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Public provision of private goods with optimal income taxation and extensive margin

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Tuomas Kosonen†

February 9, 2011

Work in Progress

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This paper studies the public provision of private goods. It presents a model where a government sets taxes optimally and provides a private good. In contrast to the earlier literature on public provision of goods, the good provided by the government affects the extensive margin of labour supply. A fixed cost not visible to the government arises when an individual participates. The publicly provided private good affects this fixed cost. I derive an optimal rule for the provision of the publicly provided private good in this set-up. The resulting rule links the optimal public provision rule with its effect on the share of the population in work and the tax differential associated with the participation decision. It can be welfare improving to have positive levels of public provision even when optimal income taxes are used to redistribute income.

1 Introduction

The degree of government-provided public services varies across countries. Health care, day care for children, care of the elderly and education are mostly

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publicly provided in the Nordic countries, while in Anglo-Saxon countries the system relies more on private markets. When is public provision at an optimal level? How does the public provision of goods depend on the tax system? These are questions that this study tries to answer.

This paper studies publicly provided private goods in a setting where income taxes have been set optimally. Therefore this study is connected to the Mirrlees tradition. One assumption in the original Mirrlees (1971) model is that there is only an intensive labour supply margin. The intensive margin describes the extent to which each individual changes his or her labour supply. This assumption is not realistic, since it is relatively uncommon to work a very small amount of hours. Recent optimal income tax literature has addressed this issue (Saez 2002, Immervoll et al. 2007 and Eissa et al. 2008). Assuming that there is also an extensive labour supply margin can make it optimal to have negative marginal tax rates. Consequently, the tax burden on the working poor and at the same time unemployment benefits can be radically reduced. The significance of these results is borne out in empirical studies that find the extensive labour supply elasticity to be larger than the intensive labour supply elasticity (Eissa and Hoynes 2004, Blundell 2006).

In this paper the publicly provided private good affects the participation decision. To create a micro-founded reason for the extensive margin, there is a fixed cost arising from participation. This assumption has been used in earlier papers studying the extensive margin (Cogan (1981), Immervoll et al. (2007) and Eissa et al. (2008)). The intuition is that to go to work, one needs to suffer or pay some fixed cost that need not be paid if the participation decision is not made. The fixed cost is not visible to the government, but the government provision affects the fixed costs and thereby the number of individuals participating in the labour force. Therefore, in addition to setting taxes optimally, the government can affect the number of taxpayers through the deployment of public expenditure. This kind of dependence between public provision and participation decisions in connection with optimal taxes has not been made in earlier public goods literature[1].

[1]Bergstrom and Blomquist (1996) studied the optimality of child care in connection with the participation decision

However a literature has studied optimal public provision with the intensive margin of labour supply in a setting where income taxes have been set optimally. Public provision should follow the Samuelson rule if consumption of public good does not convey additional information on individual ability types (Mirrlees 1976). Such is the case if labour supply and consumption are separable in the individual utility function (Christiansen 1981 and Boadway and Keen 1993). Conversely, if the consumption of public good does reveal information about the type of the individual, it can be optimal to provide such a good to a different extent than the Samuelson rule implies. When, for example, public good is complementary to working, it should be over-provided relative to the Samuelson rule because in this case public good benefits more those who work more. Then public good provision makes the distortions created by income taxation less severe.

Since the public provision in this paper is targeted at individuals who decide to participate, the good is private in nature. Publicly provided private goods have been studied in a number of papers (Blackorby and Donaldson 1988 and Besley and Coate 1991). Although there is no market failure as with pure public goods, it can be socially optimal for the state to provide these goods rather than redistribute through taxation. Usually the reason for this is related to information asymmetry. Blomquist et al. (2010) studied public provision of private goods in connection with optimal income taxation. Thus, their article is closely related to this study, although they did not have extensive labour supply margin in their model.

Including the ingredients of publicly provided private goods and non-linear income taxes with the extensive margin of labour supply, this paper provides a rule for optimal government provision of a private good. The results imply that even if the government uses optimal income taxation to redistribute income as efficiently as possible, it can be welfare improving to provide a private good publicly. This situation occurs when the public provision decreases the fixed costs of individuals. The limiting case, when public provision does not affect welfare and thus should be at zero level, is when it has no effect on the distribution of fixed costs. In all other cases the provision rule depends on direction in which public provision affects the distribution of fixed costs. The participation tax differential amplifies the effect public provision has on fixed costs.

The assumption that part of the population is excluded from consuming the good fits the case of the day care of children particularly well. Moreover the access to good quality day care to an individual clearly affects costs arising from participation. If access to day care for children is less costly to parents, it is easier for the parents to work rather than staying at home taking care of their children. Furthermore, it is entirely conceivable that the size of the net tax differential between the taxes when participating and when not affects the participation decision. The greater is the net difference in taxes from participating including the cost of day care, the more extra provision of day care can affect the participation decisions of parents.

Section 2 describes the key assumptions made in the model. It discusses the use of the fixed cost, individual optimisation problem and the features of the government objective function. It derives first-order conditions and optimal tax rates. Section 3 derives the rule for the public provision of the good in the case of one ability type and discusses the features of this simple rule. Section 4 adds a continuum of ability types to the model and presents the rule for publicly provided private goods in this case. Finally, section 5 concludes.

2 The basic set up

This section presents how participation decisions are analysed with only one ability type. There is a government that wants to redistribute income using taxation so that it's welfare functional is maximised. The welfare functional comprises individuals' utility functions, which are given uniform weight for simplicity here. The government also provides a good for individuals to consume. Lump-sum taxes are not available to finance the public provision, since the types of individuals are not observable. Therefore non-linear income taxes are used. To minimise the efficiency loss, taxes need to be set optimally. It is optimal for the government to provide the good if it reduces the efficiency loss from taxation.

There have been various ways of analysing the extensive margin of labour supply in the literature. Here I use the fixed costs of work - approach. Another way to model the participation decision would be to assume that there are discrete "working places" between which people jump if the tax incentives change (Saez 2002). This would produce much the same result, but does not give a micro-founded reason for the existence of discontinuous choice. In a similar way as the fixed cost here, Diamond (1980) uses work disutility distributed continuously throughout the population. Demand side-related reasons for discontinuous working hours are not possible options in the model analysed here, since the demand side is not properly modeled.

