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Optimal Pension Contribution Mechanism

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Keywords: Pension Contribution Mechanism, Dynamic Moral Hazard, Decentralization, Optimal Taxation, Stochastic Pontryagin Principle

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1 Introduction

Retirement program in the U.S. has witnessed a drastic change in nature since 1980s. Traditional defined benefit programs, either funded or unfunded by assets, are criticized for the lack of sustainability. For example, unfunded defined benefit programs, such as Social Security which is financed by payroll tax, a lion's share (about 40 percents) of tax revenue, are inevitably challenged by the decline of fertility after baby boom and fiscal deficit of the government. Even for the funded defined benefit program, such as traditional private employer pension plan, are doubted on financially ground. Sponsors' solvency of these plans turns out to be a substantial risk to the participants. For example, \$3,100 was received as pension after the Enron bankruptcy¹. Also, the slower growth in productivity since 1975, from 2 percents to around 1 percent, looms over returns of fund. All these received worries of defined benefit program urge the adoption of defined contribution programs, such as 401(k) and IRA. The financial soundness of defined contribution program comes from the feature that it is the contribution plan during one's working life, rather than benefit, that is specified and the retirement benefit is based on the accumulation contribution.

Under this social context, the contribution² of this paper is to provide a novel theoretical framework to design a dynamic pension contribution mechanism. We emphasize on the effect of uninsurable risks on income during one's working life, like job loss, promotion competition, obsolete of skill, location change and physical injury. Due to information asymmetry, the insurances of these shocks are mostly partial or even missing in the market, so any benevolent social planner cannot simply ignore or treating them independently. As a result of these shocks, a pension contribution mechanism

¹Another notable example, despite of its bankruptcy, General Motors is still running the largest pension fund in the country. In its issue of April 2010, Time published an article titled "GM's Pension: A Ticking Time Bomb for Taxpayers?"

²One maybe surprise, despite its importance, little has been done on the normative theory of pension contribution mechanism. For example, Diamond (2009) states,

"In particular, I think we have done too little study of the issues around tax-favored retirement savings accounts, studies that need to recognize uncertainty in future earnings, uncertainty in future spending needs, diversity in savings behavior and earnings opportunities, and uncertainty about future tax rates."

has to maintain a careful balance between smoothing contributor's consumption and pension accumulation. However, the social planner, like the market, only observes pension contribution but not one's income nor consumption, which poses a moral hazard problem to any contribution mechanism. In particular, an agent may find profitable to game the mechanism by mimicing one with very adverse income shock. So the design of an optimal pension contribution mechanism should concern agent's incentive as well. For this purpose, we formulate a dynamic optimization problem under private information of income shocks and consumption, where agents are free to contribute, consume, retire and quit the mechanism.

To see the contribution of this paper, it is helpful to appreciate the difficulty in solving the dynamic moral hazard problem. A pension contribution mechanism is a function mapping from a path of pension contribution to the level of pension available when the agent retires. The design of optimal mechanism involves two levels of optimization problem. First, agents take an arbitrary mechanism as given and solve for the path of consumption, contribution, and stopping time to retire and to quit the mechanism. These constitute the set of incentive compatible allocations, which is challenging to characterize. Second, the social planner chooses paths of incentive compatible consumption, saving, stopping time to retire and to quit the mechanism in order to solve the optimal mechanism.

The roadmap to optimal pension contribution mechanism is as follows. The definitions of concepts used can be found in the corresponding sections. First, we formulate the equivalent agent's problem in a "risk-neutral world", where the social planner always observes contribution following a Brownian motion. Second, apply stochastic Pontryagin principle (Lemma 1), the set of incentive compatible recommendations can be characterized under the risk-neutral world, then recovered under the original space (Proposition 3). Forth, we index jointly the set of incentive compatible recommendations and pension contribution mechanism by the volatility of promised continuation utility and the recommended stopping time to retire (Corollary 1). Fifth, we set up the HJB equation to solve the optimal mechanism (Proposition 4) under dynamic incentive constraints, and discuss some of it properties and implementation. Sixth, we provide a simple decentralization of the optimal mechanism through a lump tax sum and a linear

saving tax (Proposition 5). The properties of the optimal taxation are also discussed. Seventh, we compare the optimal mechanism with *laissez faire* (Proposition 6). We discuss the source of efficiency gain, which is the role of insuring pension accumulation against labor productivity shocks during one working life.

We find pension contribution should be taxed, rather than subsidized, in return for insuring pension accumulation against productivity shocks. This normative result contrasts with current practices on 401(k), IRA or others similar, where pension contribution is not taxed or subsidized by contribution match. We also find, maybe counterintuitively, the level of pension is higher under the optimal mechanism than *laissez faire*. To see the reason, we decompose pension accumulation along intensive margin, which is the level saving at each time, and extensive margin, which is the length of participating the mechanism. Because contribution is taxed, the intensive margin decreases as agents save less under the optimal mechanism. But the extensive margin increases as the insurance aspect of the optimal mechanism induces agents to stay longer in the mechanism and postpone their retirement. It turns out that the extensive margin dominates the intensive margin, thus agents retire with higher pension under the optimal mechanism.

Our key observation is that any incentive compatible contribution mechanism can be summarized by a profile of volatilities of continuation utility promised by the mechanism. The optimal allocation can be decentralized, with help of a linear saving tax and a lump-sum tax, with zero *ex ante* taxation. If there is wealth effect on consumption, agents who are old or about-to-retire (young or about-to-quit) should be lump-sum subsidized (taxed) and enjoy lower (higher) saving tax rate. In particular, under the decentralization of the optimal mechanism, both the lump-sum tax (or negative transfer) and saving tax rate are decreasing in level of saving. Under *laissez faire* agents do not consume that much at the optimal level, because they prefer to save more than the optimal level for precautionary motive. In this case, a lump sum subsidy is needed to compensate such precautionary saving. Such compensation cannot be achieved through lower saving tax rate, since the effect of labor productivity shock on saving would be exaggerated due to the substitution effect. So the social planner needs these two tax vehicles altogether to decentralize the optimal allocation. This characterizes the dy-

namic incentives of optimal taxation to decentralize optimal mechanism. As saving is accumulated toward retirement, agent's saving are more lump-sum subsidized but less insured.

1.1 Literature Review

Our framework follows the continuous-time model of variable retirement model in Diamond and Mirrlees (1978,1982). The continuous-time model is realistic since the timing of retirement is chosen from the continuum of time rather than in discrete manner. Also they focus on unobservable retirement shock, while here we focus on unobservable labor productivity shocks. For our propose to study contribution mechanism, incorporating unobservable, dynamic labor productivity shock is important, as it implies various pension contributions over time amongst the continuum of agents, and, more importantly, the risk of future income. This allows us to derive a normative theory to say how should the level of pension depend on the various contribution history under the shadow of uncertain future. Also, modeling unobservable labor productivity allows us to study how and how much risk-sharing can be achieved under incentive compatibility constraints, through a pension contribution mechanism.

The dynamic optimal taxation literature³ which emphasizes the importance of unobservable income shocks is also related to this paper. The literature has focused, almost exclusively, on fixed horizon of agent, finite or infinite, rather than allowing endogenous retirement. To see the difference, we compare the decentralization of optimal mechanism in this paper with two seminal works of Cole and Kocherlakota (2001) and Golosov and Tsyvinski (2007). The former studies an environment with hidden income, hidden consumption and hidden saving, and finds the optimal taxation is laissez faire. This is a useful reference model, since it tells us the boundary of information structure within which social planner intervention can improve efficiency. Our paper studies a less strict information structure: hidden income and consumption but observable saving. Here the observation on saving allows an additional tax vehicle on

³See Hopenhayn and Nicolini (1997), Werning (2002), Albanesi and Sleet (2006), Golosov and Tsyvinski (2006), Kocherlakota (2005), and Mitchell and Zhang (2010). On the other hand, Atkeson and Lucas (1992) and Golosov, Kocherlakota and Tsyvinski (2003) study efficient allocations in dynamic, private information economies.

saving to lump-sum tax, and the optimal mechanism can improve welfare over *laissez faire*.

