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Abstract

This paper shows that the results of Xie (1997), and Karp and Lee (2003) are strongly dependent on the specification of utility function, and that how their results are modified under the other forms of function. Unlike Long, Shimomura and Takahashi (1999), who investigate the difference between the open-loop solution and the feedback in a class of Nash game, we compare these two solutions in a class of Stackelberg game, by employing a dynamic optimal taxation problem. In this paper, we show following two points. First, depending on the value of relative risk aversion, there is a possibility that the open-loop and feedback solutions coincide. In other words, the open-loop taxation is *time-consistent* in case of logarithmic utility function. Secondarily, with the aid of numerical simulation, we show how the difference in transition path between two solution concepts varies, depending on the value of relative risk aversion.

Keywords: Differential Games; Dynamic Optimal Taxation; Time-inconsistency;
Comparison between Open-loop and Feedback Solution.

JEL Classification: C73; E61; H21.

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1 Introduction

Since the seminal work, Chamley (1986) and Judd (1985), *time-inconsistency* problem in the field of dynamic optimal taxation has attracted the interest of many researchers. They showed that the optimal capital-income taxation should be zero in the long run using the concept of open-loop solution, not feedback solution[1)]. This fashion of analysis is exposed to the risk of *time-inconsistency*, however. Despite of many studies after their work, under what kind of environment does the open-loop solution coincide with the feedback solution remains unsolved[2)].

In this paper, we show that (1) the results of Xie (1997) and Karp and Lee (2003) are strongly dependent on additive-separation of the utility function, and (2) under the general forms of function, how their result is modified[3)], and finally, (3) the transition path to the steady state of capital-income tax under each solution concept with the aid of numerical simulation.

The rest of this paper is organized as follows: The next section reviews the existing results, especially, Xie (1997), and Karp and Lee (2003). The section 3 describes the model and the results. The section 4 shows the numerical investigation and the section 5 is the conclusion.

2 Review of Existing Results

Before entering the main section, we briefly review the result of Xie (1997), and Karp and Lee (2003) and explain the drawbacks of the conventional method in this section. The model structures to Stackelberg game, in which the government and household are leader and follower, respectively. Households solve the following problem subject to the budget constraint. Here, the utility function and production function are specified as $\ln c_t + \ln g_t$ and $y_t = Ak_t$ respectively. c_t, k_t, and g_t denotes consumption, capital, and government spending, respectively.

$$\max \int_0^{\infty} \{\ln c_t + \ln g_t\} e^{-\delta t} dt, \;\; s.t.\, \dot{k} = (1-\tau_t)Ak_t - c_t, \tag{1}$$

where τ is capital-income tax. Associated with the Hamiltonian,

$$H_1 = \ln c_t + \ln g_t + \lambda_t [(1-\tau)Ak - c],$$

1) As other studies which analyze dynamic optimal taxation based on open-loop solution, see Jones, Manuelli and Rossi (1993), Lansing (1999), Coleman II (2000), for instance.

2) Although it is extremely difficult to derive the feedback solution, as one of the few exceptions, Tsutsui and Mino (1990) derive the feedback solution under the case where the value function is quadratic. Alternatively, Kemp, Long and Shimomura (1993) compare the open-loop with the feedback solution, using the model of Judd (1985). Moreover, Long and Shimomura (2000) also gave an example in which the open-loop solution is *time-consistent*. Recently, Klein, Krusell and Ríos-Rull (2008) shows the condition for derivation of Markov perfect equilibrium.