In the model in this section individuals are heterogeneous in terms of fixed costs and there is only one ability type. This produces the result that for a given number of working hours everybody has the same income. Since the important aspect of the model is that the publicly provided private good affects the participation decision, the basic mechanisms are revealed with this simpler model. In section 4, a distribution of ability types is added to the model.

2.1 Individuals' optimisation problem

Individuals consume a private consumption good, c. An individual of type n receives income y when he or she supplies h hours of labour, so that $y = hn$. At this point everybody has the same ability type n. Supplying labour consumes the individual's leisure. It is assumed that both leisure and consumption are normal goods. The utility function of an individual is

$$u(c, y, q, g) = v(c, y) - q(g)$$

$$u(c, y, q, g) = v(c_0, 0)$$

when participating in work and when not, respectively. q is a fixed cost and g a publicly provided private good. It is assumed that in the individual utility function the fixed cost is separable from consumption and labour supply. This assumption is relevant for the results. From this assumption it follows that once the participation decision is made, marginal changes in taxation affect only the working hours decision. Furthermore, it follows that public provision of the private good targets the participation decision and does not affect the working hours decision.

I assume there is a continuum of fixed costs that people suffer only when they decide to participate, $q*1(y>0)$. Individuals first pick their draw of q from the fixed costs distribution and then participate if the fixed cost is not too high. They can not themselves affect their fixed costs after it is realized. The fixed costs are distributed on a positive real axis up until some finite upper limit R: $q \in [0, R]$ with a density function $f(q|g)$ and a cumulative distribution function $F(q|g)$. The amount of workers is normalised to one, $F(R|g) = 1$.

The government can not remove any individuals from society or remove their fixed costs completely, just increase or decrease them. g only affects the shape of the distribution $f(q|g)$. I assume that the q is some well defined monotone function of $g : q(g)$ so that altering g does not change the order of people with different q's.

Conditional on participating, individuals make their working hours decision by maximising the utility conditional on the budget constraint:

$$\max v(c, y) - q(g)$$

$$\text{s.t. } c = y - T(y)$$

where $T(y)$ is tax on labour income. The following condition emerges from the FOC for the above problem:

$$\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial c}} = 1 - T'$$

which states that the indifference curves in the (c, y) space should be tangent to the marginal tax rate. It should be noted that the fixed cost does not affect the working hours decision. Individuals participate if the utility from working and suffering the fixed costs $q(g)$ is greater than the utility from not participating. If an individual does not participate, he or she consumes the transfers that government provides to non-participants, $c_0 = -T(0)$. The condition for participation can be expressed as:

$$v(c, y) - q(g) \geq v(c_0, 0) \Rightarrow \bar{q} = v(c, y) - v(c_0, 0) \tag{1}$$

where $\bar{q}$ defines the threshold value for participation, $\bar{q} \in (0, R)$. Using this threshold value, the number of individuals participating is the cumulative number of those who have a lower fixed cost than $\bar{q}$, noted $F(\bar{q}|g)$. Then the amount of non-participants is simply $1 - F(\bar{q}|g)$. For further reference, using equation (1), the following rules can be formulated: $\frac{\partial v}{\partial c} = \frac{\partial \bar{q}}{\partial c}$, $\frac{\partial v}{\partial y} = \frac{\partial \bar{q}}{\partial y}$and $\frac{\partial v}{\partial c_0} = -\frac{\partial \bar{q}}{\partial c_0}$. These simply state that changes in consumption and labour supply have an equal effect on the utility and the threshold value for participation. It is assumed that g does not affect the threshold value $\bar{q}$, since the threshold value depends on the $v()$ function which does not depend on $q(g)$. Nevertheless, g affects the distribution of q, thus it affects the number of participants. On the other hand, the threshold value $\bar{q}$ is affected by the tax differential when participating and when not. In a similar vein, $\frac{\partial y}{\partial g} = 0$, since g only affects the participation decision through its effect on the distribution of q, not the hours decision conditional on participating.

2.2 Government optimisation

The government wants to redistribute income and at the same time minimise the efficiency loss from taxation. Its tool is a non-linear income tax schedule that depends on income, denoted $T(y)$ for participants and $T(0)$ for non-participants. It provides a private good, g. The government is assumed to be benevolent and utilitarian. It optimises a welfare functional, which has individual utility functions as its argument. There is also a budget constraint on objective function of the government. To capture the idea that utility is maximised for everybody, those outside the labour force are also within the utility maximisation problem. The general formulation can be stated as: $\max_{c,y,c_0,g} W = \int_0^R u(c, y, g, q) f(q|g) dq$ s.t. budget holds. This is formulated into Lagrangian using the equation (1) to separate between participants and non-participants:

$$max_{c,y,c_0,g} W = \int_0^{\bar{q}} v(c,y)f(q|g)dq - \int_0^{\bar{q}} q(g)f(q|g)dq + \int_{\bar{q}}^{R} v(c_0,0)f(q|g)dq$$

$$+\lambda(\int_0^{\bar{q}} (y-c)f(q|g)dq - \int_{\bar{q}}^{R} c_0 f(q|g)dq - \int_0^{\bar{q}} rgf(q|g)dq) \quad (2)$$

The terms that are integrated over the interval $(0, \bar{q})$ include the participants, and the remainder includes the non-participants. λ is a Lagrange multiplier for the government budget constraint. There is no incentive compatibility constraint in this problem, since there is only one ability type. Instead, the problem of the government is to ensure that all those whose fixed costs are low enough participate. This is ensured by setting taxes optimally, since the government does not observe the fixed costs of individuals, only their distribution.

I take the first-order conditions from the government objective (2) to derive rules for optimal taxes:

$$\frac{\partial W}{\partial c_0} = \int_{\bar{q}}^{R} \frac{\partial v(c_0,0))}{\partial c_0} f(q|g)dq + \lambda[\frac{\partial \bar{q}}{\partial c_0} f(\bar{q}|g)(y-c+c_0-rg) - (1-F(\bar{q}|g))] = 0 \quad (3)$$

$$\frac{\partial W}{\partial c} = \int_0^{\bar{q}} \frac{\partial v(c,y))}{\partial c} f(q|g)dq + \lambda[\frac{\partial \bar{q}}{\partial c} f(\bar{q}|g)(y-c+c_0-rg) - F(\bar{q}|g)] = 0 \quad (4)$$

$$\frac{\partial W}{\partial y} = \int_0^{\bar{q}} \frac{\partial v(c,y))}{\partial y} f(q|g)dq + \lambda[\frac{\partial \bar{q}}{\partial y} f(\bar{q}|g)(y-c+c_0-rg) + F(\bar{q}|g)] = 0 \quad (5)$$

The government welfare functional, the first line in the equation (2), contains the individual utilities. Changes in consumption and income affect this part directly. The indirect effects come through the budget. I use Leibniz's rule in these derivations because the integral limit $\bar{q}$ depends on consumption and income. Further derivation steps are given in the Appendix A.