The optimal mechanism can be decentralized by a simple linear saving tax and a lump-sum tax (transfer if negative) in our continuous-time environment. In Golosov and Tsyvinski (2007), which also studies hidden consumption but in discrete-time environment, the optimal allocation can be decentralized by a linear saving tax as well. The insight for this simplicity is that, there are firms that takes care of all the complicate dynamic contract with agents, and the role of the government is only to correct the "externality" associated with the price-taking behavior of firms. Here the decentralization of the optimal mechanism works through agent's saving only. The reason to introduce taxes is also different. The role of saving tax is to mitigate the effect of shock and reduce precautionary saving motive. Insurance is provided by the subsidized withdrawal, as the linear saving tax applies to negative saving as well. The role of lump-sum tax is to provide incentive to affect agent retirement decision.

Our problem can be thought of a principal-agent problem, where continuous-time versions⁴ are also studied in Sannikov (2008) and Williams (2010). The elegant applications of stochastic control theories in Sannikov (2008) and Williams (2010) are important technical references for our paper. However, because of our purpose to study pension contribution mechanism, our model is different from theirs, in term of preference, information structure and choice of agents. We also focus on the implementation and decentralization of mechanism, welfare implication and the comparison with *laissez faire*, which are beyond the concern of standard principal-agent problem. Williams (2010) studies a general environment which allows persistent shocks and hidden actions in general. A useful tool suggested is stochastic Pontryagin principle, which characterizes the necessary condition of a stochastic control problem. There are wide potential applications of this technique, including ours, but with some modifications of the technique in order to apply in our problem. Here we solve the optimal indirect mechanism, so we are able to study how the pension contribution mechanism determines the level of pension according to the history of contribution.

⁴See also DeMarzo and Sannikov (2006), Piskorski (2010), Holmstrom and Milgrom (1987) and Zhang (2009).

Sannikov (2008) studies a simple environment in continuous-time model but generates a rich array of results which are helpful to explain various feature of dynamic contract. The fruitful modeling strategy of Sannikov (2008) motivates us to study the pension contribution mechanism in continuous-time model. In Sannikov (2008), the principal designs a contract which recommends efforts and specifies wages according to observable output. Agents cannot save, so consumptions are simply wages and thus observable to principal. There is no direct motive of retirement and pension benefit, but there nevertheless is the event of retirement when the principal finds too costly to motivating agent to provide hidden effort. In particular, the circumstance that agents are paid for zero recommended effort is interpreted as retirement, which is the choice of principal. The benefit scheme of pension is determined by principal to fulfill his promise on agent's continuation utility. Here to capture the event of retirement and accumulation of pension as agent's choice, we explicitly model the utility of retirement is the function of pension available. The accumulation of pension is through individual saving, or through the pension contribution mechanism provided by the social planner. Agent's consumption is hidden and the social planner only observed the path of pension contribution into the mechanism. In this environment, agents choose to retire once their pension is sufficiently high. To abstract from the benefit scheme of pension, the utility of pension is exogenous given. The implication of hidden consumption implies the social planner has to provide extra incentive to agents.

2 Benchmark Model: Laissez Faire

It would be simpler to introduce the environment where there is no pension contribution mechanism. Also it provides a benchmark to compare to effects of optimal pension contribution mechanism where we will derive in the latter section.

Time is continuous over the infinite horizon. The economy is populated by a continuum of agents over unit interval, who work, consume and save over the horizon and retire to enjoy the utility of pension. Consumption and labor productivity are private information, but saving is observable (bank account can be easily checked and verified), so a third party cannot distinguish whether an observed higher level saving

is due to higher labor productivity realized or just lower consumption.

2.1 Technology

The labor productivity is given by

$$d\theta_t = \theta dt + \sigma dB_t, \quad (1)$$

where $\theta > 0$ and labor productivity is continually hit by an idiosyncratic shock σdB_t , where $\sigma > 0$ and B_t is a standard Brownian motion. Since $d\theta_t$ is private information, there is no insurance market to cover the risk of labor productivity shock if no other relevant information available, since agents can always report $d\theta_t = -\infty$. An event in the event space $C[0, \infty)$ is a continuous path $B \equiv \{B_t, 0 \leq t < \infty\}$, which generates a filtration $\{\mathcal{F}_t^B\}_{t=0}^\infty$ with Wiener measure $\mathbb{P}$. We denote the probability space as a triple $(C[0, \infty), \{\mathcal{F}_t^B\}_{t=0}^\infty, \mathbb{P}_Y)$. All functions here and later are assumed bounded and Lipschitz continuous.

There is a saving technology with rate r . The flow of saving da_t is given by:

$$da_t = (ra_t - c_t) dt + l_t d\theta_t, \text{ given } a_0 > 0 \text{ and } \theta_0, \quad (2)$$

where l_t is $\mathcal{F}_t^B$ -measurable and adapted which denotes agent decision to work ($l_t = 1$) or to retire ($l_t = 0$) given the information $\mathcal{F}_t$ at time t . Retirement is an irreversible decision, so the work history $l \equiv \{l_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is a right-continuous jump process with at most one jump from one to zero. The timing to retire is a stopping time⁵, which is denoted as $T_R(B) : C[0, \infty) \rightarrow \mathbb{R}_+$, that is $l_t = 0$ iff $t \geq T_R(\omega)$. At time t , agents decide to work ($l_t = 1$) or retire ($l_t = 0$) after learning the idiosyncratic shock σdB_t . So it is l_t rather than l_{t-} appearing in (2).

Agents always have an outside option to quit the saving market, which is a irreversible decision and leads to continuation value $V_{\min}$. Thus $V_{\min}$ is always a lower bound for agent continuation value V_t which constitutes a participation constraint. For

⁵ T is a stopping time of filtration $\{\mathcal{F}_t\}_{t=0}^\infty$ if it is an $\mathcal{F}$ -measurable random variable on $[0, \infty]$, such that $\{T \leq t\} \in \mathcal{F}_t$ for every t . Since l_t is measurable on $\mathcal{F}_t$, T is $\mathcal{F}$ -measurable. Note under usual conditions, we only need to show $\{T < t\} \in \mathcal{F}_t$ for all t to establish T is a stopping time (Proposition 1.2.3, KS). Rewrite the event $\{T < t\} = \cup_{n=1}^\infty \{l_{t-1/n} = 0\}$, since $l_{t-1/n}$ mapping from $\mathcal{F}_{t-1/n}$ to $\{0, 1\}$, the event $\{l_{t-1/n} = 0\} \in \mathcal{F}_{t-1/n} \subseteq \mathcal{F}_t$ for all $n \geq 1$.

technical convenience, we also impose a lower bound on the level of saving, otherwise agents can maintain huge consumption over time by rolling negative saving and never retire. We assume agents are not allowed to have net debt, thus $a_t \geq 0$. Once the level of saving hits zero, we assume the agent is liquidated and forced to quit saving market forever. This also leads to continuation value $V_{\min}$. So if $V_{\min}$ is sufficiently negative, then agents never quit the saving market. We do not model what happens after the liquidation or leaving the saving market, which may depend on explicit institution and environment outside this model, so $V_{\min}$ is exogenous given. Let $T_L(B) : C[0, \infty) \rightarrow \mathbb{R}_+$ denote the stopping time to quit. For agents remained in saving market, it is necessary that their saving level is strictly positive, $a_t > 0$ and satisfies participation constraint $V_t > V_{\min}$.

2.2 Preferences and Value Function

Agents choose consumption, labor and effort, as well as stopping times to retirement and liquidation in order to maximize preferences:

$$V_0 = \max_{c, a, T_R, T_L} \mathbb{E} \left\{ \begin{aligned} & r \int_0^{T_R(B) \wedge T_L(B)} e^{-rt} u(c_t) dt \\ & + e^{-rT_R(\omega)} 1_{T_R(B) \leq T_L(B)} U(a_{T_R(B)}) \\ & + e^{-rT_L(\omega)} 1_{T_R(B) > T_L(B)} V_{\min} \end{aligned} \middle| \mathcal{F}_0^B \right\}, \text{ given } a_0 \quad (3)$$

subject to the accumulation of saving

$$da_t = (ra_t + \theta - c_t) dt + \sigma dB_t \quad (4)$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the flow of utility, which is C^3 , Lipschitz continuous⁶, increasing and strictly concave in consumption, which is a $\mathcal{F}$ -measurable and adapted process denoted as $c \equiv \{c_t, \mathcal{F}_t^B, 0 \leq t < \infty\}$. Pension $a_{T_R(B)}$ is the level of saving at the stopping time of retirement $T_R(B)$.

