset accumulation is determined by:

$$\dot{a}(t) = \int_{t-D}^{t} p(v,t)\left[\left(r(t) + \mu(t-v)\right) A(v,t) + w(t) - C(v,t)\right] dv - \int_{t-D}^{t} \left[n + \mu(t-v)\right] \cdot p(v,t) A(v,t) dv$$

so that

$$\dot{a}(t) = \left(r(t) - n\right) a(t) + w(t) - c(t), \tag{23}$$

where we have used the fact that $A(t,t) = 0$. The per capita rate of asset accumulation differs from the individual rate of asset accumulation, due to the fact that (i) the amount μA is a transfer by insurance companies from those who die to those who remain alive and thus does not add to aggregate wealth, and (ii) account has to be taken of the growing population.

2.3. Firms

Output is produced by a representative firm in accordance with the neoclassical production function having constant returns to scale:

$$Y(t) = F(K(t), L(t)), \; F_K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0, F_{LK} > 0, \tag{24}$$

where $Y(t)$ is output, $K(t)$ is capital, $L(t)$ is aggregate labor supply. In per capita terms this may be expressed as

$$\frac{Y(t)}{L(t)} \equiv y(t) = F\left(\frac{K(t)}{L(t)}, 1\right) = f(k(t)) \tag{24'}$$

Assuming that labor and capital are paid their marginal products the equilibrium wage rate and return to capital are determined by:

$$w(t) = f\left(k(t)\right) - f'\left(k(t)\right)k(t) \tag{25a}$$

$$r(t) = f'\left(k(t)\right) - \delta \tag{25b}$$

where δ is the depreciation rate of capital.

3. General Equilibrium

In equilibrium, both the capital and the labor market must clear. Labor market clearance is reflected in the fact that all agents are fully employed so that the total population equals the total labor force. Capital market equilibrium is imposed by setting aggregate assets equal to total capital $A(t) = K(t)$, so that in aggregate per capita terms, $a(t) = k(t),$ implying further that $\dot{a}(t) = \dot{k}(t).$

Substituting the factor pricing relations (25) into (23) and (21) enables us to summarize the dynamics of the macroeconomic equilibrium in the form

$$\dot{k}(t) = f\left(k(t)\right) - c(t) - (\delta + n)k(t) \tag{26a}$$

$$\dot{c}(t) = \sigma\left(f'\left(k(t)\right) - \delta - \rho\right)c(t) - \Phi(t) \tag{26b}$$

where

$$\Phi(t) = -\beta\int_{t-D}^{t} e^{-n(t-v)-M(t-v)}C_v(v,t)\,dv \tag{26c}$$

This pair of dynamic equations in $\dot{k}$ and $\dot{c}$ will be recognized as being a variant of the standard textbook neoclassical growth model. Equation (26a) is the standard aggregate per capita accumulation of capital relationship, where the normalization of individual labor supply at unity implies that that aggregate labor supply is equal to one, while (26b) is the aggregate Euler equation, determining the intertemporal allocation of consumption.

The key point to emphasize with regard to expressing the macroeconomic equilibrium in this way is that it highlights how the demographic structure impinges on the economy through the generational turnover term, $\Phi(t)$, and its impact on the aggregate Euler equation. It provides a very general representation in which various specifications of the demographic structure can be embedded. In the case of the pioneering Blanchard (1985) model, and variants such as those developed by Buiter (1988) and Weil (1989), the evolution of $\Phi(t)$ is very straightforward and the full model can be described by a three dimensional dynamic system; see e.g. Blanchard (1985, p.234).

However, the fact that $\Phi(t)$ depends upon how consumption at any instant of time varies across cohorts means that for more general demographic structures its dynamic evolution can be very complex. As we demonstrate in Section 4 below, a more realistic demographic structure leads to a much higher dynamic system, due to the fact that the marginal propensity to consume varies over the life-cycle. In general, in order to characterize the aggregate dynamics and to prevent from being totally intractable it is necessary to impose some constraints on the demography.[15]

3.1. Steady-State

In the steady-state, the distributions of consumption, asset accumulation, relative cohort size, survival and mortality, no longer depend upon calendar time but only on age ($u \equiv t - v$). As a result, with no long-run per capita growth, per capita consumption, *c,* per capita capital stock, *k,* the wage rate, *w,* the return to capital, *r,* and the generational transfer term, Φ, are all constant over time. We shall denote all steady-state quantities by tildes.

Thus, when the aggregate economy is in steady state, consumption grows at the steady rate $\sigma(\tilde{r}-\rho)$ with age, so that the consumption level of an individual of age u is equal to:

$$\tilde{C}(u) = \tilde{C}_0 e^{\sigma(\tilde{r}-\rho)u} \tag{27}$$

where, setting $t = v$ in (9a), consumption at birth, $\tilde{C}_0$, can be expressed as

$$\tilde{C}_0 = \frac{\tilde{w}\int_0^D e^{-\tilde{r}u - M(u)}du}{\int_0^D e^{-(\tilde{r}(1-\sigma)+\sigma\rho)u - M(u)}du}. \tag{28}$$

[15] Having obtained $k(t)$, one can determine the time paths for the return to capital $r(t)$ and the wage rate $w(t)$. Having obtained these one can then derive the dynamics of consumption, savings, and capital accumulation across cohorts.

In the steady-state $p(v,t) = p(u) = \beta e^{-nu-M(u)}$ implying that aggregate consumption per capita is:

$$\tilde{c} \equiv \int_0^D p(u)\tilde{C}(u)\,du = \beta\tilde{C}_0\int_0^D e^{(\sigma(\tilde{r}-\rho)-n)u-M(u)}du. \tag{29}$$

Defining the function

$$\Xi(\lambda) \equiv \int_0^D e^{-\lambda s-M(s)}ds,$$

we can combine (27)–(29) to express the steady-state per capita consumption, (29) as:

$$\tilde{c} = \tilde{w}\frac{\Xi(\tilde{r})}{\Xi(\tilde{r}(1-\sigma)+\sigma\rho)}\frac{\Xi(n-\sigma(\tilde{r}-\rho))}{\Xi(n)}. \tag{30}$$

Finally, using the demographic steady-state condition,

$$\frac{1}{\beta} = \int_0^D e^{-nu-M(u)}du = \Xi(n)$$

we can write:

$$\tilde{c} = \beta\tilde{w}\frac{\Xi(\tilde{r})\cdot\Xi(n-\sigma(\tilde{r}-\rho))}{\Xi(\tilde{r}(1-\sigma)+\sigma\rho)} \tag{30'}$$

as in d'Albis (2007, p.416).

Substituting for the steady-state factor prices, (25), the steady-state equilibrium values of per capita consumption, $\tilde{c}$, and capital, $\tilde{k}$, are jointly determined by

$$f(\tilde{k}) = \tilde{c} + (\delta+n)\tilde{k} \tag{31a}$$

$$\tilde{c} = \beta[f(\tilde{k}) - \tilde{k}f'(\tilde{k})]\frac{\Xi(f'(\tilde{k}-\delta))\cdot\Xi(n-\sigma(f'(\tilde{k})-\delta-\rho))}{\Xi([f'(\tilde{k})-\delta](1-\sigma)+\sigma\rho)} \tag{31b}$$

where the demographic characteristics are embedded in the function Ξ. Letting $\tilde{s}(\tilde{k}) \equiv \tilde{k}f'(\tilde{k})/\tilde{f}(\tilde{k})$ denote the equilibrium share of capital, d'Albis (2007) shows that the pair of equations (31a) and (31b) have a unique solution as long as $\lim_{\tilde{k}\to 0}\tilde{s}(\tilde{k}) = 1$, $\tilde{s} < \tilde{e}$, where $\tilde{e}$ is the elasticity of substitution in production and $\sigma < 1$.[16] Both conditions are mild and hold for the Cobb-Douglas production function, for example. Figure 1 illustrates this equilibrium for the calibrated model specified in Section 5.1, where AA represents (31a), BB depicts (31b), and the two intersect at the point P.

3.2. A 'non-demographic' steady state

The steady-state equilibrium discussed in the previous section is the one identified by d'Albis (2007), Lau (2009), and Gan and Lau (2010). While they argue that the solution to (31a) and (31b) is unique, there is in fact a *second* steady-state equilibrium associated with the underlying dynamic system (26). It can be identified as follows. Using (20b') we may rewrite (21) as

$$\dot{c}(t) = [\sigma(r(t) - \rho) - n]c(t) + \beta\int_{t-D}^{t} e^{-n(t-v)-M(t-v)}[nC(v,t) + C_v(v,t)]dv \qquad (21')$$

which in steady state becomes

$$[\sigma(\tilde{r} - \rho) - n]\tilde{c} + \beta\int_{0}^{D} e^{-n(t-v)-M(t-v)}[n\tilde{C}(u) - \tilde{C}'(u)]du = 0,$$

where we have use the fact that, around the steady state, $C_v(v,t) = -C_t(v,t)$.