2.3 Optimal taxes

Since there is only one ability type, conditional on participating, everybody has similar marginal utilities, as seen in the individual's maximisation problem. The government sets taxes so that the one ability type chooses hours optimally, conditional on participating.

From equations (4) and (5), I write the following rule:

$$\frac{\int_0^{\bar{q}} \frac{\partial v(c,y)}{\partial y} f(q|g)dq}{\int_0^{\bar{q}} \frac{\partial v(c,y)}{\partial c} f(q|g)dq} = \frac{\frac{\partial v(c,y)}{\partial y} F(\bar{q}|g)}{\frac{\partial v(c,y)}{\partial c} F(\bar{q}|g)} = \frac{\lambda[\frac{\partial \bar{q}}{\partial y} f(\bar{q}|g)(y-c+c_0-rg)+F(\bar{q}|g)]}{\lambda[\frac{\partial \bar{q}}{\partial c} f(\bar{q}|g)(y-c+c_0-rg)-F(\bar{q}|g)]}$$

The first equality follows from the fact that since the social welfare functional is already at optimum, the second-order effects do not affect its value. Thus the integral limit is not affected by changes in incomes and consumption in the welfare functional, and the terms $\frac{\partial v}{\partial c}$ and $\frac{\partial v}{\partial y}$ can be taken out of the integral. The integral limit is affected in the government budget constraint, which is not already at optimum. Then I can write:

$$\frac{\frac{\partial v(c,y)}{\partial y}}{\frac{\partial v(c,y)}{\partial c}} = \frac{[\frac{\partial \bar{q}}{\partial y} \frac{f(\bar{q}|g)}{F(\bar{q}|g)}(y-c+c_0-rg)+1]}{[\frac{\partial \bar{q}}{\partial c} \frac{f(\bar{q}|g)}{F(\bar{q}|g)}(y-c+c_0-rg)-1]}$$

$$\Rightarrow \frac{\frac{\partial v(c,y)}{\partial y}}{\frac{\partial v(c,y)}{\partial c}}[\frac{\partial \bar{q}}{\partial c} \frac{f(\bar{q}|g)}{F(\bar{q}|g)}(y-c+c_0-rg)-1] = [\frac{\partial \bar{q}}{\partial y} \frac{f(\bar{q}|g)}{F(\bar{q}|g)}(y-c+c_0-rg)+1]$$

$$\Rightarrow \frac{\frac{\partial v(c,y)}{\partial y}}{\frac{\partial v(c,y)}{\partial c}} = \frac{\frac{\partial v(c,y)}{\partial y}}{\frac{\partial v(c,y)}{\partial c}} \frac{\partial \bar{q}}{\partial c} \frac{f(\bar{q}|g)}{F(\bar{q}|g)}(y-c+c_0-rg) - \frac{\partial \bar{q}}{\partial y} \frac{f(\bar{q}|g)}{F(\bar{q}|g)}(y-c+c_0-rg)-1$$

$$\Rightarrow \frac{\frac{\partial v(c,y)}{\partial y}}{\frac{\partial v(c,y)}{\partial c}} = \frac{\partial v(c,y)}{\partial y}\frac{f(\bar{q}|g)}{F(\bar{q}|g)}(y-c+c_0-rg) - \frac{\partial v(c,y)}{\partial y}\frac{f(\bar{q}|g)}{F(\bar{q}|g)}(y-c+c_0-rg) - 1$$

$$\Rightarrow \frac{\frac{\partial v(c,y)}{\partial y}}{\frac{\partial v(c,y)}{\partial c}} = -1$$

This last line follows from the definition of $\bar{q}$, according to which $\frac{\partial v}{\partial c} = \frac{\partial \bar{q}}{\partial c}, \frac{\partial v}{\partial y} = \frac{\partial \bar{q}}{\partial y}$ as noted above. The marginal tax rate is:

$$MTR = \frac{v_y}{v_c} + 1 = 0 \tag{6}$$

The resulting rule states that the marginal tax rate for the participating worker is zero. There is nothing surprising about this result. Since the fixed cost from participating is separable from labour supply and consumption in the individual utility function, there is no reason to distort the hours decision after the individual has made the decision to participate. Moreover, even with two ability type settings it is normal to have the result that the highest ability type has zero marginal tax rates (Stiglitz 1982).

Another question is how the tax differential for participants and non-participants should be set optimally. Marginal tax rates are not meaningful, because the decision to participate is not continuous. Instead, I derive a rule for the participation tax rate defined as the difference between $T(y)$ (taxes faced by workers) and $T(0) = -c_0$ (subsidies government gives to non-participants) divided by the consumption difference in these two states as in Saez (2002). In this case the participation tax rate is affected by the public provision of a private good multiplied by its production cost, rg.