3) Ortigueira (2006) investigates the properties of Markov-perfect taxation by numerical simulation, but he does not focus on the case of open-loop taxation.

from first order conditions, we obtain $c_t = \frac{1}{\lambda_t}$, and

$$\dot{\lambda} = \lambda\{\delta - (1-\tau)A\}. \tag{2}$$

This equation can be also interpreted as the followers response function. Note that $c_t = \frac{1}{\lambda_t}$ shows that λ_t does not dependent on tax rate. We then obtain the sequence of consumption plan as

$$c_t = \delta k_t. \tag{3}$$

On the other hand, the government aims to maximize the same objective function, subject to the budget constraint, $g_t = \tau Y_t = \tau A k_t$. Then, we derive the path of capital-income taxation as

$$\tau_t = \frac{\delta}{2A}. \tag{4}$$

As easily verified, this taxation is *time-consistent* because it does not depend on time, t. Now, considering followers response function; eq. (2) and (3) as well as the household's budget constraint, let us define Hamiltonian for this problem as follows:

$$H_2 = \ln(\delta k_t) + \ln(\tau_t A k_t) + v_t\{(1-\tau_t)Ak_t - \delta k_t\} + \pi_t \lambda\{\rho - (1-\tau_t)A\} \tag{5}$$

v_t and π_t is co-state variable on the budget constraint and eq.(2), respectively. After some calculation, we obtain

$$\tau_t = \frac{\delta}{2A - \left(\lim_{T\to\infty} \pi_T \lambda_T e^{-\delta T}\right) e^{\delta t}}.$$

If this equation coincides with eq.(4), the following condition should hold:

$$\lim_{t\to\infty} \pi_t \lambda_t e^{-\delta t} = 0 \tag{6}$$

However, as briefly explained in Mino (2001), this result means that the following condition,

$$\frac{\partial V_1(k_0, \lambda_0)}{\partial \lambda_0} = \pi_0 = 0 \tag{7}$$

is *NOT* sufficient condition for which the solution derived by the Lagrange method is *time-consistent*[4]. In other words, Xie showed that there is a possibility that open-loop solution is *time-consistent* even if $\pi_0 \neq 0$. This is because π_t is shadow price of λ_t, and therefore this problem is uncontrollable. Correspondingly, Karp and Lee (2003) gave following answer: The problem for the government is mostly the same as in the above. The utility function is $U(c_t, g_t) = u(c_t) + v(g_t)$. Setting $J(\cdot)$ as value function of households (i.e. followers), Hamilton-Jacobi-Bellman (henceforth, HJB) equation can be written as[5],

$$\delta J(k,t) = \max_c \ \{u(c) + J_k(k,t)(f(k) - b(k)\tau(k) - c)\} + J_t(k,t). \tag{8}$$

[4] Note that $V_1(\cdot)$ is the leader's value function. Moreover, this equation is also the necessary condition for derivation of the open-loop solution.

[5] Here, the term of $v(g_t)$ is ignored in Karp and Lee (2003). $b(k)$ is the tax rule which satisfies $g(k,t) = b(k)\tau(t)$.

Under such a setting, Karp and Lee claimed that the necessary and sufficient condition for which the open-loop solution is *time-consistent* is that the value function, $J(\cdot)$ in eq. (8) is additively separable as for time and state variable, that is, $J(k,t) = W(k) + Z(t)$[6]. However, recently, Cellini and Lambertini (2007) claimed that their result is based on the following assumption of (1) additively-separable utility function and (2) the homogeneity of household. To sum up, as pointed in Shibata and Takeda (1997), the relationship between controllability and *time-inconsistency* problem remains unsolved under the general forms of utility function even at present.

3 Model and Main Results

3.1 The Model

In this section, we show the main result, using the variants of Karp and Lee (2003). In what follows, we adopt the method similar to that of Klein et al. (2008). In order to highlight the role of the non-separability of utility function, we show how their result is corrected by relaxing the assumption of additively-separable utility function by replacing with the following form,

$$U \equiv \int_0^\infty U(c_t, g_t)e^{-\delta t}dt = \int_0^\infty \frac{(c_t^\theta g_t^{1-\theta})^{1-\phi} - 1}{1-\phi} e^{-\delta t}dt,$$

where $\phi, \theta (> 0)$ are exogenous parameters[7]. Here, θ is interpreted as the importance of private consumption relative to government spending and ϕ denotes the inverse of inter-temporal elasticity of substitution (,or the value of relative risk aversion). The households are initially endowed with given amounts of capital (k_0) and public goods (g_0). Except for the modification of the utility function, our model is the same as Karp and Lee (2003). The budget constraint of households is