Recalling (27), this can be written as

$$[\sigma(\tilde{r} - \rho) - n]\left(\tilde{c} - \beta\tilde{C}_0\int_{0}^{D} e^{(\sigma(\tilde{r}-\rho)-n)u-M(u)}du\right) = 0. \qquad (32)$$

Thus in addition to (29), $\sigma(\tilde{r} - \rho) = n$ also yields $\dot{c}(t) = 0$. Hence,

$$f\left(\tilde{k}\right) = \tilde{c} + \left(\delta + n\right)\tilde{k} \qquad (31a)$$

$$\sigma(f'(\tilde{k}) - \delta - \rho) - n = 0 \qquad (31b')$$

is an alternative steady state. It is illustrated in Fig. 1 for the calibrated model by the intersection of AA and the vertical line CC, corresponding to (31b'), which intersect at Q. The key point to observe is that it is independent of the demographic structure, except insofar as this determines the overall population growth rate through the demographic steady-state relationship (15). Recalling the

[16] These conditions have been relaxed in subsequent work by Gan and Lau (2010), who show further that uniqueness is still obtained if $\sigma \geq 1$.

definition of $\tilde{r}$, (27) and (31b') imply that the steady-state growth rate of consumption across cohorts corresponding to this 'non-demographic equilibrium' equals the population growth rate.[17]

There is clearly a sharp contrast between (31b') which characterizes the 'non-demographic' steady state and (29) [and(31b)], where the demographic structure plays an important role through the impact of the mortality function on the discounting of future consumption. The key feature of the second equilibrium is that the equilibrium growth rate of consumption just equals the growth rate of population. In that case, the amount of consumption given up by the dying just equals that required to sustain the consumption of the growing population. Thus, as long as the population is growing at a constant rate, a constant steady-state per capita consumption is sustainable, independent of the time profile of the mortality function.

Finally, it is of interest to compare the two equilibria at P and Q, with the steady-state obtained in the infinitely-lived representative agent model. Denoting the corresponding steady-state per capita capital stocks by $\tilde{k}_P$, $\tilde{k}_Q$, and $\tilde{k}_R$, these three quantities are determined respectively by

$$\sigma\left(f'\left(\tilde{k}_P\right)-\delta-\rho\right)=\frac{\tilde{\Phi}}{\tilde{c}} \tag{33a}$$

$$\sigma\left(f'\left(\tilde{k}_Q\right)-\delta-\rho\right)=n \tag{33b}$$

$$\sigma\left(f'\left(\tilde{k}_R\right)-\delta-\rho\right)=\sigma n \tag{33c}$$

Recalling (20b), (33a-c) imply that if (i) the total consumption given up by the dying exceeds the consumption of the newborn, and if (ii) the intertemporal elasticity of substitution, $\sigma<1$, that $\tilde{k}_P<\tilde{k}_Q<\tilde{k}_R$.[18]

3.3. Capital maximizing birth rate

d'Albis (2007) argues that there exists a birth rate that maximizes the per capita capital stock. He defines the measure: $\alpha_x \equiv \int_0^D up(u)x(u)du \Big/ \int_0^D p(u)x(u)du$ where α_x measures the average of the quantity $x(u)$ across cohorts. He shows that the capital stock-maximizing birth rate occurs where

[17] This equilibrium satisfies the transversality condition (5c) so it cannot be ruled out as being unsustainable.
[18] In that case all steady states are dynamically efficient, in that the capital stocks would be less than at the golden rule.

the average age of workers equals the average age of asset holders, i.e. $\alpha_w = \alpha_A$. In our case it is straightforward to show that:

$$\operatorname{sgn}(\alpha_A - \alpha_w) = \operatorname{sgn}\left(\int_0^D u\tilde{A}(u)p(u)du - \int_0^D up(u)du \cdot \int_0^D \tilde{A}(u)p(u)du\right). \tag{34}$$

Using the fact that $p(u)$ may be interpreted as a probability density function we conclude that for $\alpha_w = \alpha_A$ the covariance between $\tilde{A}(u)$ and u must be zero.[19] For the covariance to be zero assets either have to be constant over the life-cycle or that its profile has to be linearly independent of the age profile.

To see that assets are actually hump-shaped over the life cycle, rather than constant, note that in the steady state, agents accumulate assets according to:

$$\dot{\tilde{A}}(u) = (\tilde{r} + \mu(u))\tilde{A}(u) + \tilde{w} - \tilde{C}(u) \tag{35a}$$

so that starting with zero initial endowment, $\tilde{A}(0) = 0$, the agent's wealth at age u is:

$$\tilde{A}(u) = \int_0^u \left[\tilde{w} - \tilde{C}(u)\right] e^{-\tilde{r}u - M(u)} du \tag{35b}$$

with the transversality condition implying:

$$\int_0^D \left[\tilde{w} - \tilde{C}(u)\right] e^{-\tilde{r}u - M(u)} du = 0. \tag{35c}$$

Under weak conditions, d'Albis shows that in steady state $\tilde{r} > \rho$, so that agents' consumption grows uniformly over their lifetimes. Using this fact, in conjunction with (27), (28), and (35a), one can show that because $\tilde{A}(0) = \tilde{A}(D) = 0$, $\dot{\tilde{A}}(0) > 0$, $\dot{\tilde{A}}(D) < 0$ and that the agent's assets reaches a maximium at an age $\hat{u}$:

$$\tilde{A}(\hat{u}) = \frac{\tilde{C}(\hat{u}) - \tilde{w}}{\tilde{r} + \mu(\hat{u})}.$$

Thus the time profile of the agent's wealth over the life cycle is hump shaped as illustrated in Panel (iii) of Figures 3-5.

[19] The key result that is being employed is that in general, $E(xy) = E(x)E(y) + \operatorname{cov}(x, y)$.

As the asset profile is hump shaped over the life-cycle it may be that there exists a unique value of the birth rate such that the asset profile and the age profile are not linearly dependent. In that case the average age of asset holders equals the average age of the workers. But as savings are primarily used to finance consumption later in life it is fair to suppose that the average age of the capital owner is higher than the average age of the worker. Indeed, in our simulations we show that for a realistic mortality function $\alpha_A = 52.20,\ \alpha_w = 44.22$. Thus, we find that an increase in the population growth rate associated with an increase in the birth rate leads to a reduction in the per capita stock of capital. In contrast, our simulations also show that if the increase in the population growth rate is the result of a reduction in mortality it will result in an increase in the per capita capita capital stock; see Table 2 and Section 5.[20] This contrast in the two ways of increasing the growth rate of population is consistent with the empirical evidence on this issue obtained by, inter alia, Kelley and Schmidt (1995), Bloom, Canning and Graham (2003) and Erlandsen and Nymoen (2008).

4. Specific Demographic Models

Hitherto we have not imposed any restrictions on the exact form of the survival function. To proceed further, we focus on the functional form proposed by Boucekkine, de la Croix, and Licandro (2002) (BCL), which is tractable, amenable to numerical simulations and fits the data well; see Table 1 and Figure 2. For comparative purposes, and to show how it fits into our analytical framework, we also discuss the familiar demographic structure proposed by Blanchard (1985), Buiter (1988) and Weil (1989) (BBW).[21]

4.1. BCL demographic structure

The survival function is specified by

$$S(t-v) \equiv e^{-M(t-v)} = \frac{\mu_0 - e^{\mu_1(t-v)}}{\mu_0 - 1}, \qquad \text{(for } 0 \le t-v \le D\text{)}, \qquad \mu_0 > 1,\ \mu_1 > 0, \tag{36a}$$

[20] This result was obtained in a numerical simulation of the Diamond (1965) model by Sinha (1985).
[21] Alternatively, Bruce and Turnovsky (2010) use the de Moivre function which has the advantage of including both the DS and BBW specifications as special cases, but is less tractable than the BCL function.

where the maximum age D, determined where $S(t-v)=0$, implies

$$\mu_0 = e^{\mu_1 D} = 0; \; i.e. \; D = \frac{\ln \mu_0}{\mu_1}. \tag{36b}$$

Following BCL we refer to μ_0 as "youth mortality" and μ_1 as "old age mortality". The BCL specification yields the following forms for the functions we have used in the general specification:

(i) Hazard rate:

$$\mu(t-v) \equiv \frac{-S'(t-v)}{S(t-v)} = \frac{\mu_1 e^{\mu_1(t-v)}}{\mu_0 - e^{\mu_1(t-v)}}. \tag{37a}$$

(ii) Relative cohort size:

$$p(v,t) \equiv \beta e^{-n(t-v)-M(t-v)} = \beta \frac{\mu_0 - e^{\mu_1(t-v)}}{\mu_0 - 1} e^{-n(t-v)} \tag{37b}$$

(iii) Demographic steady state:

$$\frac{1}{\beta} = \frac{1}{\mu_0 - 1}\left[\mu_0 \frac{1-e^{-nD}}{n} + \frac{1-e^{(\mu_1 - n)D}}{\mu_1 - n}\right] \tag{37c}$$

(iv) Average age of workers:

$$\alpha_W = \frac{\beta\mu_0}{\mu_0 - 1}\frac{1}{n^2}\left[e^{-nD}(-nD-1)+1\right] - \frac{\beta}{\mu_0 - 1}\frac{1}{(\mu_1 - n)^2}\left[e^{(\mu_1 - n)D}\left((\mu_1 - n)D - 1\right)+1\right] \tag{37d}$$