From equation (3):

$$\frac{\partial v(c_0,0))}{\partial c_0}(1 - F(\bar{q}|g)) + \lambda[\frac{\partial \bar{q}}{\partial c_0}f(\bar{q}|g)(y - c + c_0 - rg) - (1 - F(\bar{q}|g))] = 0$$

$$\Rightarrow \lambda \frac{\partial \bar{q}}{\partial c_0} f(\bar{q}|g)(y - c + c_0 - rg) = \left(-\frac{\partial v(c_0, 0))}{\partial c_0} + \lambda\right)(1 - F(\bar{q}|g))$$

$$\Rightarrow (y - c + c_0 - rg) = \left(-\frac{\partial v(c_0, 0))}{\partial c_0} + \lambda\right) \frac{(1 - F(\bar{q}|g))}{\lambda \frac{\partial \bar{q}}{\partial c_0} f(\bar{q}|g)}$$

$$\Rightarrow \frac{T(y) - T(0) - rg}{(c - c_0)} = \left(\frac{\partial \bar{q}}{\partial c_0} + \lambda\right) \frac{1 - F(\bar{q}|g)}{\lambda \frac{\partial \bar{q}}{\partial c_0} f(\bar{q}|g)(c - c_0)}$$

$$\Rightarrow \frac{T(y) - T(0) - rg}{(c - c_0)} = \left(\frac{\partial \bar{q}}{\partial c_0} \frac{1}{\lambda} + 1\right) \frac{1}{\eta}$$

where $\eta = \frac{\partial(1-F(\bar{q}|g))}{\partial c_0} \frac{(c-c_0)}{1-F(\bar{q}|g)} = -\frac{\partial \bar{q}}{\partial c_0} f(\bar{q}|g) \frac{c-c_0}{1-F(\bar{q}|g)}$ is the participation elasticity. This rule states that the size of the optimal participation tax rate depends inversely on the absolute value of the participation elasticity, which is similar to Saez (2002) and Eissa et al. (2008). The optimal participation tax rate also depends on how the benefits for non-participants affect the threshold value for participation and the shadow price of the government budget, λ. Both the participation elasticity and the change in $\bar{q}$ are negative numbers. It is proved in the Appendix B that the $T(y) - T(0) - rg$ term is positive in the optimal allocation. Therefore the participation tax rate is positive as long as the consumption of participants is higher than that of non-participants. The larger the participation elasticity is as an absolute value, the smaller the participation tax rate becomes. It is also proved in the Appendix B that c_0 is positive when taxes are set optimally. Thus the government is providing subsidies to non-participants.

In solving the optimisation problem, the public provision of g is first kept constant and the government sets taxes optimally. Nevertheless, the amount of public provision g evaluated at its cost r affects the participation tax rate. This is natural, since the public provision is targeted only at the participating share of the population. Thus the public provision affects the net tax differential when working and when not. Large values of rg could offset an otherwise large difference between the tax rates $T(y) - T(0)$.

3 Public provision of private goods

Let us now consider the public provision of private goods. The aim is to develop a welfare maximising rule for optimal government provision. The interesting part is to look at what kind of impact unobserved fixed costs have on the government provision rule. If public provision does not have any effect on participation behaviour through fixed costs, the government could still provide the good. In that case it might not be welfare improving to do so. The government redistributes through optimal income taxation. The question is whether public provision can increase social welfare even if the taxes have been set optimally.

The aim is to derive a rule where the sum of marginal rates of substitution between public and private goods is equated to the rate of transformation to produce the publicly provided private good. In this model the good g only affects the fixed cost distribution $f(q|g)$, while consumption c affects the $v(c, y)$ part of the individual's utility function. Moreover, non-participants do not suffer the fixed cost and therefore they do not benefit from government provision. If the public provision of g does not affect the fixed costs of individuals in any way, the derivative of g with respect to q and its distribution are zero $:\frac{\partial q(g)}{\partial g} = 0 \Rightarrow \frac{\partial f(q|g)}{\partial g} = 0$

I assume here that the government has set taxes optimally. To derive a rule for optimal provision of g, I take the FOC for g from the government objective function (2):

$$
\begin{aligned}
&\frac{\partial W}{\partial g} = \\
&\int_0^{\bar{q}} v(c,y)\frac{\partial f(q|g)}{\partial g}dq - \int_0^{\bar{q}} \left(q(g)\frac{\partial f(q|g)}{\partial g} + \frac{\partial q(g)}{\partial g} f(q|g) \right) dq \\
&+ \int_{\bar{q}}^{R} v(c_0,0)\frac{\partial f(q|g)}{\partial g}dq \\
&+ \lambda \left(\int_0^{\bar{q}} \frac{\partial y}{\partial g} f(q|g)dq + \int_0^{\bar{q}} (y-c)\frac{\partial f(q|g)}{\partial g}dq - \right) \\
&+ \lambda \left(\int_{\bar{q}}^{R} c_0 \frac{\partial f(q|g)}{\partial g}dq - \int_0^{\bar{q}} \left(\frac{\partial rg}{\partial g} f(q|g) + rg\frac{\partial f(q|g)}{\partial g} \right) dq \right) = 0 \qquad (7)
\end{aligned}
$$

This equation is modified to obtain a simpler rule for the optimal provision of g:

$$
\int_0^{\bar{q}} \frac{-\partial q(g)}{\partial g} f(q|g))dq + \int_0^{\bar{q}} (v(c,y) - q(g))\frac{\partial f(q|g)}{\partial g}dq + \int_{\bar{q}}^{R} v(c_0,0)\frac{\partial f(q|g)}{\partial g}dq
$$

$$
= -\lambda(\int_0^{\bar{q}} (y-c)\frac{\partial f(q|g)}{\partial g}dq - \int_{\bar{q}}^{R} c_0\frac{\partial f(q|g)}{\partial g}dq - \int_0^{\bar{q}} rg\frac{\partial f(q|g)}{\partial g}dq - rF(\bar{q}|g))
$$

$$
\Rightarrow \int_0^{\bar{q}} \frac{\frac{-\partial q(g)}{\partial g}}{\frac{\partial v(c,y)}{\partial c}} \frac{\partial v(c,y)}{\partial c} f(q|g))dq = -\lambda((y - c + c_0 - rg)\frac{\partial F(\bar{q}|g)}{\partial g} - rF(\bar{q}|g))
$$

In the last equation the first two distribution effects ($\frac{\partial f(q|g)}{\partial g}$) were enveloped out, since the government welfare functional was already at optimum. The distribution effects work through the government budget, however. Also, $\frac{\partial y}{\partial g} = 0$ because in this simple case g is separable from y in the individual utility function. Below I use the government welfare weights which are $\omega = \frac{\partial v}{\partial c} f(q|g)/\lambda$ for workers. The rule for providing g then becomes:

$$\int_0^{\bar{q}} MRS_{gc}\omega dq = rF(\bar{q}|g) - (T(y) - T(0) - rg)\frac{\partial F(\bar{q}|g)}{\partial g}$$

The MRS_{gc} term describes the marginal valuation of g compared to c for each participant. It is weighted by the government welfare weights. It is imminent from the derivations of the above rule that if g does not have any effect on the fixed costs, then the optimal public provision level is zero. On the other hand, if public provision has effect on the fixed costs, the marginal rate of substitution is equated with marginal rate of transformation $(rF(\bar{q}|g))$ and the distribution effect $\left(\frac{\partial F(\bar{q}|g)}{\partial g}\right)$ multiplied by the size of the participation tax differential $(T(y) - T(0) - rg)$. If the public provision produces negative MRS_{gc} it is not welfare improving and should not be carried out. This is because if in the above rule MRS_{gc} is negative, $\frac{\partial q(g)}{\partial g} > 0 \Rightarrow \frac{\partial F(\bar{q}|g)}{\partial g} > 0$. This means that there are no benefits from providing g in terms of welfare or distributional gains.