In this model, agents retire because they can enjoy the value of pension and not working. Once the agent retires, his continuation value is explicitly specified by the utility of pension $U : \mathbb{R} \rightarrow \mathbb{R}$ which is twice differentiable, increasing and strictly concave in the level of pension a_{T_R} . The utility of pension U may depend on pension

⁶ u is Lipschitz continuous if there exists M such that $|u(c') - u(c)| \leq M|c' - c|$, for all c' and c . Lipschitz continuity implies bounded first derivative.

benefit arrangement and institution, which are again beyond our model and thus are summarized by an exogenous function of pension. A technical convenience is that we deport the stochastic transversality condition of a_t to U , so we do not need to worry about the possibility of ponzi game. Define the continuation value V_t as

$$V_t = \max_{c, a, T_R, T_L} \mathbb{E} \left\{ \begin{array}{l} r \int_t^{T_R(B) \wedge T_L(B)} e^{-r(s-t)} u(c_s) ds \\ + e^{-r(T_R(B)-t)} \mathbf{1}_{T_R(B) \leq T_L(B)} U(a_{T_R(B)}) \\ + e^{-r(T_L(B)-t)} \mathbf{1}_{T_R(B) > T_L(B)} V_{\min} \end{array} \middle| \mathcal{F}_t^B \right\} \text{ given } a_t \text{ and (4).} \quad (5)$$

Continuation value V_t is agent's value from time t given the information $\mathcal{F}_t^B$. At time t , the expected remaining times to retire S_t^R and to quit the saving market S_t^L are

$$S_t^R \equiv \mathbb{E}\{T_R(B) - t | \mathcal{F}_t^B\}, S_t^L \equiv \mathbb{E}\{T_L(B) - t | \mathcal{F}_t^B\}. \quad (6)$$

We maintain an assumption on U , which is the proof of Proposition 1 to establish finite a_T and retirement in finite time almost surely:

$$V_{\min} + \bar{\beta}a \geq U(a), \quad (7)$$

where $\bar{\beta} \in (0, \infty)$ solves $rV_{\min} = \bar{\beta}\theta + \max_{c \geq 0}\{ru(c, 1) - \bar{\beta}c\}$. Note the above assumption implies $V_{\min} \geq U(0)$, so there is a threshold of saving such that agents prefer quit than retirement iff the level of saving is less than such threshold. In order words, both quit and retirement are not strictly dominated and can be triggered.

The saving problem is not standard precautionary saving problem as agents can retire and quit the saving market. We first summarize some properties of continuation value V_t and the choice of consumption, retirement and quit under laissez faire:

Proposition 1 *Given maintained assumptions and the level of saving a_t ,*

(a) *The continuation value given by (5) is a continuously differentiable function of saving $V_t = V(a_t)$, where $V : [0, a_{ret}] \rightarrow \mathbb{R}$ and a_{ret} solves*

$$rV(a) = V_a(a)(ra + \theta) + \frac{\sigma^2}{2}V_{aa}(a) + \max_{c \geq 0}\{ru(c) - V_a(a)c\}, \quad \forall a \in [0, a_{ret}], \quad (8)$$

$$\text{s.t. } V(a_{ret}) = U(a_{ret}), V_a(a_{ret}) = U_a(a_{ret}), V(0) = V_{\min}. \quad (9)$$

(b) *The solution $V(a)$ and such a_{ret} exist and the continuation value V_t is the solution $V(a_t)$ with greatest $V_a(0)$. $V(a)$ is increasing over $a \in [0, a_{ret}]$ and strictly concave.*

(c) The stopping times to retire and to quit the saving market are:

$$T_R = \inf \{t : a_t = a_{ret}\}, T_L = \inf \{t : a_t = 0\}. \quad (10)$$

(d) Consumption is given by a strictly increasing $C(a_t) \equiv u_c^{-1}(V_a(a_t)/r)$ if well defined, and $C(a_t) = 0$ otherwise.

(e) $V_a(a_t) \in (0, \bar{\beta})$ is a bounded martingale. The drifts of saving a_t and continuation value V_t point in the direction in which $V_{aa}(a_t)$ is increasing. The drift of consumption $C(a_t)$ point in the direction in which $u_{cc}(C(a_t))$ is increasing.

(f) The expected remaining time to retire $S^R(a)$ and to quit $S^L(a)$ given promised continuation utility $a_t = a$ solves

$$-1 = rS_a^R(a)(ra - C(a)) + \frac{\sigma^2}{2}S_{aa}^R(a), S^R(0) = \infty, S^R(a_{ret}) = 0, \quad (11)$$

$$-1 = rS_a^L(a)(ra - C(a)) + \frac{\sigma^2}{2}S_{aa}^L(a), S^L(0) = 0, S^L(a_{ret}) = \infty. \quad (12)$$

Proof. See Appendix. ■

The constraint $V_a(a_{ret}) = U_a(a_{ret})$ on the HJB is also called smooth-pasting condition. We proof the necessity of smooth-pasting condition for optimality in the Appendix.

Verbally, V_a is the shadow price of saving, which equals to marginal utility of consumption, as saving and consumption are free to transform into each other. The price of saving is strictly increasing in consumption, because of the strictly concavity of u . Note that there is always wealth effect on consumption, captured by the fact that $C(a_t)$ is strictly increasing in a_t . However, because of the presence of shock, it is possible to have positively drift of saving a_t but negative drift of consumption C_t , if $u_{ccc}(C_t)$ and $V_{aaa}(a_t)$ are in opposite sign. This is a result of Ito Lemma, since there is an additional drift in consumption driven by shock, associated with the second derivative of $C(a)$. Agents choose to retire once a_t hit the pension target a_{ret} , and quit the saving market once a_t hits zero.

To see the effect of uncovered labor productivity shocks, we compare with the first best where labor productivity shocks can be completely insured, that is $\sigma = 0$. Then

we have a closed-form solution to (8). Assume agents retire in finite time definitely⁷. Then we can verify that the continuation value V with $a_t \in [a_0, a_{ret}^{FB}]$ units of saving is linear in a_t :

$$V^{FB}(a_t) = U(a_{ret}^{FB}) + (a_t - a_{ret}^{FB}) U_a(a_{ret}^{FB}), \quad (13)$$

where the first-best level of pension a_{ret}^{FB} and consumption⁸ are given by

$$U(a_{ret}^{FB}) - u(c^{FB}) = \left(a_{ret}^{FB} + \frac{\theta - c^{FB}}{r} \right) U_a(a_{ret}^{FB}), c^{FB} = u_c^{-1} \left(\frac{U'(a_{ret}^{FB})}{r} \right)$$

The (deterministic) remaining time to retire and to quit with a_t units of saving are

$$S^R(a_t) = \frac{1}{r} \ln \left(\frac{ra_{ret} + \theta - c^{FB}}{ra_t + \theta - c^{FB}} \right), S^L(a_t) = \infty.$$

Essentially, under the first best, agents consume constant c^{FB} over time and save the rest for pension. Agents retire once their saving reaches a_{ret}^{FB} . Continuation value is linearly increasing in a_t since higher a_t does not increase consumption but implies less remaining time to retire, in a linear fashion. Also agents never quit the saving market, as the continuation value is increasing over time and never falls below $V_{\min}$.

We can draw some comparisons. Under the benchmark economy, there is strictly positive probability that agents will quit the saving market in finite time. Since agents dissave when they are hit by negative labor productivity shocks. If negative shocks happens so frequent or severe that their saving becomes sufficiently low, then quitting the saving market becomes a preferred outside option. The following proposition compares first-best consumption and pension with those in benchmark economy:

Proposition 2 *Given $\sigma > 0$. First-best consumption c^{FB} is always higher than the consumption $C(a_t)$ in the benchmark economy for any $a_t \in (0, a_{ret})$. First-best pension a_{ret}^{FB} is higher than pension a_{ret} in the benchmark economy.*

Proof. See Appendix ■

⁷This is the case when

$$U(a_{ret}^{FB}) + (a_0 - a_{ret}^{FB}) U'(a_{ret}^{FB}) > u(ra_0 + \theta).$$

If this is violated, then agents will consume $ra_0 + \theta$ and never retire.

⁸We assume a_0 is high enough such that $ra_0 + \theta > c$, so c is always feasible.

One might expect we can conclude that agents have shorter expected remaining time to retire in the benchmark, since in the first best, saving for pension have higher target, $a_{ret}^{FB} > a_{ret}$, but agents save less, $c^{FB} > C(a_t)$. This is not true in general, because in the benchmark there is strictly positive probability that agents will quit in finite time, which implies $S^R = \infty$.