4.1.1. Aggregate dynamics

While the general macrodynamic equilibrium is summarized by the system (26a-c), the evolution of $\Phi(t)$ may in fact be complex, requiring one to consider the dynamics of its components. To this end it is practical to begin with the alternative definition of $\Phi(t)$, given in (20b), which for the BCL function becomes:

$$\Phi(t) = \frac{\beta\mu_1}{\mu_0 - 1}\int_{t-D}^{t} e^{(\mu_1 - n)(t-v)} \cdot C(v,t)\,dv - \beta C(t,t) + nc(t)$$

Using (7) and (9) we can write:

$$\Phi(t) = \frac{\beta\mu_1}{\mu_0 - 1}\Gamma(t) - \beta\frac{H_B(t)}{\Delta_B(t)} + nc(t) \tag{38}$$

where:

$$\Gamma(t)=\int_{t-D}^{t}C(v,v)e^{\sigma R(v,t)+(\mu_1-n-\sigma\rho)(t-v)}dv \tag{39a}$$

$$\frac{H_B(t)}{\Delta_B(t)}=C(t,t), \tag{39b}$$

and

$$H_B(t)\equiv H(t,t)=\int_{t}^{t+D}w(\tau)e^{-R(t,\tau)-M(\tau-t)}d\tau, \tag{39c}$$

$$\Delta_B(t)\equiv\Delta(t,t)=\int_{t}^{t+D}e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-M(\tau-t)}d\tau \tag{39d}$$

That is, $H_B(t)$ and $\Delta_B(t)$ are, respectively, the amounts of human wealth and the inverse marginal propensity to consume *at birth.*

Differentiating (39a)-(39d), imposing the factor prices (25a), (25b), and recalling the dynamics of consumption and capital (26a), (26b), the full dynamic system can then be expressed as:[22]

$$\dot{c}(t)=\Big(\sigma\big(f'(k(t))-\delta-\rho\big)-n\Big)c(t)-\frac{\beta\mu_1}{\mu_0-1}\Gamma(t)+\beta\frac{H_B(t)}{\Delta_B(t)} \tag{40a}$$

$$\dot{k}(t)=f(k(t))-(\delta+n)k(t)-c(t) \tag{40b}$$

$$\dot{\Gamma}(t)=\frac{H_B(t)}{\Delta_B(t)}-\frac{H_B(t-D)}{\Delta_B(t-D)}e^{\sigma R(t-D,t)+(\mu_1-n-\sigma\rho)D}+\Big(\sigma\big(f'(k(t))-\delta-\rho\big)+\mu_1-n\Big)\Gamma(t) \tag{40c}$$

$$\dot{\Delta}_B(t)=-1-\Big((\sigma-1)\big(f'(k(t))-\delta\big)-\sigma\rho+\mu_1\Big)\Delta_B(t)+\frac{\mu_1\mu_0}{\mu_0-1}\int_{t}^{t+D}e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)}d\tau \tag{40d}$$

$$\dot{H}_B(t)=-f(k(t))+f'(k(t))k(t)+\big(f'(k(t))-\delta-\mu_1\big)H_B(t)+\frac{\mu_1\mu_0}{\mu_0-1}\int_{t}^{t+D}w(\tau)e^{-R(t,\tau)}d\tau \tag{40e}$$

This comprises a fifth order system in: (i) per capita consumption, (ii) per capita capital stock, (iii) the consumption given up by the dying, (iv) the initial human wealth of the new born, and (v) the (inverse) of the marginal propensity to consume out of wealth by the newborn. In principle, the dynamics can be analyzed using numerical simulations. We should note that with H_B and Δ_B being evaluated both at time t and at time $t-D$ this involves the analysis of mixed differential-difference

[22] In determining (40d), (40e) we have used $e^{\mu_1(\tau-t)}=\mu_0-e^{-M(\tau-t)}(\mu_0-1)$

equations, which presents a computational challenge that is beyond the scope of the present paper.[23] We conjecture that since the only "sluggish" variable is $k(t)$, the stable manifold will be one dimensional, implying monotonic transitional adjustment in response to any structural change, including the demographic characteristics.[24]

4.1.2 Steady state

Defining $\varphi(x,D) \equiv \left(1-e^{-xD}\right)/x$, the steady state can be summarized by the following system[25]

A. Demographic Variables

$$\frac{1}{\beta}=\frac{1}{\mu_0-1}\left[\mu_0\varphi(n,D)-\varphi(n-\mu_1,D)\right] \tag{41a}$$

$$D=\frac{\ln\mu_0}{\mu_1} \tag{41b}$$

B. Economic Variables

$$\tilde{C}_0=\frac{\tilde{w}\left[\mu_0\varphi(\tilde{r},D)-\varphi(\tilde{r}-\mu_1,D)\right]}{\left[\mu_0\varphi(\sigma\rho+(1-\sigma)\tilde{r},D)-\varphi(\sigma\rho+(1-\sigma)\tilde{r}-\mu_1,D)\right]} \tag{41c}$$

$$\tilde{c}=\frac{\beta\tilde{C}_0}{\sigma(\tilde{r}-\rho)-n}\left\{\frac{\mu_1}{\mu_0-1}\varphi\left(\sigma(\rho-\tilde{r})+n-\mu_1,D\right)-1\right\} \tag{41d}$$

$$f(\tilde{k})=\tilde{c}+(\delta+n)\tilde{k} \tag{41e}$$

where $\tilde{r}$ and $\tilde{w}$ are defined in (25a), (25b). Equations (41a), (41b) define the demographic structure, summarized by the four parameters, β,μ_0,μ_1, and n. Given the demographic parameters and the definitions of $\tilde{r}$ and $\tilde{w}$, equations (41c-e) determine the economic variables, $\tilde{c},\tilde{C}_0$, and $\tilde{k}$. By

[23] Naturally, by imposing constant returns and a rectangular survival function it becomes possible to characterize the equilibrium dynamics. See, for instance, d'Albis and Augeraud-Véron (2009) and the references therein.

[24] Of course, the nature of the underlying transitional path will depend upon the eigenvalues of the dynamic system (40). A unique solution will require that there be just one stable eigenvalue, and that its attainment will involve initial jumps in c,Γ,Δ_B, and H_B.

[25] We are focusing on the 'demographic equilibrium' at which $\sigma(\tilde{r}-\rho)\neq n$.

combining (41c) and (41d) this can be reduced to a pair of equations in $\tilde{c}$ and $\tilde{k}$, which is analogous to (31a) and (31b). Having determined the aggregates, the steady-state age profiles of consumption and asset accumulations can be obtained by substituting into (27), (28), and (35).

System (41) provides the basis for our numerical simulations in Section 5. We use this system to examine the effects of a number of economic and demographic structural changes on both the aggregate behavior of the economy and on the patterns of consumption and asset accumulation over the life cycle.

4.2 BBW demographic structure

For comparative purposes it is useful to show how the BBW model fits into this framework. Blanchard (1985) assumes the birth rate to be equal to the mortality rate ($\beta = \mu$), so that the net population growth rate is zero. Buiter (1988) relaxes this assumption and extends the model to the case where $\mu \neq \beta$, effectively combining the Blanchard model with that of Weil (1989).

The survival function is specified by:

$$S(t-v) \equiv e^{-M(t-v)} = e^{-\mu(t-v)}. \tag{42}$$

from which we immediately infer that the hazard rate, μ, is constant, while the relative cohort size is $p(v,t) = \beta e^{-\beta(v-t)}$. The demographic steady-state holds by definition, life expectancy equals $1/\mu$ and is constant over the life cycle, while the average age of workers is $1/\beta$.

The key variable in the dynamics, the generational turnover term, $\Phi(t)$, now simplifies drastically to

$$\begin{aligned}\Phi(t) &= \int_{-\infty}^{t} \mu \cdot \beta e^{\beta(v-t)} \cdot C(v,t)\,dv - \beta C(t,t) + nc(t) \\ &= (\mu+n)c(t) - \beta C(t,t)\end{aligned} \tag{43}$$

Introducing the BBW structure into (9a) leads to:

$$C(v,t) = \frac{A(v,t) + \int_t^\infty w(\tau) e^{-R(t,\tau)-\mu(\tau-t)} d\tau}{\int_t^\infty e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-\mu(\tau-t)} d\tau} = \frac{A(v,t)+H(t)}{\Delta(t)} \tag{44}$$

The crucial characteristic that renders the model so tractable is that all agents have the same planning horizon (i.e. ∞) and mortality rate (i.e. μ). Therefore, human wealth, $H(t)$, (future discounted income from labor) is the same for all agents, irrespective of their age. The same applies to $\Delta(t)$, the (inverse of) the marginal propensity to consume out of human wealth:

$$\Delta(t) = \int_t^{\infty} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-\mu(\tau-t)} d\tau \tag{45}$$

Differentiating (45), its dynamics are governed by:

$$\dot{\Delta}(t) = -1 - \left((\sigma-1)r(t) - \sigma\rho - \mu\right)\Delta(t). \tag{46}$$

Aggregate per-capita consumption is:

$$c(t) \equiv \int_{-\infty}^{t} p(v,t)C(v,t)dv = \int_{-\infty}^{t} p(v,t)[\Delta(t)]^{-1}(A(v,t) + H(t))dv$$

$$= \left[\Delta(t)\right]^{-1}\left(a(t) + H(t)\right) = \left[\Delta(t)\right]^{-1}\left(k(t) + H(t)\right) \tag{47}$$

From (9a') consumption of a new-born, $C(t,t)$, is:

$$C(t,t) = \left[\Delta(t)\right]^{-1} H(t) = c(t) - \left[\Delta(t)\right]^{-1} k(t) \tag{48}$$

Hence, using (43), and recalling that $n = \beta - \mu$, we can write the aggregate dynamic system as:

$$\dot{k}(t) = f\left(k(t)\right) - (\delta + n)k(t) - c(t) \tag{49a}$$

$$\dot{c}(t) = \sigma\left(f'\left(k(t)\right) - \delta - \rho\right)c(t) - \beta\left[\Delta(t)\right]^{-1} k(t) \tag{49a}$$

$$\dot{\Delta}(t) = -1 - \left((\sigma-1)\left(f'\left(k(t)\right) - \delta\right) - \sigma\rho - \mu\right)\Delta(t), \tag{49c}$$

thus reducing it to a tractable third order system; see also Blanchard (1985, p.234). The steady state follows readily by setting $\dot{k}(t) = \dot{c}(t) = \dot{\Delta}(t) = 0$.