The significance of the additional terms to the MRT on the right hand side is that if they are negative (positive), g should be over-provided (under-provided) relative to the simple rule where MRS_{gc} is equated with MRT. If the government provision of g shifts the weight of the fixed cost distribution towards workers (and at the same time away from non participants), the distribution effect is positive and the total effect is negative. Increasing government provision shifts the distribution towards workers if it decreases the fixed costs. The tax differential is always positive, which is proved in the Appendix (B). The role of the participation tax rate is to amplify the distribution effect; the greater the net income differential is between participation and non-participation, the more it amplifies the effect through changes in distribution.

A real-world example that fits this theoretical model best is the day care of children. It is sometimes provided by the government, although in nature it is a private good. When public provision makes day care more accessible, participation becomes easier for parents. Thus public provision of day care lowers the fixed cost for participation. The added provision shifts the fixed costs for a small number of parents below the threshold value $\bar{q}$ and consequently they find it optimal to participate. This effect is amplified if the existing participation tax differential is at a high level. In the tax differential the level of public provision affects the participation tax rate. If the provision of day care is already at a high level, increasing government provision even further does not have as great an impact on the provision rule as when existing provision is at a lower level.

4 Adding a distribution of ability types

From here on I assume that for each fixed cost there is a distribution of ability types. This assumption makes the heterogeneity of the population two-dimensional. The idea here is that some individuals are more able to produce and some more eager to participate. The two characteristics do not depend on each other. In this setting it is possible that an individual of a higher ability type (who is productive) does not participate because the fixed cost of participation is too high for that individual. I first set up the individual optimisation problem, then the government problem and then study the optimal public provision of goods.

The individual utility function can now be written:

$$u(c, y, g, q, n) = v(c, y, n) - q(g)$$

Thus the fixed cost q is separable from the ability type n. If an individual does not participate the utility function is written:

$$u(c, y, g, q, n) = v(c, 0, n)$$

Conditional on participating, an individual chooses working hours according to the utility maximization problem:

$$\max \ u(c, y, g, q, n)$$

$$s.t. \ c_n = y_n - T(y_n)$$

where c_n and y_n denote the consumption and labour income of type n, respectively. It is additionally assumed that given q the only difference in preferences is that more productive individuals earn more with the same number of working hours: $y = hn$. The government only sees total income y, not h or n separately.

At the participation limit, the utility from participating and not participating must be equal:

$$v(c, y, n) - q(g) = v(c, 0, n)$$

$$\Rightarrow \bar{q}_n = v(c, y, n) - v(c, 0, n) \tag{8}$$

where $\bar{q}_n$ defines the threshold value for participation of type n.

4.1 Government optimisation

The government aims to maximise the welfare functional where the arguments are the individual utility functions. There is a budget constraint which balances the revenue collected in taxes and the expenditure on the government provision of g.

Now an incentive compatibility constraint needs to be introduced into the model. This is to cope with the possibility that more able workers pretend to be less able if the tax incentives are not set correctly. This kind of problem with an ability type distribution was first introduced by Mirrlees (1971) and followed by a sizeable literature. Although there is another dimension of heterogeneity in the present model - fixed costs - the incentive compatibility constraint can be formulated in a normal way. I demonstrate how the fixed cost does not affect the derivation of the normal incentive compatibility constraint in the Appendix C. The intuition is simply that since the fixed cost q and the ability type n are separable in the individual utility function, the fixed cost does not affect the choice of working hours, which depends on ability type.

For later purposes I denote $\frac{dv}{dn} = \gamma$. This is the incentive compatibility constraint used in the derivations below. It requires that the first-order condition for the utility with respect to ability type is zero. If this condition is fulfilled, type n chooses the allocation intended for type n. Before inserting this into the government objective function, it is integrated by parts.

The government objective is integrated over the distribution of ability types. For each ability type, it is separated between participants and non-participants using equation (8), as was done in the one ability type case. Now that there is a continuum of fixed costs for each ability type, I write the fixed cost distribution conditional on ability type, $f(q|n,g)$. Then there is a different share of participants for each ability type, denoted by the cumulative distribution function $F(\bar{q}|n,g)$. Additionally, there is a distribution for the ability types denoted $h(n)$.

The Lagrangian for this problem is formulated as follows:

$$
\begin{aligned}
& max_{c,y,c_0,g} W = \\
& \int_{\underline{n}}^{\bar{n}} \left(\int_0^{\bar{q}_n} v(c,y,n) f(q|n,g) dq - \int_0^{\bar{q}_n} q(g) f(q|n,g) dq \right) h(n) dn \qquad (9) \\
& + \int_{\underline{n}}^{\bar{n}} \left(\int_{\bar{q}_n}^{R} v(c_0,0,n) f(q|n,g) dq \right) h(n) dn \\
& + \lambda \int_{\underline{n}}^{\bar{n}} \left(\int_0^{\bar{q}_n} (y-c) f(q|n,g) dq - \int_{\bar{q}_n}^{R} c_0 f(q|n,g) dq \right) h(n) dn \\
& - \lambda \int_{\underline{n}}^{\bar{n}} \left(\int_0^{\bar{q}_n} rg f(q|n,g) dq \right) h(n) dn \\
& - \int_{\underline{n}}^{\bar{n}} (\alpha' v + \alpha\gamma) dn + \alpha(\underline{n}) v(\underline{n}) + \alpha(\bar{n}) v(\bar{n}) \qquad (10)
\end{aligned}
$$

The last two terms are the transversality constraints. For them to work without problems, I need to assume that the distribution of ability types covers all the fixed cost types. The incentive compatibility constraint needs to work for each n. There is no need to integrate it over the fixed cost distribution, since fixed costs do not affect the working hours decision.