Consumption and remaining time to retire become stochastic when labor productivity shock cannot be insured. Agents retire when there is sufficient saving for pension. When labor productivity shock cannot be insured, retirement time and pension can be fluctuated as well, so agents have to strike the balance between smoothing consumption and smoothing the accumulation of saving for pension. It turns out both consumption and pension are lower. The former is because agents save more for precautionary motive, and the latter is because, the continuation value of not retiring is lower under the benchmark, so agents prefer to retire even with lower level of pension. All these represent efficiency loss as agents can be better-off if both consumption and pension can be covered. With presence of asymmetric information the first best is no longer available. It calls for the second best allocation where incentive compatibility is concerned, which is studied in the next section.

3 Pension Contribution Mechanism

Suppose instead of doing nothing, the social planner collects contribution during the working life and offers pension when agents retire. The main issue of this paper is to characterize an optimal pension contribution mechanism in which agents are free to contribute, save, consume, retire with pension and quit the mechanism under private information of labor productivity shocks and consumption. The information asymmetry is our starting point as it implies incomplete market in the first place, which calls for a pension contribution mechanism. In our model, a pension contribution mechanism is the function $A(Y, T) : C[0, \infty) \times [0, \infty) \rightarrow \mathbb{R}_+$ which describes the level of pension under contribution Y when the agent retires at time T .

Let dY_t denote the flow of pension contribution, which is normalized to volatility

of labor productivity σ and given by

$$\sigma dY_t = -c_t dt + l_t d\theta_t, \quad Y_0 = 0, \quad \text{given } \theta_0. \quad (14)$$

Agents stop contributing after retirement or quit, so $Y_t = 0$ for $t \geq \min(T_R, T_L)$. The social planner only observes history of pension contribution $Y \equiv \{Y_t, 0 \leq t < \infty\}$, but not its decomposition which involves agent consumption c and labor productivity θ . If c is observed and thus θ , then the first best can be achieved as the social planner can force agents to contribute according to θ . So we assume hidden consumption in our model.

We first focus on the mechanism where agents do not save individually and thus pension contribution mechanism is the only source of pension. A standard argument in the literature of optimal taxation is that, under observable saving, for any incentive compatible allocation under a mechanism there is another mechanism such that the incentive compatible allocation is the same except that agents do not save. To save space we do not show the construction, but a standard procedure can be found in Cole and Kocherlakota (2001). Intuitively, the social planner can always save for agents, so agents are indifferent between saving individually or by the mechanism. This is shown in the later section of decentralization. On the other hand, if individual saving is hidden instead, then we can show the only incentive compatible mechanism is laissez-faire, as in Cole and Kocherlakota (2001). What the social planner can observe is crucial. We provide further analysis on the information structure in the later section.

Given a pension contribution mechanism $A(Y, T)$, agent's continuation value v_t in the mechanism given history of contribution $\mathcal{F}_t^Y$ is:

$$v_t = \max_{c \geq 0, Y, T_R, T_L} \mathbb{E} \left\{ \begin{array}{l} r \int_t^{T_R \wedge T_L} e^{-r(s-t)} u(c_s) ds \\ + e^{-r(T_R-t)} 1_{T_R \leq T_L} U(A(Y, T_R)) \\ + e^{-r(T_L-t)} 1_{T_R > T_L} V_{\min} \end{array} \middle| \mathcal{F}_t^Y \right\}, \quad \text{s.t. (14)}. \quad (15)$$

Agent's continuation value v_t is greater than $V_{\min}$ or $U(A(Y^t, t))$, otherwise, he would quit the mechanism if $v_t \leq V_{\min}$ or would retire immediately if $v_t \leq U(A(Y^t, t))$. We denote the set of maximizers $\mathcal{D}$ as a function of pension contribution mechanism

$A(Y, T)$:

$$\mathcal{D}(A) \equiv \{(c, Y, T_R, T_L) : c, Y, T_R, T_L \text{ solves (15) at } t = 0 \text{ given } A(Y, T_R)\} \text{ and } a_0 > 0. \quad (16)$$

We say recommended consumption $c(Y) \equiv \{c_t, \{\mathcal{F}_t^Y\}_{t=0}^\infty, 0 \leq t < \infty\}$ and recommended stopping times to retire $T_R(Y)$ and to quit $T_L(Y)$ *incentive compatible* under pension contribution mechanism $A(Y, T)$ if there is process Y such that $(c(Y), Y, T_R(Y), T_L(Y)) \in \mathcal{D}(A)$. The social planner recommends $c(Y)$, $T_R(Y)$ and $T_L(Y)$ according to the history of pension contribution Y , the only thing observed by the social planner in this economy.

By duality argument and assuming the law of the large number applies, which we will provide more detail in later section, the objective of a social planner can be stated in two levels. First, given a initial promised lifetime utility V_0 , the social planner solves $A(Y, T)$ that maximizes the net revenue G such that $c(Y)$, $T_R(Y)$ and $T_L(Y)$ are incentive compatible:

$$G(V_0) \equiv \max_{c, Y, T_R, T_L} \mathbb{E} \left\{ \begin{array}{l} \int_0^{T_R(Y) \wedge T_L(Y)} e^{-rt} dY_t \\ -e^{-rT_R(Y)} 1_{T_R(Y) \leq T_L(Y)} A(Y, T_R(Y)) \end{array} \middle| \mathcal{F}_0^Y \right\} \quad (17)$$

$$\text{s.t. } (c(Y), Y, T_R(Y), T_L(Y)) \in \mathcal{D}(A),$$

$$V_0 = v_0.$$

The net revenue function $G(V_0)$ represents the most efficient pension contribution mechanism to deliver lifetime utility V_0 . Second, the social planner maximizes V_0 subject to $a_0 \geq -G(V_0)$, that is the optimal-incentive pension contribution mechanism which is feasible.

To see the contribution of this paper, it is helpful to appreciate the difficulty in the first step. There are two level of maximization in the design of optimal mechanism. The social planner choose some functions in a constrained set to maximize agent's lifetime value. Characterizing the set of incentive compatible $c(Y)$, $T_R(Y)$ and $T_L(Y)$ is also challenging, since $\mathcal{D}$ is a mapping from a function $A(Y, T)$ to a quadruple functions $(c(Y), Y, T_R(Y), T_L(Y))$ which involves another maximization problem (15). Without pension contribution mechanism, Proposition 1 has shown agents' continuation value

can be simply summarized by HJB equation with a as state variable, so agent's decision at time t depends on a_t only. Under pension contribution mechanism, such HJB equation no longer applies. As the whole path of pension contribution Y affects the pension level through an explicit mechanism $A(Y, T)$, agent's decision at time t is path-dependent as well. We will characterize agent's decision by stochastic Pontryagin principle instead.

With the presence of pension contribution mechanism, we change the probability space to $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$, where $\mathcal{F}_t^Y$ is the augmented filtration generated by Y ⁹. Labor productivity $d\theta_t$ is continuously hit by idiosyncratic shock dB_t as represented in (1), but right now B is a standard Brownian motion on $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$ ¹⁰. The reason for new filtration $\{\mathcal{F}_t^Y\}_{t=0}^\infty$ is that Y and t are the only common knowledge between the social planner and agents, so the social planner determines recommended actions and pension contribution mechanism with respect to Y .

3.1 Risk-Neutral Measure

In the indirect mechanism described here, the recommended consumption $c(Y)$, recommended stopping time to retire $T_R(Y)$ and to quit $T_L(Y)$ as well as pension contribution mechanism $A(Y, T)$ depend on the contribution Y , so it is convenient to avoid Y endogenously determined. This can be done by applying Girsanov theorem.

Consider the point of view of the social planner in a "risk-neutral world", where Y is a standard Brownian motion on some probability space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$. The social planner can always recommend agents consumption $c(Y)$ such that c is progressive measurable¹¹. A process B^0 can be constructed from an arbitrary recommendation

⁹The augmented filtration $\mathcal{F}_t^Y$ generated by Y is the product σ -field of the filtration generated by Y^t and $\mathbb{P}$ -null sets. The reason to consider augmented filtration generated by Y rather than filtration generated by Y is that the latter is not right-continuous, hence usual conditions fail to be held. Usual conditions are needed for Martingale Representation Theorem, the existence RCLL modification and the existence of the weak solution to backward stochastic differential equation, which are applied in this paper.