5. Numerical Simulations

To obtain further insights we simulate the steady-state equilibrium using the BCL survival function. To do this, we first estimate its two parameters, μ_0 and μ_1, by nonlinear least squares, using US cohort data[26] for 1960 and 2006. The estimation results reported in Table 1 highlight that in both cases we obtain a tight fit with highly significant parameter estimates. The resulting estimated survival function for 1960 is illustrated in Fig. 2.A. Since we do not consider childhood and education, we normalize the function so that birth corresponds to age 18. As can be seen in the figure it tracks the actual survival data for the United States closely from age 18 until around 85. Beyond that age its concavity does not match the data particularly well. However, we do not view that as serious since only 1.5% of the US population exceed 85 and these individuals are generally retired and are relatively inactive in the economy.[27] Fig 2.B illustrates the outward shift in the estimated BCL surival function between 1960 and 2006. Its implied increase in the life expectancy to around 77 is generally consistent with the actual increase in life expectancy of around 6 years that occurred over that period. For comparative purposes we also estimate and illustrate the BBW survival function in Tabel 1 and Fig. 2.A. Being convex, rather than concave, it does does not match the data well.

Table 2 summarizes the key structural parameters for the baseline economy, all of which are quite standard. Output is produced by a Cobb-Douglas function, $y = Ak^{\alpha}\bar{l}^{1-\alpha}$, where $\bar{l}$ denotes inelastically supplied labor, with the elasticity of capital $\alpha = 0.35$ and depreciation rate $\delta = 0.05$. With respect to preferences, we set the intertemporal elasticity of substitution to 0.5, consistent with the consensus estimates reported by Guvenen (2006). As noted, the rate of time preference increases with age. Hence we take $\rho = 0.035$ to be the rate of time preference at birth, implying a discount rate of 0.0405 for the individual of average age.

The baseline calibration adopts the demographic parameters of 1960. Thus, the estimates of the BCL function imply a maximum attainable age of 88.74 and life expectancy at age 18 (birth) of 71.70. These are a little low, reflecting the fact that, as Fig 2.A illustrates, the function fails to capture the outliers beyond age 85. We take the birth rate to be 2.37%, which given the survival function, implies a population growth rate of 0.87%. Again, this is a little low, but does not take

[26] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research, Rostock (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 12/10/2010).
[27] With this in mind, it might be more appropriate to refer to D as the maximum attainable economic age.

account of the role of immigration. The implied equilibrium economic variables include an equilibrium capital-output ratio of 2.97 and a real net return on capital of 6.78%. The marginal propensity to consume at birth out of wealth is approximately 0.057%, and the each cohort's consumption grows at 1.65% with age.[28] The corresponding parameters and implied equilibrium values for the BBW model are also reported in Table 2. It yields a much higher life expectancy, due to the fact that the maximum attainable age in that model is infinite.

From this initial baseline equilibrium we analyze the steady-state effects of two types of structural changes: (i) an increase in productivity; (ii) changes in the demographic structure.

5.1. Increase in productivity

We consider a neutral technological change, where *A* increases by 25% from 1 to 1.25. As seen from Row 2 in Table 3, this leads to a proportionate increase in capital and output, causing the capital-output ratio to remain unchanged.

Fig. 3.A illustrates the aggregate and the distributional effects for the BCL survival function. The locus BB in panel (i) depicts the pre-shock growth in consumption with age (eq. (6)). The increase in productivity raises the wage rate, while the rate of return on capital remains unchanged. This causes the BB locus to shift up to B'B', implying a uniformly higher consumption level for all ages, but growing at the unchanged rate. The AA locus presents the average per capita consumption, which correspondingly jumps up to A'A'. Panel (ii) illustrates the long-run distributional changes across the cohorts. Its mildly hump-shaped locus reflects the fact that the increase in consumption with age is offset by the increasing mortality with age, leading to declining cohort-weighted consumption.

Panel (iii) illustrates the distribution of assets along the life cycle. Starting with zero assets at birth (18), agents accumulate wealth until around 65, after which they decumulate until assets run out at the maximum attainable age. This is reflected in the inverted-U locus EE which shifts out to E'E' with the increase in productivity. The figures indicate that the greatest impact on wealth of the

[28] The corresponding key values of the 'non-demographic equilibrium' are $\tilde{k} = 6.620$ and $\tilde{c} = 1.559$. As a computational point we note that the proximity of the two equilibria, P and Q, as illustrated in Fig. 1, requires care in solving the non-linear steady-state system so as to ensure that the correct equilibrium is in fact identified.

productivity increase accrues to individuals aged around 65. The upward shift in the distributional locus is also reflected in the horizontal line DD which illustrates the average per capita wealth, and which shifts up to D'D' following the technological increase. Panel (iv) reflects assets weighted by the size of the cohorts. Due to the decline in survival with age the greatest share of the benefits is enjoyed by the 55 year old cohort.

Fig. 3.B illustrates the same exercise for the BBW demographic structure. It contrasts sharply, and is much less plausible, as a result of the convex surival function and the fact that agents may potentially live indefinitely (albeit with an arbitrarily low probability). For example, the perpetual upward slope of the assets accumulation locus EE in panel (iii) is unsatisfactory. However, with the dwindling cohort size the implications for distributions across cohorts, as illustrated in Panel (iv) is closer to the pattern implied by the more plausible BCL survival function.

5.2. Changes in the demographic structure

We contrast the impact of an increase in the population growth rate of 0.5 percentage points driven by either an increase in the birth rate, a decrease in mortality, or a combination of the two. Table 3 summarizes the various scenarios and shows how the economic consequences differ dramatically, depending on the source of the increase in the population growth rate.

5.2.1. Increase in the birth rate

In order to increase the population growth rate by 0.5 percentage point from 0.87% to 1.37% the birth rate must increase from 2.37% to 2.69%. Table 3, line 4 reveals that this leads to a 1.31% reduction in the per capita capital stock (from 5.337 to 5.268). This is illustrated by the slight downward shift of the line DD in Fig. 4, Panel (iii). This response is consistent with the characterization of the steady state provided in Section 3 and the fact that the average age of wealth owners (52.20) exceeds that of workers (44.22). It is also consistent with the view emphasized by Kelley and Schmidt (1995) that an increase in the population growth rate resulting from a higher birth rate will have a negative effect on the level of economic activity. This is because it increases the relative number of young who have not accumulated any capital stock to contribute to the productive capacity of the economy. This reduction in aggregate assest accumulation has several consequences. It leads to a 0.46% reduction in the wage rate (from 1.168 to 1.163) and an increase in the rate of return on capital from 6.78% to 6.89%. It also leads to a 0.46% decline in per-capita output (from 1.797 to 1.789) and a 2.04% decline in per capita consumption (from 1.484 to 1.453), the latter being illustrated by the downward shift in the AA line in Fig. 4(i).

The distributional consequences are also modest, as Fig. 4 illustrates. The life cycle path for consumption, illustrated by BB in Panel (i), remains virtually unchanged. The slight reduction in the wage, with the anticipation of the future higher return to capital causes a very slight reduction in consumption at birth. However, the increase in the rate of return on capital increases the consumption growth rate over the life-cycle. Hence, toward the end of their life-cycle agents experience an increase in their consumption while average per capita consumption declines. The distributional consequences across cohorts are more substantial and in fact opposite to those experienced by individuals, as illustrated by the roatation of the CC curve to CC' in Panel (ii). Thus, the increase in the relative size of the younger cohorts, due to the higher birth rate, implies that they enjoy a larger share of the overall consumption, while the decline in the relative size of older cohorts means that their share of consumption declines, even though each surviving member's consumption level has increased.

The hump shaped locus EE in Panel (iii), which reflects that the accumulation of assets over the life cycle, shifts out, albeit slightly. This is a consequence of the increased rate of return on capital. Panel (iv) illustrates how, with the increase in the relative size of the young cohorts due to the higher birth rate, the share of wealth each existing cohort owns increases. This also explains why, even though at each age each individual has a slightly higher level of wealth, per capita wealth is nevertheless smaller. This is because with a higher birth rate a relatively larger share of the agents is young and as young agents posses relatively little capital, this leads to lower aggregate per-capita capital (see Panel (iii) locus DD and D'D').