The tax properties are derived in earlier literature (Saez 2002, Immervol et al. 2007 and Jacquet, Lehmann and Van der Linden 2010), so I do not derive them here. The general properties of the optimal tax rates are that there is a non-linear tax schedule for participants. For each ability type there is different participation rate. Depending on the size of the extensive and intensive labour supply elasticities, the tax schedule may incorporate negative marginal tax rates. The higher the extensive labour supply elasticity in lower income groups, the more likely the tax schedule is to have in-work benefits for the working poor.

4.2 Publicly provided private goods with many ability types

Here I derive the provision rule for g_n in the case of many ability types. I take the first order conditions with respect to the publicly provided private good, g_n, for each n separately.

$$
\begin{aligned}
&\frac{dW_n}{dg_n} = \\
&\left(\int_0^{\bar{q}_n} v(c,y,n)\frac{\partial f(q|n,g)}{\partial g}dq\right)h(n) \\
&-\left(\int_0^{\bar{q}_n}\left(q(g)\frac{\partial f(q|n,g)}{\partial g}+\frac{\partial q(g)}{\partial g}f(q|n,g)\right)dq\right)h(n) \\
&+\left(\int_{\bar{q}_n}^{R} v(c_0,0,n)\frac{\partial f(q|n,g)}{\partial g}dq\right)h(n) \\
&+\lambda(\int_0^{\bar{q}_n}(y_n-c_n)\frac{\partial f(q|n,g)}{\partial g}dq-\int_{\bar{q}_n}^{R}c_0\frac{\partial f(q|n,g)}{\partial g}dq \\
&-\int_0^{\bar{q}_n}\left(\frac{\partial rg}{\partial g}f(q|n,g)+rg\frac{\partial f(q|n,g)}{\partial g}\right)dq)h(n)=0
\end{aligned}
$$

where y_n and c_n indicate the income and consumption of type n. This first-order condition is very similar to that in the one ability type model. The only difference is that since this is taken separately for each n, the conditional density $f(q|n,g)$, the publicly provided good, labour income and consumption are all functions of n. Thus there is a specific provision level for participants of each ability type.

The above FOC can be derived into a simpler form as follows:

$$\begin{aligned}
&\int_0^{\bar{q}_n} \frac{-\partial q(g)}{\partial g} f(q|n,g))dqh(n) = \\
&-\lambda \left(\int_0^{\bar{q}_n} (y_n - c_n) \frac{\partial f(q|n,g)}{\partial g} dq - \int_{\bar{q}_n}^{R} c_0 \frac{\partial f(q|n,g)}{\partial g} dq \right) h(n) \\
&-\lambda \left(- \int_0^{\bar{q}_n} rg \frac{\partial f(q|n,g)}{\partial g} dq - rF(\bar{q}|n,g) \right) h(n)
\end{aligned}$$

$$\Rightarrow \int_0^{\bar{q}_n} \frac{\frac{-\partial q(g)}{\partial g}}{\frac{\partial v(c,y,n)}{\partial c_n}} \omega_n dq = rF(\bar{q}|n,g) - (y_n - c_n + c_0 - rg_n) \frac{\partial F(\bar{q}|n,g)}{\partial g}$$

where $\omega_n = \frac{\partial v_n}{\partial c_n} f(\bar{q}|n,g)/\lambda$ is the welfare weight that the government puts on a worker of type n. The term describing the effect of providing g_n on the distribution of workers coming from the government welfare functional is again enveloped out. This equation can be written as follows:

$$\int_0^{\bar{q}_n} MRS_{gc} \omega_n dq = rF(\bar{q}|n,g) - (T_n(y_n) - T(0) - rg_n) \frac{\partial F(\bar{q}|n,g)}{\partial g} \tag{11}$$

where $T_n(y_n)$ is taxes paid by a participating worker of type n. This is the optimal provision rule for a publicly provided private good of type n. It states that the public provision of g_n is welfare improving, as long as it has a positive effect on welfare in terms of MRS_{gc} for type n. The more the provision of g_n increases the number of participants of type n, the more welfare improving it is. There should be no government provision of goods in the opposite case. The greater the tax differential is between participation and non-participation, the greater the impact of the whole additional term is to the provision rule.

The optimal provision rule depends on the ability type, in this case through the participation rate and the government welfare weights. Also, the size of the additional terms depends on the ability type. Since this rule holds for every n, the effect is different for different ability types. The responsiveness of the participation rate to changes in government provision $\left(\frac{\partial F(\bar{q}|n,g)}{\partial g}\right)$ depends on the shape of the cumulative distribution function. With bell-shaped distributions, if the participation rate is already close to it's maximum, it is difficult to increase it even further. Across different ability types, it is possible that high n types participate more than low n types. In that case, high ability types would have a smaller additional term in the absolute value than lower ability types.

The interaction with the participation tax differential adds interesting interaction to the rule. The greater the participation tax differential for a given income group, the greater impact it has on additional terms for that group. For example, the participation tax differential could be smaller for lower income groups than for higher income groups. The pre-existing provision level of g_n affects the tax differential. Even if the participation tax differential is high without public provision, large values of rg_n can offset this.

The leading real-world application for this model is publicly provided day-care for children. The general reasons for this were explained in the one ability type case. The added implications of many ability types fits the example of child care as well. The greater effect public provision of child care has on an income group, the more it should greater welfare effect it has on an income group. For example parents on low incomes could be more responsive to the added provision of child care. They could find that a significant share of their fixed cost of participation is removed by the extra provision of day care. This effect is stronger, the higher the participation tax differential is for this group. These effects could be lower for higher-income groups if they gain more from participation in any case.

5 Conclusion

This study has investigated a government provision rule for a private good in a setting where the good affects the extensive margin of labour supply. The good is connected to the extensive margin through it's effect on a fixed cost arising from participation. First, the optimal tax and the public provision rule are formulated in a simple case where there is heterogeneity only in the fixed costs. The optimal marginal tax rate for those participating is zero. A participation tax rate affecting the participation decision depends inversely on the participation elasticity. The rule for government provision of the private good states that a good that increases the participation in the labour force is welfare improving to provide publicly even if the income taxes have been set optimally. The size of the participation tax differential amplifies this effect.