¹⁰Note B is still Brownian motion under the augmented filtration $\{\mathcal{F}_t^Y\}$ (Theorem 2.7.9 KS). In general B may no longer be Brownian motion under a "larger" filtration, see Example B.5 of Medvedev (2007).

¹¹Such progressive measurable c always exists. Since c is $\mathcal{F}^Y$ -measurable and adapted, c has a progressive measurable modification (Proposition 1.1.12 KS). Y is a modification of X if for every $t \geq 0$ we have $\Pr[X_t = Y_t] = 1$. X is progressive measurable (with respect to the filtration $\{\mathcal{F}_t^Y\}$) if for every $t \geq 0$ and Borel set $A \in \mathcal{B}(\mathbb{R})$, the set $\{(s, \omega) ; 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$ belongs to the

$c(Y)$, not necessarily incentive compatible, as

$$B_t^0 = Y_t - \int_0^t \frac{\theta - c_s}{\sigma} ds.$$

By Girsanov theorem, we can construct a probability measure $\mathbb{P}_0$ such that B^0 is a standard Brownian motion on $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_0)$, which can be done as follows. First, define a continuous local martingale on $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$:

$$\Gamma_t(c) \equiv \exp \left(\int_0^t \frac{\theta - c_s}{\sigma} dY_s - \frac{1}{2} \int_0^t \left(\frac{\theta - c_s}{\sigma} \right)^2 ds \right), \quad (18)$$

with $\Gamma_0 = 1$. Indeed $\Gamma_t(c)$ is martingale if c satisfies Novikov condition (Corollary 3.5.13 KS):

$$\mathbb{E}_Y \left\{ \exp \left(\frac{1}{2} \int_0^t (c_s)^2 ds \right) \right\} < \infty, \quad (19)$$

for all t . The expectation $\mathbb{E}_Y$ is taken with respect to probability measure $\mathbb{P}_Y$. A sufficient condition is that c is finite a.s. We assume recommended consumption c is constructed such as to satisfy Novikov condition throughout this paper. So given Novikov condition, by Girsanov theorem (Theorem 3.5.1 KS), there is a unique probability measure $\mathbb{P}_0$ such that the probability of path $B^{0t} \in \mathcal{F}_t^Y$, where $0 \leq t < \infty$, is

$$\mathbb{P}_0(B^{0t}) = \mathbb{E}_Y[1_{B^{0t}} \Gamma_t(c)], \quad (20)$$

and B^0 becomes a standard Brownian motion on $(\Omega, \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_0)$. In particular, the Radon-Nikodym derivative connects the constructed measure $\mathbb{P}_0$ with risk neutral measure $\mathbb{P}_Y$ under $(\Omega, \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$:

$$d\mathbb{P}_0 = \Gamma_t(c) d\mathbb{P}_Y.$$

Agent's problem (15) under $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_0)$ is equivalent to the following problem under risk-neutral space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$:

$$v_t = \max_{c \geq 0, T_R, T_L} \mathbb{E}_Y \left\{ \begin{array}{l} r \int_t^{T_R(Y) \wedge T_L(Y)} e^{-r(s-t)} \Gamma_s(c_s) u(c_s) ds \\ + e^{-r(T_R(Y)-t)} 1_{T_R(Y) \leq T_L(Y)} \Gamma_{T_R(Y)}(c_{T_R(Y)}) U(A(Y, T_R(Y))) \\ + e^{-r(T_L(Y)-t)} 1_{T_R(Y) > T_L(Y)} \Gamma_{T_L(Y)}(c_{T_L(Y)}) V_{\min} \end{array} \right| \mathcal{F}_t^Y \right\}, \quad (21)$$

product σ -field $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t^Y$.

s.t. the density $\Gamma_t(c)$ evolves according to

$$d\Gamma_t = \frac{\Gamma_t}{\sigma} (\theta - c_t) dY_t. \quad (\Gamma)$$

We want to stress the expectation $\mathbb{E}_Y$ is taken with respect to risk-neutral measure $\mathbb{P}_Y$ instead, and more importantly, Y is a Brownian motion under $\mathbb{P}_Y$. The objective of agent problem under risk-neutral space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$ is adjusted by Γ_t such that agents' behaviors look the same by a third party from either the constructed world $\mathbb{P}_0$ or the risk-neutral world $\mathbb{P}_Y$.

If recommended consumption $c(Y)$ solves (21) and thus incentive compatible, then we have $B^0 = B$ and $\mathbb{P}_0 = \mathbb{P}^{12}$. The problem stated in (21) becomes (15) when $\mathbb{P}_0 = \mathbb{P}$. So the maximizer (c, T_R, T_L) to (21) with respect to Y constitutes the set of incentive compatible action $\mathcal{D}(A)$, which is derived in the next section.

3.2 Stochastic Pontryagin Principle and Incentive Compatibility

As in any Ramsey problem, we want to obtain the first order conditions of agent problem, which will constitute the set of incentive compatible recommendation $\mathcal{D}(A)$ for the social planner problem. Applying stochastic Pontryagin principle detailed in Yong and Zhou (1999), we can define a system of co-state equations to support the maximum. With slight abuse of notations, we denote $v' \equiv \{v'_t, \{\mathcal{F}_t^Y\}_{t=0}^\infty, 0 \leq t < \infty\}$ and $r\sigma q$, where $q \equiv \{q_t, \{\mathcal{F}_t^Y\}_{t=0}^\infty, 0 \leq t < \infty\}$, as the co-states of drift and volatility to Γ_t in the agent's problem (21) under risk-neutral space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$. We verify the optimality of stochastic Pontryagin principle in the proof of the following lemma, so one can take the following as guess and verify. We will show $v' = v$, at least in the sense of modification, so the co-state of drift to Γ_t turns out to be agent's continuation value under the original space. Co-states can be interpreted as the supporting prices for agent's optimal choices. The current value Hamiltonian is

$$\mathcal{H}(\Gamma_t, q_t) \equiv \max_{c_t} \{r\Gamma_t u(c_t) + rq_t \Gamma_t (\theta - c_t)\}. \quad (22)$$

¹²One can also interpret B^0 as the report of labor productivity shocks and construct a direct mechanism from B^0 , which is essentially the same. In this case, the social planner's allocation is incentive compatible if $B^0 = B$.

The co-state of drift v'_t is the solution to the following BSDE under $\mathbb{P}_Y$

$$\begin{cases} dv'_t = \left(rv'_t - \frac{\partial \mathcal{H}_t}{\partial \Gamma_t} \right) dt + r\sigma q_t dY_t, \\ v'_{T_R} = U(A(Y, T_R)), v'_{T_L} = V_{\min}. \end{cases} \quad (23)$$

Yong and Zhou (1999) show that the weak solution $v'(Y)$ to the above BSDE exists and is unique under $(\mathcal{F}^Y, \mathbb{P}_Y)$. Given $v'(Y)$, the stopping time to retire and to quit can be determined by v' :

$$T_R(v') = \inf \{t : v'_t(Y) = U(A(Y, t))\}, T_L(v') = \inf \{t : v'_t(Y) = V_{\min}\}. \quad (24)$$

The first order conditions for c_t is

$$u_c(c_t) = q_t. \quad (25)$$

Given q_t , we can express consumption in (25) and utility as $\mathcal{C}(q_t)$ and $\mathcal{U}(q_t) \equiv u(\mathcal{C}(q_t))$ respectively.

Verbally, stochastic Pontryagin principle essentially states if $(c(Y), Y, T_R(Y), T_L(Y)) \in \mathcal{D}(A)$ under risk-neutral space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$, then it is necessary in general, and sufficient in particular our model, that there exists a unique supporting prices v' and q on drift and volatility respectively to support $(c(Y), Y, T_R(Y), T_L(Y))$ to be optimal to the Hamiltonian (22). Furthermore, the evolution of v' can be determined by (23). In sum, given a pension contribution mechanism $A(Y, T)$, we can characterize the set of incentive compatible recommendation $\mathcal{D}(A)$ with the help of q under risk-neutral space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$.