$$\dot{k} = (1-\tau_t) f(k_t) - c_t - g_t.$$

Next, let us describe the firms' behavior. It is assumed that factor markets are perfectly competitive and that firms maximize their profits. Labor and capital stock are used for production; production technology yields constant returns to scale. Therefore, production functions are expressed as $Y_t = F(K_t, L_t) : \Re_+^2 \to \Re_+$, where K_t, L_t, and Y_t respectively represent capital stock, labor, and output in aggregate terms. Setting k_t as $\frac{K_t}{L_t}$, the maximizing process yields

$$w_t = f(k_t) - k_t \cdot f'(k_t) \equiv w(k_t), \;\; R_t = f'(k_t) \equiv R(k_t).$$

In that equation, R_t and w_t respectively indicate the rental rate of capital and the real wage rate. Then, the government's expenditure is financed by the following balanced-budget rule:

$$g_t = \tau_t [w_t + R_t k_t] = \tau_t f(k_t), \tag{9}$$

[6] As for this result, see also Mino (2001), who offers the similar result.

[7] Note that if $\phi = 1$, this case approximately coincides with the Xie (1997)'s model.

which means that the rate of capital-income tax equals to that of labor-income tax.

3.2 Comparison Open-loop with Feedback solution

In this subsection, we then investigate the condition for the coincidence of open-loop and feedback solution.

Open-loop Solution.

Considering the households' response function[8)] is

$$U_c(c_t, g_t) = \delta U_c(c_{t+1}, g_{t+1})(1-\tau_t) f'(k_{t+1}) \tag{10}$$

the Hamiltonian for the leader's problem is written as

$$J_G = \frac{(c_t^{\theta} g_t^{1-\theta})^{1-\phi} - 1}{1-\phi} \\ + \mu_{2t}\{-U_c(c_t, g_t) + \delta U_c(c_{t+1}, g_{t+1})(1-\tau_t) f'(k_{t+1})\}, \tag{11}$$

where μ_{2t} is co-state variable. The necessary conditions are following first order conditions:

$$\frac{\partial J_G}{\partial \tau} = U_c + \mu_{2t}\{-U_{cc}(c_t, g_t) + \delta U_{cc}(c_{t+1}, g_{t+1})(1 + (1-\tau_t) f'(k_{t+1}))\} = 0 \tag{12a}$$

$$\frac{\partial J_G}{\partial g} = U_g + \mu_{2t}\{-U_{cg}(c_t, g_t) + \delta U_{cg}(c_{t+1}, g_{t+1})(1 + (1-\tau_t) f'(k_{t+1}))\} = 0. \tag{12b}$$

$$\dot{k} = \frac{\partial J_G}{\partial \mu_{2t}} = -U_c(c_t, g_t) + \delta U_c(c_{t+1}, g_{t+1})(1-\tau_t) f'(k_{t+1}), \tag{12c}$$

together with the transversality condition is

$$\lim_{t\to\infty} g_t \mu_{2t} e^{-\delta t} = \lim_{t\to\infty} k_t \mu_{1t} e^{-\delta t} = 0.$$

From these equations, we obtain the open-loop solution; $\{c_t^*, g_t^*, k_t^*, \tau_t^*\}_{t=0}^{\infty}$.

Feedback Solution.

On the other hand, let us derive the feedback solution. The feedback solution is derived through dynamic programming and we employee the HJB approach. Let $V(\cdot)$ be the value function of government[9)]

$$\delta V(k,t) = \max_{\tau_t, g_t} \left[U(c_t, g_t) + V_k(k,t)\{f(k_t) - f(k_t)\tau_t - c_t - g_t\}\right] + V_t(k, t+1) \\ s.t. \text{ eqs. (9) and (**??**)}. \tag{13}$$

[8)] For derivation of these equations, see the appendix.

[9)] In the model of Karp and Lee (2003), the term $v(g_t)$ can be ignored because the utility function is additively-separable, while, in our model the same step cannot be applied. Moreover, note that by the definition of Markov property, the feedback solution meets the property of *time-consistency*.