5.2.2 Decrease in the mortality rate

The two alternative ways to increase the population growth rate from 0.87% to 1.37% are either to decrease youth mortality, μ_0, to 82.751 or old age mortality, μ_1, to 0.0438. As the economic consequences are similar, we restrict attention to the latter.

From Table 3 we see that this leads to a 9.63% increase in the per capita stock of capital (from 5.337 to 5.851). This is illustrated by the upward shift of the line DD in Fig. 5, Panel (iii). It is consistent with the characterization of the steady state provided in Section 5 and the fact that the average age of wealth owners (52.20) exceeds that of labor (44.22). It is also consistent with the view emphasized by Kelley and Schmidt (1995) that an increase in the population growth rate resulting from a reduction in mortality will have a positive effect on the level of economic activity. This is because it increases the relative number of old people who have accumulated capital stock to contribute to the productive capacity of the economy. This increase in aggregate asset accumulation has several consequences It leads to a 3.3% increase in the wage rate (from 1.168 to 1.206) and a decrease in the rate of return on capital from 6.68% to 6.10%. It also leads to a 3.3% increase in per-capita output (from 1.797 to 1.856) and a negligible (0.03%) decline in per capita consumption with the increased population, the latater being illustrated by the imperceptible shift in the AA line in Fig. 5(i).

The distributional consequences are illustrated in Fig. 5 and are seen to be non-monotonic. Panel (i) shows that the increase in the wage rate coupled with the anticipation of the future lower return to capital causes a slight increase in consumption at birth. However, the decrease in the rate of return on capital decreases the consumption growth rate over the life cycle. Hence, after a few years agents experience a decrease in their consumption and since this is the experience of most cohorts, average per capita consumption declines. In Panel (ii) we see that the increase in longevity and associated increase in old age cohorts, coupled with the upward shift and flattening of the BB curve causes the CC curve to move out to C'C'. Thus, the increase in consumption of the very young causes their share of overall consumption to increase. However, the decline in the growth rate of consumption for people between around 30 and 80 causes their share of consumption to decline, while the increase in longevity leads to an increase in consumption share of the very old.

Panel (iii) reveals that the increase in longevity causes the EE locus to shift up and to the right. In early stages the life cycle the rate of asset accumulation declines very slightly, reflecting the decline in the rate of return on capital. As a result, the decline in mortality causes relatively young agents' wealth to decline slightly. However, the increase in longevity induces them to save for a longer period and to accumulate more assets in light of their increased longevity. Finally, Panel (iv) illustrates how with the increase in the relative size of old cohorts tilts the share of wealth significantly in their direction.

These patterns are consistent with the empirical evidence. For example, the fact that consumption declines for all but the youngest cohorts, while the wealth of older agents increase is consistent with the empirical findings of Fair and Dominguez (1991), Attfield and Cannon (2003), and Erlandsen and Nymoen (2008) all of whom find that the effect of an ageing population is to lead to a decline in overall per capita consumption for all equivalent income levels. The pattern we obtain of asset accumulation increasing with life expectancy agrees with the findings of Bloom, Canning, and Graham (2003).

5.2.3. Increase in birth rate versus decrease in mortality

Comparing Figs. 4 and 5 we see that achieving a specified increase in the population growth rate by increasing the birth rate or decreasing the mortality rate has dramatically different consequences for the economy. First, whereas only a mild increase in the birth rate of 0.44% will raise the population growth rate by 0.5%, to achieve the same objective by reducing mortality would require increasing longevity by around 12 years, which would seem to be a much more formidable task. Second, whereas a 0.5 percentage point increase in the population growth rate resulting from an increase in births will have only a slight negative effect on the productive capacity of the economy (measured by its per capita capital stock), the same increase in the population growth rate brought about by reduced mortality will have a significant expansionary effect. This contrast in magnitudes agrees exactly with the empirical results obtained by Blanchet (1988), thus emphasizing the importance of the form in which population growth occurs.

Finally, our results can be reconciled with the cross-country empirical evidence cited by Kelley and Schmidt (1995) who found that whereas population growth had had a negligible effect on growth during the 1960s and 1970s, it had a negative effect in the 1980s. This can be explained by comparing line 7 of Table 3 with line 4. Increasing the birth rate to only 2.65% and reducing old age mortality to 0.0524 causes the economic effects to be largely offsetting so that the per capita capital stock, output, wage rate, return to capital all remain unchanged. In summary, the changing mix between increased birth rate and decreased mortality can very naturally account for the different empirically estimated long-run effects of population growth rates at different stages of development.

6. Conclusions

This paper has introduced a realistic age-dependent demographic structure into a neoclassical growth model. In doing so, we have had two primary objectives. The first is to provide a general characterization of how the demographic structure impinges on the macrodynamic equilibrium. We show how this depends on the generational turnover term, which is an integral component of the intertemporal consumption allocation decision. Setting up the aggregate dynamics as a generalization of the conventional neoclassical growth model, provides two major insights. Not only does it enable us to view alternative demographic specifications in a unified way, but also we are able to identify two, rather than just one, viable steady-state equilibria. The first is highly sensitive to the demographic structure, whereas in the second equilibrium demographic factors play but a minor role.

The second objective is to analyze the effect of structural changes – most importantly demographic structural changes – on both the aggregate macro equilibrium, as well as the distributional life-cycle implications. This is done numerically using the very general survival function proposed by Boucekkine, de la Croix, and Licandro (2002). The most striking result is the sharp contrast, both qualitatively and quantitatively, in the effects of changes in the population growth rate on the macro economy. Whether an increase in population occurs because of an increase in births or a decrease in mortality is crucially important, and in this regard our results corroborate the empirical findings obtained in the demographic literature.

While we view our paper as being canonical, it clearly can be extended in various directions. First, the contrast between births and mortality in influencing the population growth rate and the resulting consequences for distribution across cohorts and for the aggregate economy raises interesting policy issues for a country seeking to influence its population growth rate. Second, it is straightforward to extend the framework to allow for retirement and to address issues pertaining to social security and retirement benefits, issues that are of crucial importance for the US and other countries with their ageing populations. Finally, while we have focused on the long-run (steady-state) implications of demographic structural changes, the nature of the transition from one steady-state to another is also important.

Table 1
Estimated Survival Functions

A. Boucekkine *et al.* (2002) survival function

$S(u) = I(u \le D)\dfrac{\mu_0 - e^{\mu_0 u}}{\mu_0 - 1} + \varepsilon$ where* $\varepsilon \sim i.i.d.(0, \sigma^2)$		
US Cohort	1960	2006
μ_0 (st. dev.)	43.9817 (3.4183)	78.3618 (6.0193)
μ_1 (st. dev.)	0.0535 (0.0012)	0.0566 (0.0011)
Adj. R^2	0.9957	0.9961

* $I(u \le D)$ is an indicator function that is 1 for $u \le D$ and 0 otherwise.

B. Blanchard (1985) survival function

$S(u) = e^{\mu u} + \varepsilon$ where $\varepsilon \sim i.i.d.(0, \sigma^2)$		
US Co0068ort	1960	2006
μ (st. dev.)	0.0141 (0.0011)	0.0112 (0.0011)
Adj. R^2	0.6708	0.6157

Table 2
Baseline Parameters and Benchmark Equilibrium

Baseline Model			
Structural Parameters		BCL[1]	BBW[2]
Total factor productivity	A	1	1
Capital share of output	α	0.35	0.35
Depreciation rate	δ	5%	5%
Inter-temporal substitution elasticity	σ	0.5	0.5
Time preference rate	ρ	3.5%	3.5%
Time preference of average individual	$\rho+\mu(\bar{u})$	4.05%	4.91%
Demographic Parameters			
Youth mortality	μ_0	43.9817	N/A
Old age mortality	μ_1	0.0535	0.0141
Birth rate	β	2.37%	2.37%
Life-expectation at 18 (Age)	L_{18}	71.70	88.85
Average age of workers	α_W	44.22	42.19
Average age of asset holders	α_A	52.20	79.75
Maximum attainable age (implied)	D	88.74	∞
Population growth rate (implied)	n	0.87%	0.96%
Implied Economic Variables			
Per capita capital stock	$\tilde{k}$	5.3372	7.2002
Per capita output	$\tilde{y}$	1.7970	1.9956
Capital/Output ratio	$\tilde{k}/\tilde{y}$	2.9700	3.6081
Real interest rate	$\tilde{r}$	6.78 %	4.7%
Wage rate	$\tilde{w}$	1.1681	1.2671
Average per capita consumption	$\tilde{c}$	1.4836	1.5665
Marginal propensity to consume at birth	$[\Delta_B]^{-1}$	0.0573	0.0551
Coefficient of variation of assets	θ_A	0.4954	0.6535

[1]Boucekkine *et al.* (2002)
[2]Blanchard (1985)-Buiter (1988)-Weil (1989)