We simply refer promised continuation utility as the continuation utility v described in (15) when agents follow social planner's recommendations $(c(Y), Y, T_R(Y), T_L(Y))$ under a pension contribution mechanism $A(Y, T)$. The following proposition is a helpful technical result, which constructs the set of incentive compatibility under original space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$ and provides an economic interpretation of co-states v' and $r\sigma q_t$, which are the promised continuation utility and its volatility if social planner's recommendations are incentive compatible:

Proposition 3 (a) *Given pension contribution mechanism $A(Y, T)$. There exists a bounded process q which is positive and progressively measurable, such that $(\mathcal{C}(q), T_R(v'), T_L(v'))$ is incentive compatible with respect to $A(Y, T)$ under $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$, where recommended stopping times $T_R(v')$ and $T_L(v')$ are given by (24), and v' is given by*

$$\begin{cases} dv'_t = r[v'_t - \mathcal{U}(q_t)]dt + r\sigma q_t dB_t^0, \\ v'_{T_R} = U(A(Y, T_R)), v'_{T_L} = V_{\min} \end{cases} \quad (26)$$

(b) *Given promised lifetime utility $v_0 = v'_0 > V_{\min}$, then v' given by (26) is the promised continuation utility v . The volatility of promised continuation utility v_t is $r\sigma q_t$*

Proof. See Appendix. ■

We denote B^0 implied Brownian motion here since under incentive compatible recommendation, the social planner can recover the unobservable labor productivity shock B from observed contribution Y , by setting $B = B^0$ and $\mathbb{P} = \mathbb{P}_0$.

The co-state $v'_t = v_t$ is the promised continuation utility with volatility $r\sigma q_t$. Higher q_t always implies lower c_t . The above lemma concludes that the supporting price along time dimension turns out to be the continuation utility promised by a public pension mechanism. It confirms a celebrated result of Abreu, Pearce and Stacchetti (1990) that agents' promised continuation utility are sufficient to summarize the solution agent's optimization problem. At time 0, the mechanism promises a lifetime utility v_0 . If there were none of any flow of utility from consumption, the drift of promised continuation utility has to be rv_t in order to keep the level of promised lifetime utility at v_t . So the drift of promised continuation utility has to be reduced by $r\mathcal{U}(q_t)$ if the mechanism induces agent to choose of consumption at $\mathcal{C}(q_t)$. That is why the drift of promised continuation utility is $r[v_t - \mathcal{U}(q_t)]$. On the other hand, co-state $r\sigma q_t$ can be interpreted as the volatility of promised continuation utility with respect to implied shock dB_t^0 . A higher $r\sigma q_t$ implies the promised continuation utility would be more sensitive to income shock. An economy with perfect insurance on continuation utility implies $r\sigma q_t = 0$. So q_t can also be interpreted as the degree of risk exposure allowed by a pension contribution mechanism. We would elaborate the insurance function of a mechanism in next section.

3.3 Volatility Implied by Pension Contribution Mechanism

The above section shows that, given any pension contribution mechanism $A(Y, T)$, there exists a unique process q that characterizes the set of incentive compatible actions. However, the explicit formulation of q in the stochastic Pontryagin principle is far from known. So instead of finding q given a pension contribution mechanism $A(Y, T)$, we construct $A(Y, T)$ from any arbitrary process q which is adapted and $\mathcal{F}^Y$ -measurable. In other words, we want to represent a pension contribution mechanism by q rather than $A(Y, T)$. The following corollary summarizes this idea:

Corollary 1 *Given $v_0 > V_{\min}$, for any bounded process q and any recommended stopping time to retire T_R , which are positive, adapted and $\mathcal{F}^Y$ -measurable, there exists a pension contribution mechanism $A(Y, T)$ and recommended stopping time to quit T_L such that $(\mathcal{C}(q), T_R(Y), T_L(Y))$ is incentive compatible under $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$. The evolution of continuation value v under such incentive compatible recommendations is*

$$dv_t = r(-\theta + v_t - \mathcal{U}(q_t) + \mathcal{C}(q_t))dt + r\sigma q_t dY_t, \text{ given } v_0 > V_{\min}. \quad (27)$$

In particular, recommended stopping time to quit $T_L(Y)$ is:

$$T_L(Y) = \inf \{t : v_t(Y) = V_{\min}\}. \quad (28)$$

The supporting $A(Y, T)$ is:

$$A(Y, T) = \begin{cases} U^{-1}(v_T(Y)) & \text{if } T = T_R(Y), \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

Proof. See Appendix ■

The above corollary allows us to represent a pension contribution mechanism as a triple:

$$\mathcal{M} \equiv \{q(Y), T_R(Y), v_0\}. \quad (30)$$

Given $\mathcal{M}$, a recommended consumption $c(Y)$ satisfied (25) is incentive compatible, and the above corollary constructs an incentive compatible recommended stopping time to quit as the stopping time when v_t hits $V_{\min}$. Essentially, by representing a pension contribution mechanism as (30), we can characterize the set of incentive compatible

recommendation as the social planner's choice of q and stopping time T_R with given v_0 . It allows us to use standard HJB approach to solve the social planner problem (17) with v_t as state variable.

Remark. In the original mechanism $A(Y, T)$, the promised continuation utility v_t is the solution to BSDE (26) under some process q . The solution of BSDE (26) solves for an initial condition v_0 . Changing to a recursive mechanism implies v_t becomes a solution to forward stochastic differential equation, FSDE, (27) with a given initial condition v_0 . Having state variable to be FSDE, rather than BSDE, we can apply the standard HJB approach to solve for the optimal mechanism. Also we no longer need to find such q appeared in the original mechanism, which is mentioned in the beginning of this section. It is a technical convenience, since a great deal of additional assumptions are needed to guarantee a well-behaved stochastic control problem for a couple FSDE-BSDE system. See Ma and Yong (2007) for further discussion on this issue.

3.4 Optimal Mechanism

This section outlines the design of pension contribution mechanism. Recall that the evolution of continuation values under incentive compatible recommendations and mechanism $\mathcal{M}$ is given by (27). They become constraints to mechanism design of a dual problem, which solves for incentive compatible recommendations to maximize the continuation revenue of pension contribution mechanism subject to a level of promised lifetime utility v_0 . Under $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$, the social planner problem (17) can be rewritten under representation $\mathcal{M}$:

$$G(v_0) \equiv \max_{q, T_R, T_L} \mathbb{E} \left\{ \int_0^{T_R \wedge T_L} e^{-rt} [\theta - \mathcal{C}(q_t)] dt - e^{-rT_R} 1_{T_R \leq T_L} U^{-1}(v_{T_R}) \middle| \mathcal{F}_0^Y \right\}, \quad (31)$$

subject to¹³

$$dv_t = r(v_t - \mathcal{U}(q_t)) dt + r\sigma q_t dB_t, \text{ given } v_0 > V_{\min}, v_{T_L} = V_{\min}. \quad (32)$$

Note G satisfies boundary conditions:

$$G(V_{\min}) = 0, G(v_{T_R}) = -U^{-1}(v_{T_R}) \quad (33)$$

¹³Note Γ is not needed as state variable as it does not show up in any agent's first order conditions, promised continuation utility, continuation revenue, nor boundary conditions.

as agents quit immediately if the social planner delivers value $V_{\min}$. The continuation revenue of the mechanism after the agent walks away becomes zero.

Since the set of incentive compatible recommendation can be completely characterized by the evolution of promised continuation utility, the revenue of the optimal public pension mechanism G solves the following HJB with v as state variable:

$$rG(v) = \max_q \left\{ \theta - \mathcal{C}(q) + rG_v(v)[v - \mathcal{U}(q)] + \frac{(r\sigma q)^2}{2} G_{vv}(v) \right\}. \quad (34)$$

Let V_{ret}^* denote the level of promised continuation value when the social planner recommends agents to retire, and a_{ret}^* denote level of pension released. The revenue function $G(v)$ satisfies the smooth-pasting condition if:

$$G_v(V_{ret}^*) = \frac{-1}{U_a(a_{ret}^*)}. \quad (35)$$

Then the optimal feasible mechanism promises to deliver maximal v_0 such that $G(v_0) + a_0 \leq 0$.