We then obtain the following first order conditions for an interior maximum as,

$$\frac{\partial V(k,t)}{\partial c_t} = U_c(c_t, g_t) + V_k(k,t)\cdot(-1) + V(k,t)\{f'(k_t) - f'(k_t)\tau_t\} = 0, \tag{14a}$$

$$\frac{\partial V(k,t)}{\partial g_t} = U_g(c_t, g_t) + V_k(k,t)\cdot(-1) = 0. \tag{14b}$$

Moreover, Benveniste=Scheinkman equation using the envelope theorem is obtained as,

$$\delta\cdot\frac{\partial V(k,t)}{\partial k_t} = V_k(k,t)\cdot(-1) + V(k,t)\{f'(k_t) - f'(k_t)\tau_t - g_t\} = 0. \tag{14c}$$

Using these equations, we have,

$$\dot{k}_t = R_t k_t - U_c \delta c_t. \tag{15}$$

By showing that the open-loop solution coincides with the feedback, the open-loop solution is *time-consistent*. Then, we have following:

Proposition 1 *Depending on the value of relative risk aversion (ϕ), whether the open-loop solution is time-consistent is determined as follows:*

case 1. In case of $\phi = 1$, these two solutions coincide.
case 2. In case of $\phi \neq 1$, these solutions does not coincide.

Proof *If $\phi = 1$, eq.*(12c) *is rewritten as*

$$\phi_t \dot{k}_t = \phi_t \cdot f(k_t)\cdot k_t - 1$$

So, together with (12a)*, this equation is rewritten as,*

$$c_t = \delta\frac{d\phi}{dt}U_{cg} \tag{16}$$

Comparing eq.(12c) *and eq.*(14c)*, the following condition should hold:*

$$(\phi - \frac{d\phi}{dt})U_{cg} = 0$$

To make these two solutions quadrate, we find the term $U_{cg} = 0$, which means that the utility function is additively separable. Therefore, this case corresponds to the case, $\phi = 1$. ∎

Remark 1. This theorem says, in case of $\phi = 1$, the open-loop solution is *time-consistent*. Otherwise, it is *time-inconsistent*. From another angle, this result shows that the results of Xie (1997) and Karp and Lee (2003) are strongly dependent on the assumption that the utility function is additively-separable.
Remark 2. Let us explain the relationship between this result and Controllability[10)]. How the relationship between the co-state variable on the households' response function, and the equation, $\lambda(0) = 0$?

[10)] As for this issue, see Dockner, Jorgensen, Long and Sorger (2000, Ch.5).

Before explaining it, let us define the concept of controllability.

Definition

Uncontrollability means the co-state variable of the follower is independent on the leader's choice, while Controllability means that the co-state variable of the follower is dependent on the leader's choice.

In the models of Xie (1997) and Karp and Lee (2003), their problem is uncontrollable. In other words, the leader (i.e. the government) cannot the co-state variable by controlling the policy variable in their model. On the other hand, in our model, the co-state variable at initial point is...

4 Numerical Analysis

In the previous section, we have showed the condition for which the open-loop solution is *time-consistent* at the steady state, but it remains undone to clear up how these two solutions differ in the paths to the steady state. This section presents how the open-loop solution differs from the feedback with the aid of numerical simulation[11]. Here, in order to push through numerical simulation, let us translate the settings described in the previous section into discrete time version. Then, we set the utility function is $U(\cdot) = \sum_{t=0}^{\infty} \left(\frac{1}{1+\delta}\right)^t \frac{(c_t^{\theta} g_t^{1-\theta})^{1-\phi} - 1}{1-\phi}$, where $\delta = 0.6$, $\phi = 0.3$, and $\theta = 0.4$. Moreover, we set the production function as $f(k_t) = k_t^{\alpha}$ $(\alpha = 0.3)$.

The algorithm for derivation of the open-loop and feedback solution is respectively summarized as[12],

- Open-loop Solution
  1. Determine the initial point and the value of criterion, $\varepsilon > 0$.
  2. Linearize the FOCs around the steady state.
  3. Derive the difference equation.
  4. Repeat until the convergence.
- Feedback Solution
  1. Determine the value function $V_0(\cdot)$ at the initial point by guess-and-verify method.
  2. Determine the initial point and the value of criterion, $\varepsilon > 0$.
  3. By the method of self-generation, seek the the value function $V_1(\cdot)$.
  4. Repeat until $|V_{t+1} - V_t| < \varepsilon$.
  5. If the above condition holds, Stop.