Table 3
Structural Changes

		Demography		Economic Variables						
		L_{18}	n	$\tilde{k}$	$\tilde{y}$	$\tilde{k}/\tilde{y}$	$\tilde{r}$	$\tilde{w}$	$\tilde{c}$	$[\Delta_B]^{-1}$
Baseline Model		71.20	0.87%	5.337	1.797	2.970	6.78%	1.168	1.484	0.0568
Increase in productivity	$A \to 1.25$	71.20	0.87%	7.523	2.533	2.970	6.78%	1.647	2.091	0.0568
Demographic Shocks										
Increase in birth rate	$\beta \to 2.69\%$	71.20	1.37%	5.268	1.789	2.945	6.89%	1.1627	1.453	0.0573
Decrease in youth mortality	$\mu_0 \to 82.751$	82.86	1.37%	5.748	1.844	3.117	6.23%	1.199	1.478	0.0521
Decrease in old age mortality	$\mu_1 \to 0.0438$	83.57	1.37%	5.851	1.856	3.153	6.10%	1.206	1.483	0.0518
Off-setting change in birth rate and old age mortality	$\beta \to 2.65\%$ $\mu_1 \to 0.0524$	72.37	1.37%	5.337	1.797	2.970	6.78%	1.168	1.457	0.0566
US Demographic change										
2006 Mortality and birth rate	$\beta \to 1.4\%$ $\mu_0 \to 78.3618$ $\mu_1 \to 0.0566$	78.38	-0.51%	5.812	1.852	3.139	6.15%	1.203	1.590	0.0524

Figure 1. Steady State Equilibrium

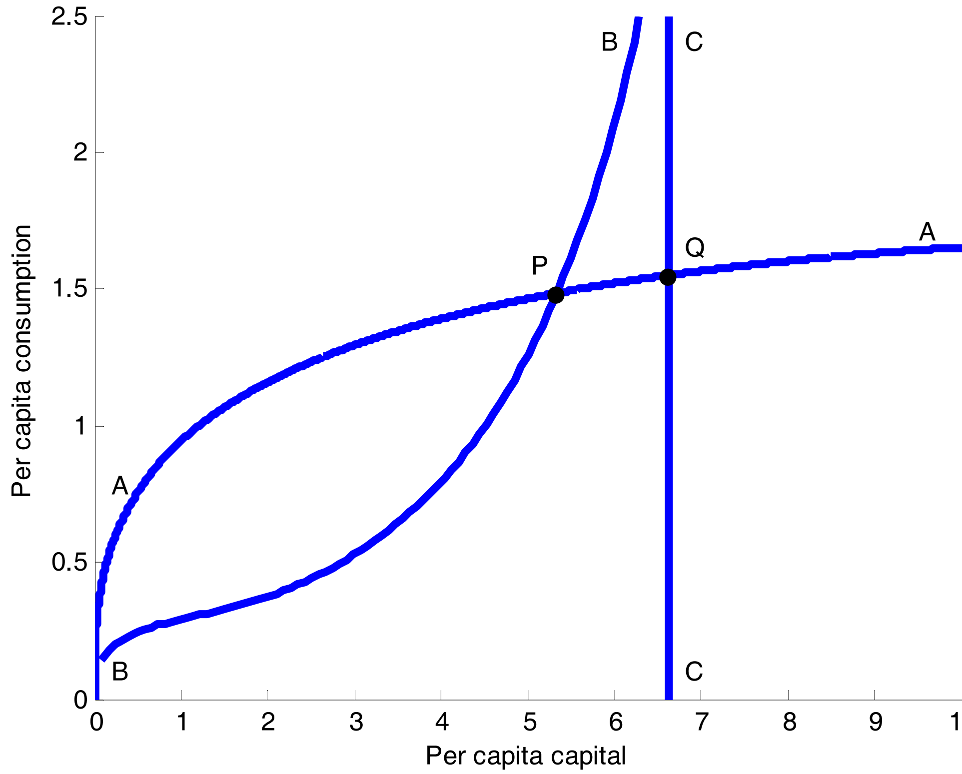
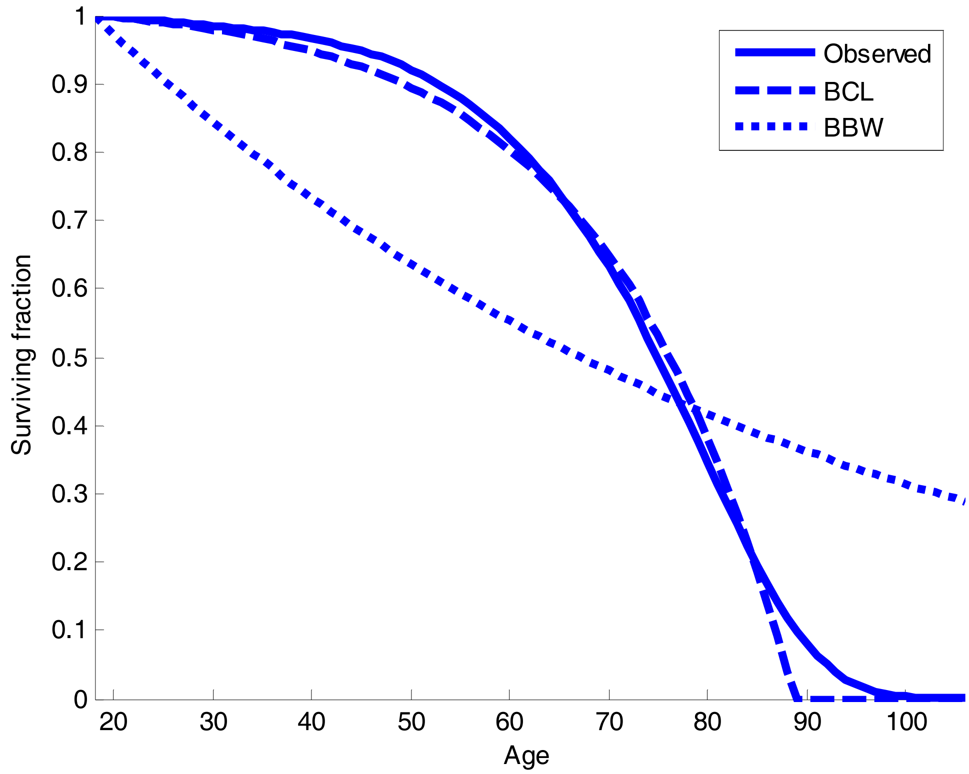