Having introduced all the relevant conditions, we are ready to solve for the optimal (indirect) pension contribution mechanism, which is stated in the following proposition:

Proposition 4 (a) *There are unique continuous differentiable solution G , unique V_{ret}^* and a_{ret}^* that solve the second-order ODE (34) such that (33) and (35) are satisfied. Under the optimal pension contribution mechanism, the continuation revenue (17) promising continuation utility v is $G_t = G(v)$, which is decreasing and strictly concave in v .*

(b) *Lifetime utility under optimal mechanism is $v_0 = G^{-1}(-a_0)$. Given contribution Y , the evolution of continuation utility promised by the optimal mechanism, v_t^* , is given by*

$$dv_t^* = [-\theta + \mathcal{C}_t^* + r(v_t^* - \mathcal{U}_t^*)] dt + r\sigma q_t^* dY_t, \text{ given } v_0^* = v_0 \quad (36)$$

where $\mathcal{U}_t^* = \mathcal{U}(q_t^*)$, $\mathcal{C}_t^* = \mathcal{C}(q_t^*)$ is the optimal incentive compatible recommended consumption, and $q_t^* = q^*(v_t^*)$ solves

$$q_t^* = \mathcal{C}_q^*(q_t^*) \frac{rG_v(v_t^*)q_t^* + 1}{(r\sigma)^2 G_{vv}(v_t^*)} = u_c(\mathcal{C}_t^*) > 0. \quad (37)$$

(c) The optimal incentive compatible recommended stopping time to retire T_R^* and to quit T_L^* are:

$$T_R^*(Y) = \inf\{t : v_t^*(Y) = V_{ret}^*\}, T_L^*(Y) = \inf\{t : v_t^*(Y) = V_{\min}\}. \quad (38)$$

(d) The drift of v_t^* points in the direction in which $G_{vv}(v_t^*)$ is increasing.

(e) Given the optimal mechanism to promise continuation utility v_t^* . The pension gradient is $-rq_t^*G_v(v_t^*) \in (0, 1)$, so pension is always insured rather than subsidized.

(f) If $G_v(V_{\min}) \leq -1$, then $rq^*(v_t^*) < 1$ for all v_t^* .

(g) If $G_{vvv}(v_t^*)$ is sufficiently high, then volatility $q^*(v_t^*)$ is strictly decreasing in promise continuation utility v_t^* .

(h) The expected remaining time to retire $S^{R*}(v_t^*)$ and to quit $S^{L*}(v_t^*)$ given $v_t = v_t^*$ solve

$$\frac{S_{vv}^{R*}(v_t^*)}{2} = -\frac{rS_v^{R*}(v_t^*)(v_t^* - \mathcal{U}_t^*) + 1}{(r\sigma q_t^*)^2}, S^{R*}(V_{\min}) = \infty, S^{R*}(V_{ret}^*) = 0, \quad (39)$$

$$\frac{S_{vv}^{L*}(v_t^*)}{2} = -\frac{rS_v^{L*}(v_t^*)(v_t^* - \mathcal{U}_t^*) + 1}{(r\sigma q_t^*)^2}, S^{L*}(V_{\min}) = 0, S^{L*}(V_{ret}^*) = \infty. \quad (40)$$

Proof. See Appendix. ■

We are interested in the insurance of pension provided by a public pension mechanism. Interpret $-G_t = -G(v_t^*)$ as the balance in the "pension account", which we will provide further discussion in the section of implementation. The corresponding counterpart in laissez faire is saving a_t , where both are the accumulation for pension. By Ito lemma, the evolution of the balance is

$$d(-G_t) = [-rG_t + (1 + rq_t^*G_v(v_t^*))(\theta - \mathcal{C}_t^*)]dt - rq_t^*G_v(v_t^*)\sigma dY_t.$$

Given a pension contribution Y , let's call $d(-G_t)/d(\sigma Y_t)$ the pension gradient, where there is pension subsidy when pension gradient is greater than unity. Under perfect insurance in the first best, then the pension gradient is zero. Since agents are subject to income shock, so do their pension contributions, a perfect insurance implies the continuation revenue is completely insensitive to agent's current contribution. Under laissez faire, the pension gradient is unity. Under the optimal mechanism, by Proposition 4(e), the pension gradient is strictly less than one, so it is optimal to insure pension rather than to subsidize so.

Why not agents withdraw arbitrarily high from the optimal mechanism? It is because the mechanism is history-dependent. The social planner keeps track of the promised continuation utility, if agents withdraw arbitrarily high from optimal mechanism, then there will be a arbitrarily large drop in the continuation utility promised by the mechanism, which is not optimal to agent. In particular, from (36), the volatility of promised continuation utility q_t^* poses a trade-off between the amount withdrawn and the level of promised continuation utility. From (34), the optimal q_t^* is the level that balances amongst three factors: the benefit of increment in pension accumulation $\theta - \mathcal{C}(q_t^*)$, the benefit of postponing consumption and effort $v_t - \mathcal{U}(q_t^*)$, and the benefit of increment in volatility of promised continuation utility, $r\sigma q_t^*$. The shadow prices for the second and the third are G_v and G_{vv} respectively.

It is optimal to recommend agents to retire when promised continuation value v_t^* is sufficiently high to hits V_{ret}^* . Due to income effect, maintaining promised continuation value higher than V_{ret}^* is too costly to the social planner. Given v_t^* is promised, the marginal revenue of promised continuation utility is $G_v(v_t^*)$. If the agent retires, the social planner has to provide $U^{-1}(v_t^*)$ units of pension in order to keep the promise. So the marginal cost of retiring at v_t^* is $-1/U_a(U^{-1}(v_t^*))$. At the optimal stopping time to retire, the marginal revenue equal to the marginal cost, which is given by the smooth-pasting condition (35). It is never optimal to retire if smooth-pasting condition never satisfied.

Why it is optimal to recommend agents to quit the mechanism? The expected remaining time to quit is given by $S^{L*}(v_t^*)$ in Proposition 4(g). Under the optimal mechanism, agents quit when promised continuation value v_t^* hits V_{min} . This happens when there is a series of withdrawal ($dY_t < 0$) which are frequent or high enough such that v_t^* becomes sufficiently low. The series of withdrawal is allowed because this is to protect agent's consumption to series of negative labor productivity shocks. But this must be done as the cost of redistributing resource from other agents. So there is a level that further insurance is not optimal to be provided when the withdrawal accumulated is sufficient large. In this case the social planner recommend the agent to quit the mechanism rather than draining resource from other agents.

3.5 Implementation of Optimal Mechanism

How does the optimal mechanism work? Imagine there is a public pension account $-G_t$ maintained by the social planner. At time 0, the balance in the account is a_0 . Agents can contribute some of their income into the account. Agents can withdraw all the balance as pension when they retire. In particular, the evolution of pension account balance under the optimal pension contribution mechanism is:

$$d(-G_t) = [-rG_t + (1 + rq_t^* G_v(v_t^*))(\theta - \mathcal{C}_t^*)] dt - rq_t^* G_v(v_t^*) \sigma dY_t. \quad (41)$$

The first term on the right hand side represents the autonomous drift of the balance even without any contribution, which consists of the accumulation through the saving technology with rate r , and social planner's contribution. Since $\theta - \mathcal{C}_t^*$ is the expected contribution and $-rq_t^* G_v(v_t^*) \in (0, 1)$ by proposition 1(e), the social planner is contributing into the pension account if agent's expected contribution is positive. The second term is agent's actual contribution. Note agent's contribution does not lead to a change in balance quid pro quo, essentially, contributing (withdrawing) one dollar will add (subtract) less than one dollar to the balance, where the ratio is measured by pension gradient, which is $-rq_t^* G_v(v_t^*) \in (0, 1)$, and can be completely kept track by v_t^* , whose evolution (36) depends of pension contribution Y . Since the pension gradient is less than unity, agent can insure against negative income shock by withdrawing from the pension mechanism. It is the representation of pension account by contribution. Another representation is by promised continuation utility v_t : the balance in pension account is simply $-G(v_t^*)$.

To see the welfare effect of optimal public pension mechanism, expanding promised continuation utility v_t^* as a stochastic integral, we have

$$v_t^* = e^{rt} G^{-1}(0) - r \int_0^t e^{r(t-s)} [\mathcal{U}_s^* - q_s^* (\theta - \mathcal{C}_s^*)] ds + r\sigma \int_0^t e^{r(t-s)} q_s^* dY_s. \quad (42)$$

Since the volatility of promised continuation utility q_t^* is positive, so higher pension contribution dY_t increases promised continuation utility v_t , hence sooner to hit V_{ret}^* and retire. The pension is fixed as $a_{ret}^* = -G(V_{ret}^*)$. So from the social point of view, retired agents are different in their retirement age only. It is never optimal to provide perfect insurance through public pension mechanism, as the volatility of promised

continuation utility q_t^* is never zero. It is because of the dynamic moral hazard problem: agents would keep drawing $dY_t = -\infty$ from the public pension mechanism and never retire if $q_t^* = 0$.