The right-side of the above figure (fig. 1) depicts the transition paths of open-loop solution and the left side (fig. 2) depicts the feedback solution. From the above figures, we find that the open-loop

[11] Regarding the Lucas (1990)-type, see Grüner and Heer (2000).

[12] These algorithms of open-loop and feedback are based on Judd (1998, Ch.12) and Novales, Dominguez, Perez and Ruiz (1999), or Chari, Christiano and Kehoe (1995), respectively. As a simulation software, we use Matlab 2010b.

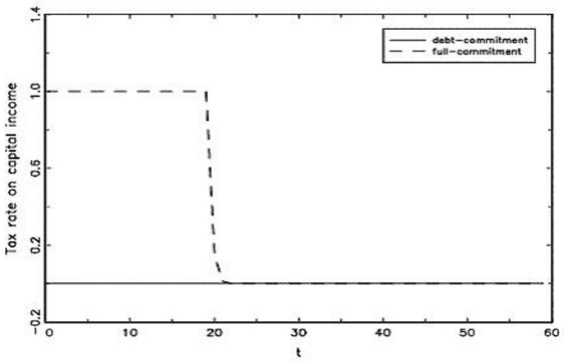

Figure 1 Open-loop Solution

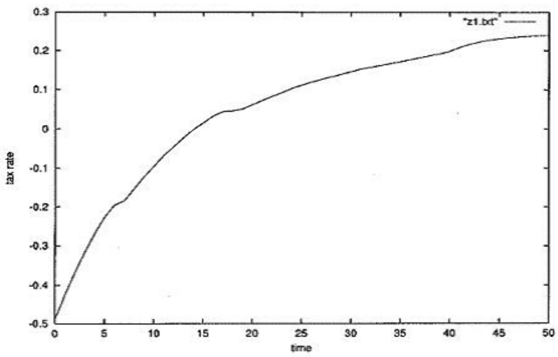

Figure 2 Feedback Solution

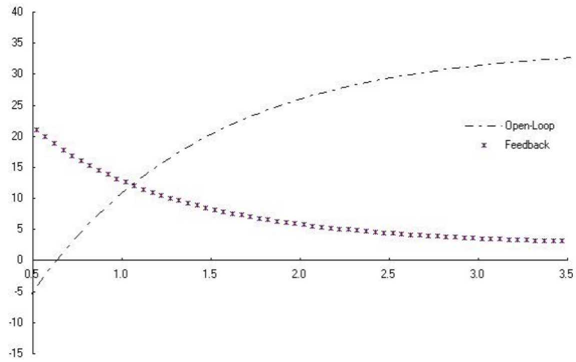

Figure 3 The difference between two solutions

solution decreases as time passes, while the feedback increases as time passes. It should ne noted that the feedback solution is less than zero at the early stage, which means subsidy from the government, and after that the tax rate becomes positive. Intuitively, this is because to promote capital accumulation, the government set tax rate less than zero.

■The Effect of the Difference in Relative Risk Aversion We then show that depending on the value of relative risk aversion, how these two solutions differ at the steady state by numerical simulation. By repeating simulation from the case $\phi = 0.1, 0.2$, to $\phi = 3.0$ by every 0.1, we obtain the fig. 3. From this figure, we find that at the case $\phi = 1$, these two solution coincide. As the value of ϕ is larger, the difference between two solutions larger.

To summarize, we have:

Proposition 2 *As the value of ϕ is larger, the difference of capital-income tax rate under two solutions at the steady state becomes larger. When $\phi = 1$, these two solutions coincide.*

5 Conclusion

We analyze the properties of both open-loop and feedback solution in this paper. Future research is to show (1) the difference between inelastic labor supply case and elastic case and (2) derivation of feedback solution explicitly under the more general form of function.

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