Figure 2. Demography

A. Actual and Estimated Survival Functions



B. Shift in BCL Survival Function

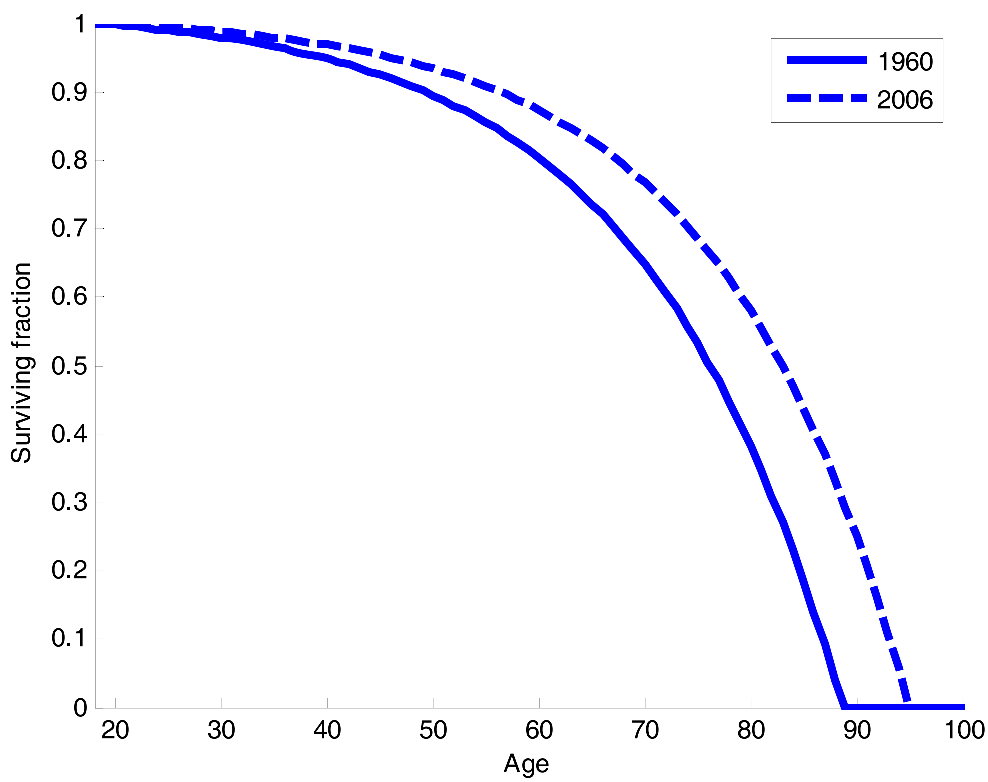


Figure 3A. Increase in Productivity: BCL

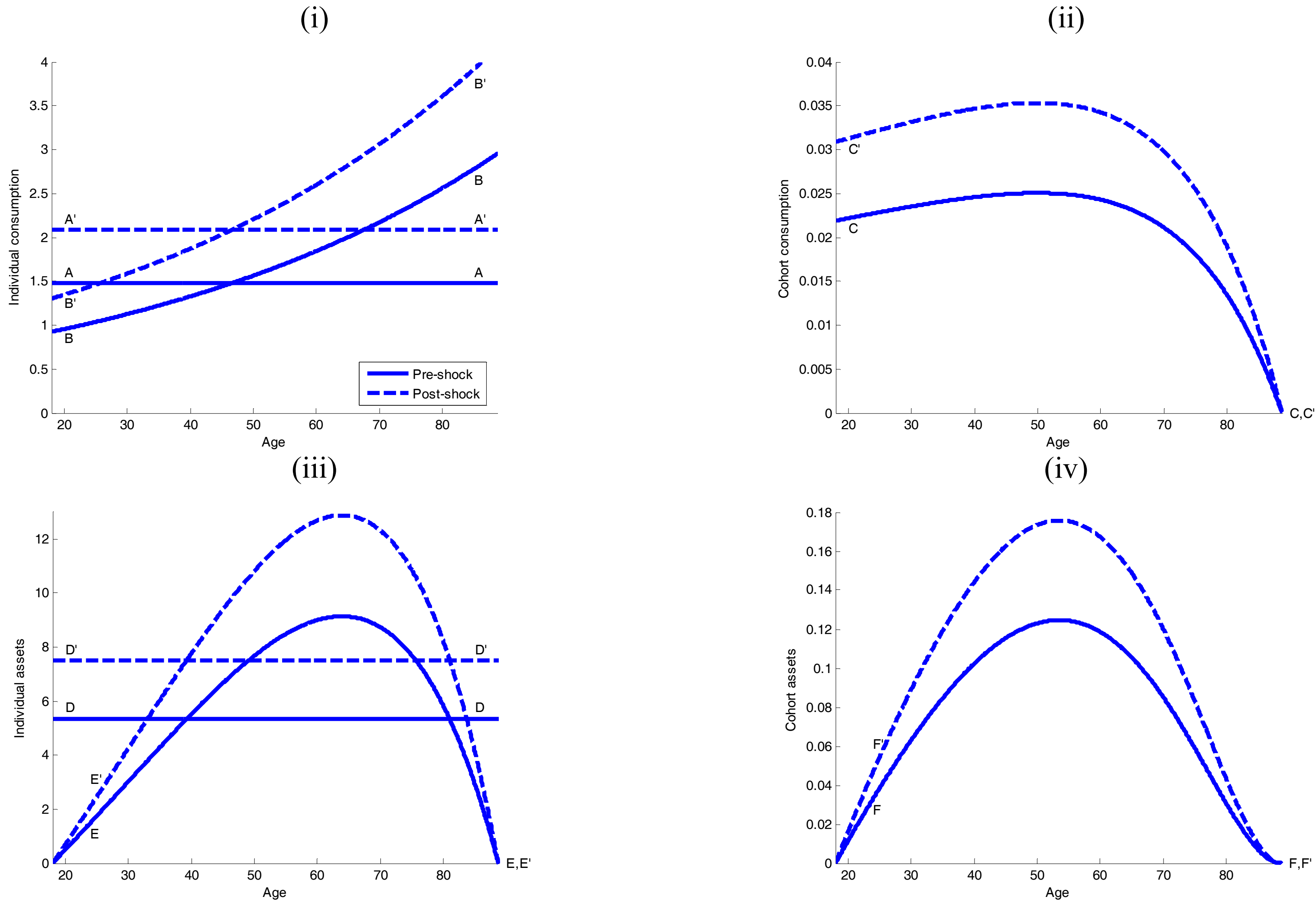


Figure 3B. Increase in Productivity: BBL

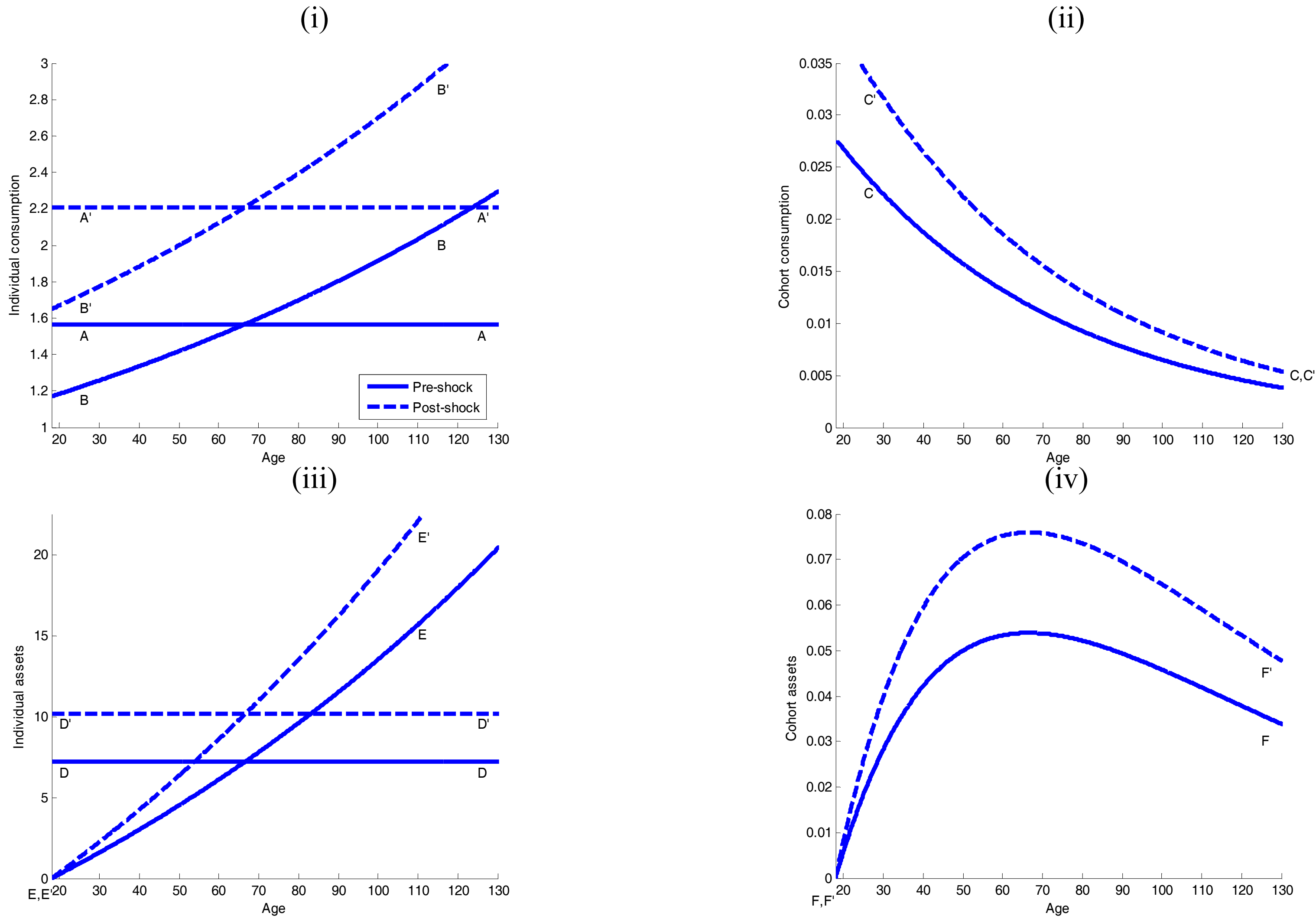