Second, a more general model incorporated another source of heterogeneity: ability types. The public provision rule in this case is similar to the simple case of just one ability type. The difference is that there is a separate rule for each ability type. The government provision rule depends on ability types through the distribution effects and the participation tax differentials that vary across ability types. With these statistics available, the rule provides a clear intuition about cases in which public provision should be extended and those in which it should not. The defining factors in the rule are the direction in which participation changes and the extent of the net income differential between participation and non-participation.

The earlier literature studying government provision of goods in connection with optimal taxes has not taken the extensive margin of labour supply into account. Thus the results of Mirrlees (1976), Christiansen (1981), Boadway and Keen (1993) and Boadway et al. (1998) state that public projects should differ from the simple cost-benefit analysis only if they convey some information on the productivity of individuals. This does not occur if consumption and leisure are weakly separable in the individual utility function. The model studied in this paper differs from this result. Indeed, even if consumption and leisure are separable, there could be reasons to publicly provide private goods. If the good that is provided by the government lowers the fixed cost of participation, it could be welfare improving to provide it even if redistribution is taken care of optimal income taxes.

Another line of earlier research has studied the implications of welfare reforms with the extensive margin of labour supply taken into consideration (Saez 2002, Immervoll et al. 2007 and Eissa et al. 2008). The extensive margin has an impact on the optimal tax and transfer schedule. While the present study does not attempt to simulate the tax schedule for welfare reform purposes, the results have some significance here. It is shown that public provision of goods affects the government revenue requirement. Empirically, government expenditure on education, child care, health care and other services can be sizeable. The results here show that this kind of provision should be taken into account when considering welfare reforms, if the provision affects the participation decisions of individuals.

References

[1] Bergstrom, T. and Blomquist, S. 1996. The political economy of subsidized day care. European Journal of Political Economy, Vol. 12, 443-457.

[2] Besley, T. and Coate, S., 1991. Public provision of private goods and the redistribution of income. The American Economic Review 81, 979-84.

[3] Blackorby, C. and Donaldson, D., 1988. Cash versus Kind, Self-Selection, and Efficient Transfers. The American Economic Review, 78, 691-700.

[4] Blomquist, S., Christiansen, V. and Micheletto, L. 2010. Public Provision of Private Goods and Nondistortionary Marginal Tax Rates. American Economic Journal: Economic Policy, 2(2): 1–27.

[5] Blundell, R., 2006. Earned income tax credit policies: Impact and optimality. The Adam Smith Lecture, 2005. Labour Economics 13, 423-443.

[6] Blundell, R. and MaCurdy, T., 1999. Labour Supply: A Review of Alternative Approaches., in Ashenfelter, O. and Cards, D., eds., Handbook of Labour Economics, Volume IIIA. Amsterdam: North-Holland.

[7] Boadway, R. and Keen, M., 1993. Public Goods, Self-Selection and Optimal Income Taxation. International Economic Review, Vol.34, No. 3, pp. 463-478.

[8] Boadway, R., Marchand, M, and Sato, M., 1998. Subsidies versus Public Provision of Private Goods as Instruments for Redistribution. Scandinavian Journal of Economics, 100, 545-564.

[9] Christiansen, V., 1981. Evaluation of Public Projects under Optimal Taxation. Review of Economic Studies 48 (1981), 447-457.

[10] Cogan, J., F., 1981. Fixed Costs and Labour Supply. Econometrica, Vol. 49, No. 4, pp. 945-963.

[11] Diamond, P., 1980. Income taxation with fixed hours of work. Journal of Public Economics, Volume 13, Issue 1, February 1980, Pages 101-110.

[12] Eissa, N. and Hoynes, H., 2004. Taxes and the labor market participation of married couples: the earned income tax credit. Journal of Public Economics, 88, pp. 1931- 1958.

[13] Eissa, N., Kleven, H. and Kreiner, C., 2004. Evaluation of Four Tax Reforms in the United States: Labour Supply and Welfare Effects for Single Mothers, NBER Working Paper 10935.

[14] Eissa, N., Kleven, H. and Kreiner, C., 2008. Evaluation of Four Tax Reforms in the United States: Labour Supply and Welfare Effects for Single Mothers. Journal of Public Economics, 92, 795-816.

[15] Eissa, N. and Liebman, J., B., 1996. Labour Supply Response to the Earned Income Tax Credit. The Quarterly Journal of Economics, Vol. 111, No. 2, 605-637.

[16] Immervoll, H., Kleven, H., Kreiner, C. and Saez, E., 2007. Welfare Reform in European Countries: A Microsimulation Analysis. The Economic Journal, 117, 1-44.

[17] Jacquet, L., Lehmann, E. and Van der Linden, B., 2010. Optimal Redistributive Taxation with both Extensive and Intensive Responses. IZA Discussion Papers, No. 4837.

[18] Mirrlees, J.,A., 1971. An Exploration in the Theory of Optimum Income Taxation. Review of Economic Studies, Vol.38, No. 2, (Apr. 1971) pp. 175-208.

[19] Mirrlees, J.A., 1976. Optimal tax theory : A synthesis. Journal of Public Economics, Volume 6, Issue 4, 327-358.

[20] Saez, E., 2002. Optimal Income Transfer Programs: Intensive Versus Extensive Supply Responses. The Quarterly Journal of Economics, pp 1039-1073.

[21] Stiglitz, J., E., 1982. Self-Selection and Pareto Efficient Taxation. Journal of Public Economics, Vol. 17, No. 2, pp. 213-240.