Figure 4. Increase in Birth Rate

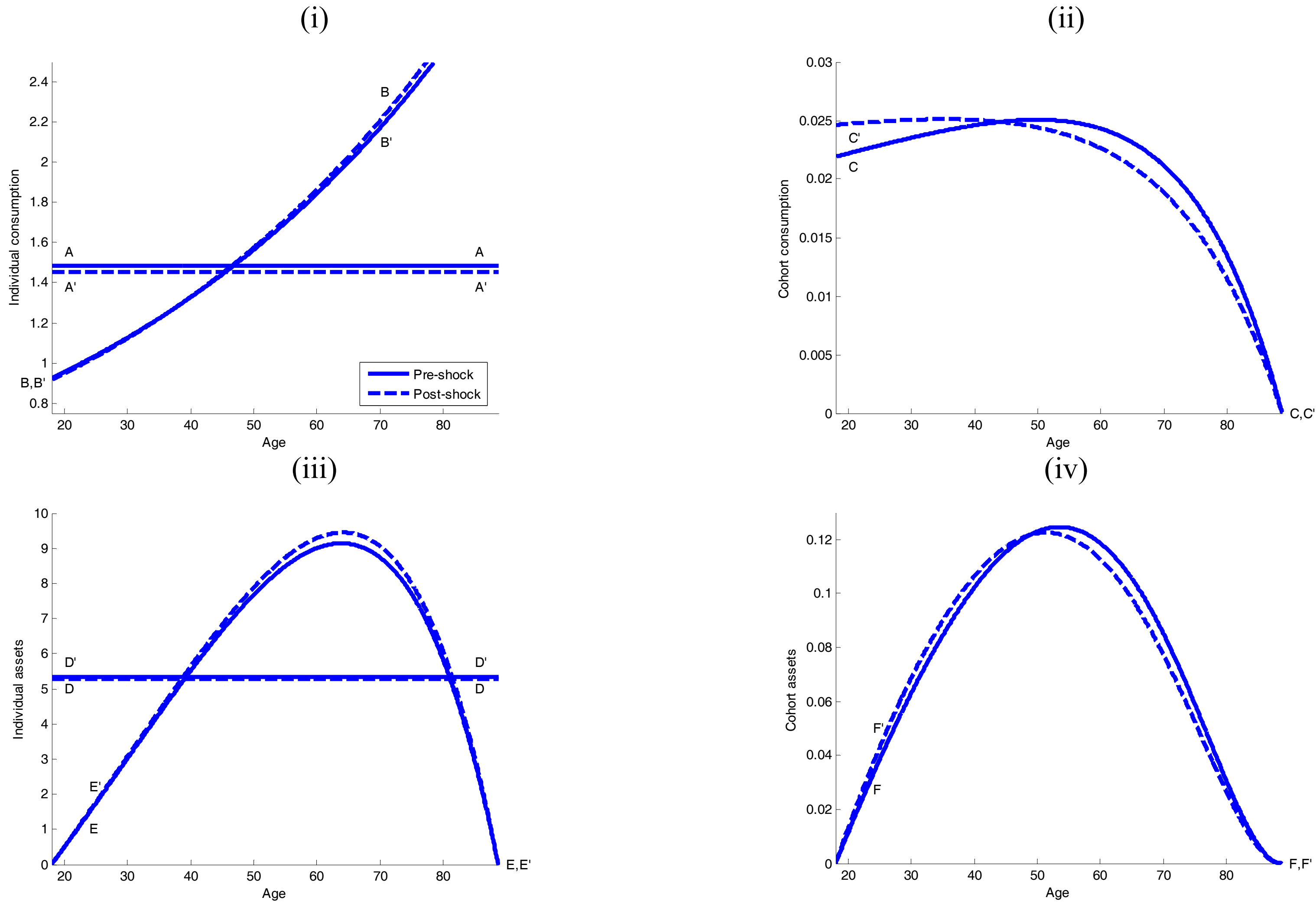


Figure 5. Decrease in Old Age Mortality

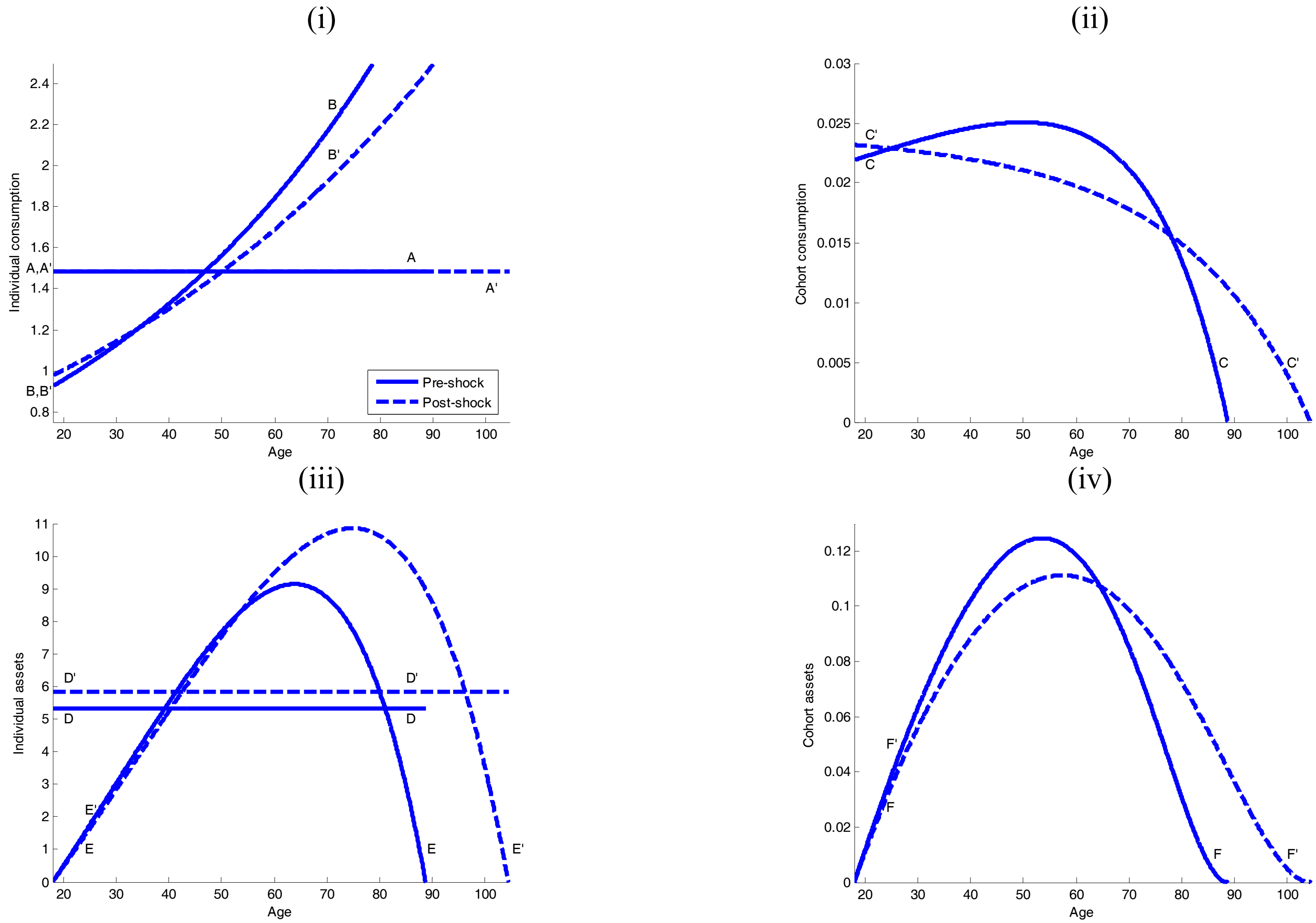
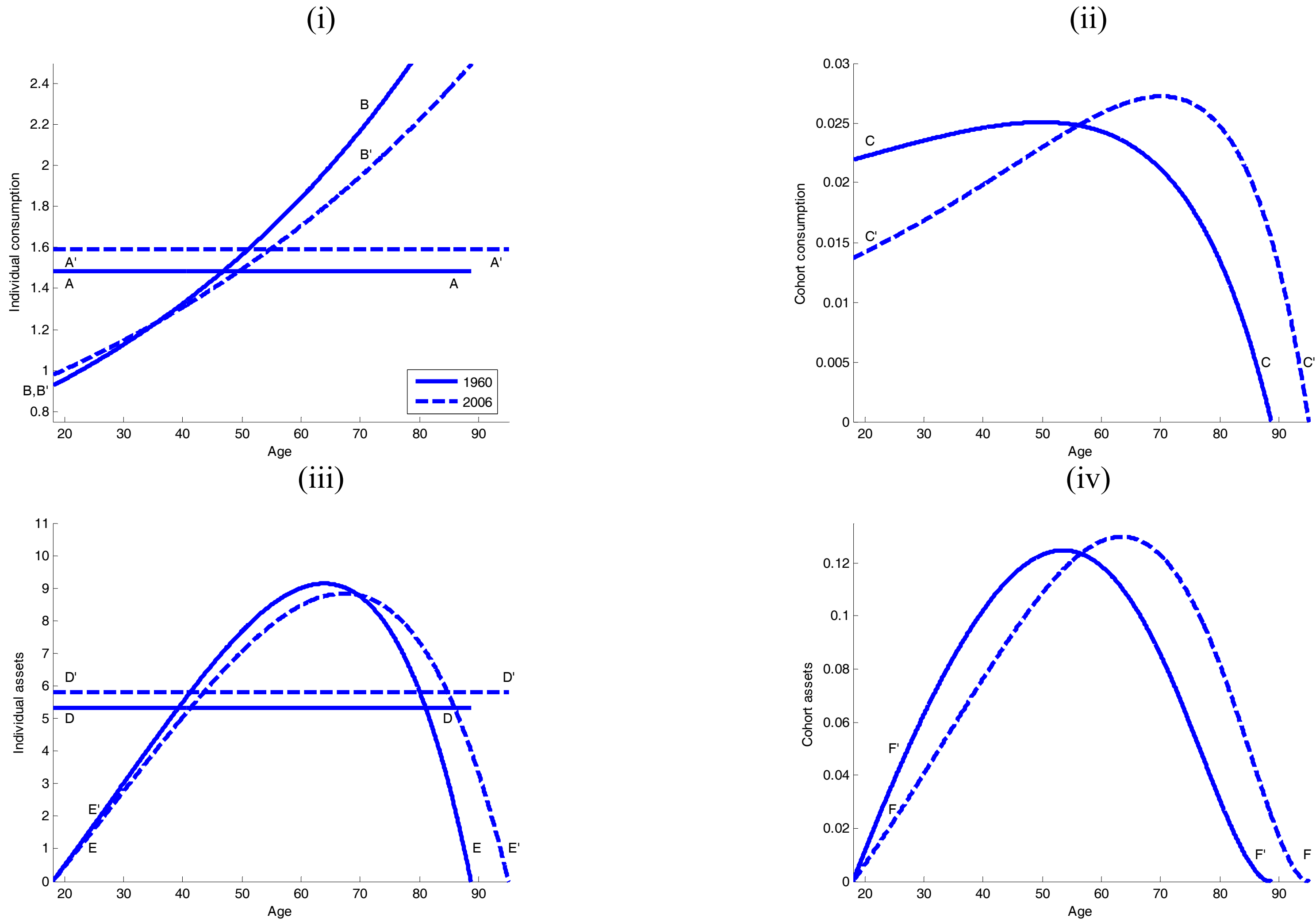


Figure 6. US Demographic Change 1960-2006



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