A Appendix Derivation of FOC

The derivation of the first-order conditions (3)-(5) from the government objective function (2) is shown. The objective function consists of the utility part and the budget part. Since the individual utility is already at an optimum, marginally changing consumption has only a direct effect on individual utility. From the budget part I need to take the indirect effects to private consumption into account as well. I need to use Leibniz's rule when deriving an integral where the limit depends on the derivative. In this case $\bar{q}$ depends on consumption and income. This can be seen from equation (1). I take the derivative $\frac{\partial W}{\partial c_0}$ in equation (3) as an example of how the remainder of the derivations are performed. First there is the partial derivation of the inner function of the integral term in the objective function,

$$\int_{\bar{q}}^{R} \frac{\partial v(c_0, 0))}{\partial c_0} f(q|g)dq$$

The limit is enveloped out, since in the objective function individual utility is already at optimum and the derivation only directly affects the utility for this part. The same does not hold for the government budget, since there is no utility function that is already optimised, just the amount of taxes and transfers paid by everybody and the government provision of g. The derivation of the budget becomes:

$$\lambda \frac{\partial}{\partial c_0}(\int_0^{\bar{q}} (y-c) f(q|g)dq - \int_{\bar{q}}^{R} c_0 f(q|g)dq - \int_0^{\bar{q}} rgf(q|g)dq)$$

$$\begin{aligned}
&\iff \lambda \left(\int_0^{\bar{q}} \frac{\partial}{\partial c_0}(y-c)f(q|g)dqf(q|g)dq + \frac{\partial \bar{q}}{\partial c_0}(y-c)f(\bar{q}|g) \right) \\
&+\lambda \left(+ \int_{\bar{q}}^{R} \frac{\partial}{\partial c_0} c_0 f(q|g)dq - \frac{\partial \bar{q}}{\partial c_0} c_0 f(\bar{q}|g) \right) \\
&+\lambda \left(- \int_0^{\bar{q}} \frac{\partial}{\partial c_0} rgf(q|g)dqf(q|g)dq - \frac{\partial \bar{q}}{\partial c_0} rgf(\bar{q}|g) \right)
\end{aligned}$$

The first and third integrals above are zero and in the second integral the density function is not affected by the changes in c_0. Only the threshold value $\bar{q}$ is affected. The above expression then simplifies to

$$\lambda \left[\frac{\partial \bar{q}}{\partial c_0}(y-c-rg)f(\bar{q}|g) + (1-F(\bar{q}|g)) - \frac{\partial \bar{q}}{\partial c_0} c_0 f(\bar{q}|g) \right]$$

which gives the first order condition (3) after combining with the direct effect from the utility part.

B Appendix The Sign of Taxes on Non-participants

I prove here the sign of certain terms in the optimum allocation. The FOC of 2 with respect to λ, the Lagrange multiplier for the government budget constraint, is:

c_{0}

$$\begin{aligned} \frac{\partial W}{\partial \lambda} &= (y-c)F(\bar{q}|g) - c_0(1-F(\bar{q}|g)) - rgF(\bar{q}|g) = 0 \\ &\Rightarrow (y-c+c_0-rg)F(\bar{q}|g) - c_0 = 0 \end{aligned}$$

From here on, I denote the terms $(y-c+c_0-rg) \equiv A$, $f(\bar{q}|g) \equiv \bar{f}$ and $F(\bar{q}|g) \equiv \bar{F}$. The term A is part of the participation tax rate and is also in the rule for optimal public provision of g. Then using FOC for c_0 and y from the government objective function it follows:

$$\begin{aligned} \frac{-\frac{\partial v}{\partial y}\bar{F}}{\frac{\partial \bar{q}}{\partial y}\bar{f}A + \bar{F}} &= \frac{-\frac{\partial v}{\partial c_0}(1-\bar{F})}{\frac{\partial \bar{q}}{\partial y}\bar{f}A - (1-\bar{F})} \\ \Rightarrow -\frac{\partial v}{\partial y}\bar{F}\left(\frac{\partial \bar{q}}{\partial c_0}\bar{f}A - (1-\bar{F})\right) &= -\frac{\partial v}{\partial c_0}(1-\bar{F})\left(\frac{\partial \bar{q}}{\partial y}\bar{f}A - \bar{F}\right) \\ \Rightarrow \frac{\partial v}{\partial c}\frac{\partial v}{\partial y}\bar{f}A &= \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial c_0}\right)\bar{F}(1-\bar{F}) \end{aligned}$$

where the last line follows from $\frac{\partial \bar{q}}{\partial c_0} = -\frac{\partial v}{\partial c_0}$ and $\frac{\partial \bar{q}}{\partial y} = \frac{\partial v}{\partial y}$ as noted when the participation threshold was defined in equation (1). The last equality proves that A is positive, since consumption and leisure are normal goods and the distribution terms are positive by definition. Using the sign of A again proves that $c_0$is positive from the following equation:

$$A\bar{F} = c_0$$

which was modified from the FOC for λ from the Lagrangian (2).

C Appendix Incentive Compatibility Constraint

I first write the indirect utility function of type n individual as $e(n)$. Then a given allocation is incentive compatible if everyone of type n receives the greatest utility from choosing an allocation intended for type n:

$$e(n) - v(c, y, n) - q(g) = 0 \leq e(n') - v(c, y, n') - q(g) \tag{12}$$

where n' refers to some ability type different from n. This equation takes the difference between the indirect utility and the direct utility. The equation reflects the fact that this difference should be minimised when acting according to one's own type. Choosing the allocation intended for any other type n' means that the utility achieved is smaller than when choosing an allocation intended for one's own type.

It can be seen already from this that the fixed cost drops out from this expression, since it separable from the rest of the utility function. From here on, it is possible to follow a normal derivation of the incentive compatibility constraint. I totally differentiate the expression in equation (12) to get:

$$\frac{de}{dn} - \frac{\partial v}{\partial c}\frac{dc}{dn} - \frac{\partial v}{\partial y}\frac{dy}{dn} - \frac{\partial v}{\partial n} = 0$$

$$\Rightarrow -\frac{\partial v}{\partial c}\frac{dc}{dn} - \frac{\partial v}{\partial y}\frac{dy}{dn} = 0$$

From here it is obvious that the fixed cost does not affect the marginal utility with respect to productivity. The intuition here is that people participate when the utility from participation is greater than the utility from not participating. Ability type affects this difference only as regards the participation part. Thus if the allocation is incentive-compatible so that everyone chooses according to their own type, they also participate if their fixed cost is low enough to do that. For this reason participation is affected through average taxes and the hours decision through marginal tax rates.

$$\frac{dv}{dc}\frac{dc}{dn} + \frac{dv}{dh}\frac{dh}{dn} = 0$$

$$\frac{dv}{dc}\frac{dc}{dy} + \frac{dv}{dh}\frac{dh}{dy} = 0$$

$$\frac{v_h}{n}\frac{dcdy}{dndc} + \frac{v_h}{n}\frac{dvdc}{dcdy}\frac{dh}{dn} = 0 \Rightarrow \frac{dv}{dn} = -\frac{v_h h}{n}$$