Note that the above suggest G_t to be interpreted as the continuation revenue from pension contribution from an agent promised continuation utility v_t^* . Assume the law of the large number applies, then total revenue of the optimal public pension mechanism at t is the sum of the continuation revenue from all agents

$$\mathbb{E} \left\{ \begin{array}{l} \int_0^t e^{rs} dY_s + e^{rt} a_0 + 1_{t < \min(T_R, T_L)} G(v_t^*) \\ -e^{r(t-T_R)} 1_{T_R \leq \min(t, T_L)} a_{ret}^* \end{array} \middle| \mathcal{F}_0^Y \right\} = a_0 + G(v_0^*) = 0,$$

where the first equality follows from the fact that the expectation is a martingale. So total revenue of the optimal pension contribution mechanism is always zero, thus the mechanism is always feasible.

3.6 Decentralization

Implementation of optimal pension contribution mechanism is not unique, as the allocation without individual saving is just one of them. In fact there is a continuum of optimal mechanism that leads to the same allocation but with different path of individual saving. So optimal mechanisms are different in the mixing level of pension supported by the mechanism and individual saving. In this section we are interested in other extreme: how the optimal mechanism can be decentralized by individual saving only. This can be done with help of a linear saving tax and a lump-sum tax. The former taxes any flow of new saving deposited into individual saving account. Let τ_t and $\tau_{sav,t}$ denote the lump-sum tax and linear saving tax rate respectively, so the accumulation of individual saving is

$$da_t = (ra_t - \tau_t) dt + (1 - \tau_{sav,t}) \sigma dY_t. \quad (43)$$

The following proposition describes how to decentralize the allocation of optimal mechanism through individual saving:

Proposition 5 *Given individual saving a_t . Optimal lump-sum tax τ_t and linear saving tax rate $\tau_{sav,t}$ are given by:*

$$\tau_t = (1 - \tau_{sav,t}) (\theta - C_t^*), \tau_{sav,t} = 1 + r q_t^* G_{v,t}, \quad (44)$$

where $G_{v,t} \equiv G_v(G^{-1}(-a_t))$. Furthermore,

(a) Agents' continuation value with saving a_t attains $V_t = G^{-1}(-a_t) = v_t^*$, with the evolution following (36). Consumption is $c_t = \mathcal{C}_t^*$. Agents retire when $a_t = a_{ret}^*$; quit when $a_t = 0$.

(b) Saving tax rate is always strictly positive, $\tau_{sav,t} > 0$.

(c) The ex ante taxation is zero under optimal $c_t = \mathcal{C}_t^*$. The ex ante accumulation of individual saving is

$$\mathbb{E}_t da_t = (ra_t + \theta - \mathcal{C}_t^*) dt.$$

(d) Assume $\mathcal{C}_t^*$ is increasing in a_t . Then the volatility of continuation utility $r\sigma q_t^*$ is decreasing in saving a_t . Also, there is lump-sum subsidy, $\tau_t < 0$, iff saving a_t is sufficiently high. Both τ_t and $\tau_{sav,t}$ are decreasing in a_t .

Proof. See Appendix. ■

The proof of proposition involves matching components of (41) with τ_t and $\tau_{sav,t}$ in (43). Let σdY_t denote the flow of new saving, and the evolution of private saving becomes

$$da_t = [ra_t + \theta - \tau_{sav,t}\mathcal{C}_t^* - (1 - \tau_{sav,t})c_t] dt + (1 - \tau_{sav,t})\sigma dB_t. \quad (45)$$

The optimal saving tax is linear because the saving tax rate $\tau_{sav,t}$ does not depend on the flow of new saving σdY_t .

The optimal saving tax rate is always strictly positive. The social planner taxes saving and subsidizes withdrawal linearly in order to provide insurance on pension accumulation. This contrasts with current practices on 401(k), IRA or others similar, where pension contribution is not taxed or subsidized by contribution match and withdrawal is subject to penalty. Note that complete insurance implies $r q_t^* G_{v,t} = 0$ and hence $\tau_{sav,t} = 1$, which is complete confiscation of net saving. As discussed above, complete insurance through saving tax is not feasible, sine if saving tax rate $\tau_{sav,t} = 1$ then all agents will always dissave infinite amount until quit the saving market.

On the other hand, the lump-sum tax τ_t can be positive or negative. By Proposition 5, the social planner subsidizes agents in lump sum if when the optimal level of consumption is higher than the average labor productivity, $\mathcal{C}_t^* > \theta$. This is the

case it is optimal to induce the agent to consume higher than θ . Under laissez faire he may not consume that much at the optimal level, because he prefers to save more than the optimal level for precautionary motive. In this case, a lump sum subsidy $\tau_t < 0$ is needed to compensate such precautionary saving. Such compensation cannot be achieved through lower saving tax rate $\tau_{sav,t}$, since the effect of labor productivity shock dB_t on saving da_t would be exaggerated on the contrary. So the social planner needs these two tax vehicles to decentralize the optimal allocation.

If there is wealth effect on consumption, that is C_t^* is increasing in a_t , then by Proposition 5(d) the social planner should subsidize pension in lump sum when a_t is sufficiently high, which more likely to be agents who are old or about to retire, and vice versa. This implies on average, the social planner should lump-sum subsidize the old or about-to-retire agents and should lump-sum young or about-to-quit agents. On the other hand, by Proposition 5(d) the saving tax rate is decreasing in a_t , which, from (45), implies pension insurance is also decreasing in a_t , as da_t is more sensitive to dB_t . This characterizes the dynamic incentives of optimal taxation to decentralize optimal mechanism. As saving is accumulated toward retirement, agent's saving are more lump-sum subsidized but less insured.

The ex ante taxation is zero under optimal allocation may echo the celebrated result of Kocherlakota (2005). The reason in our model is that, any ex ante distortion with the presence of saving market will distort the level of pension as accumulated saving available when agents retire. This violates the optimality that agents should retire with the same level of pension. At the time of retirement, previous history should not matter. Rather, the decision to retire is based on purely the comparison between the continuation value and utility of pension. Under decentralization, such comparison is always the same as long as agents have the same level of saving. So it is always optimal for agents to retire at the same level of pension. So ex ante distortion of saving is not optimal.

Under decentralization it is helpful to see why under hidden saving the only optimal mechanism is laissez faire. If net saving dY_t in (43) are hidden, then the saving tax rate must be zero since the social planner cannot tax on hidden saving. So the only available tax vehicle is lump-sum tax. But as argued before, it is never optimal to

distort ex ante saving, so lump-sum tax is not optimal as well. Hence laissez faire is the only optimal mechanism. This result is in the spirit of Cole and Kocherlakota (2001), though our model features retirement and quit.

3.7 Comparison between Optimal Mechanism and Laissez Faire

To see the role of pension contribution mechanism, it would be helpful to compare the allocation under optimal pension contribution mechanism with laissez faire. This is essentially the comparison of two solutions $V(a)$ and $G(v)$. The following proposition characterizes main difference:

Proposition 6 *Under optimal pension contribution mechanism,*

(a) *Pension is higher (lower) than the one under laissez faire (the first best), $a_{ret} < a_{ret}^* < a_{ret}^{FB}$.*

(b) *Comparing balance in pension account $-G(v_t)$ with balance of saving a_t under laissez faire. Given the level of promised continuation utility v_t , the balance is less (more) than the one under laissez faire (the first best), $(V^{FB})^{-1}(v_t) < -G(v_t) < V^{-1}(v_t)$. The lifetime utility is higher (lower) than laissez faire (the first best), $V(a_0) < G^{-1}(-a_0) < V^{FB}(a_0)$.*

(c) *Given the level of promised continuation utility v_t , optimal mechanism provides strictly more insurance on pension than laissez faire, as its volatility is $-\sigma q_t^* G_v(v_t) < \sigma$, but lower drift, $-rG(v_t) + \theta - C(q^*(v_t)) < rV^{-1}(v_t) + \theta - C(-G(v_t))$. Agents enjoy more (less) consumption than laissez faire (the first best), but less than, $C(-G(v_t)) < C(q^*(v_t)) < C^{FB}(-G(v_t))$.*

(d) *Given the level of promised continuation utility v_t , the expected remaining time to quit is less than the one under laissez faire, $S^{L*}(v_t) > S^L(-G(v_t))$.*

(e) *Marginal utility of consumption $u_c(C_t)$ is a martingale under laissez faire; under optimal mechanism, marginal utility of consumption is the volatility of promised continuation utility, $u_c(C_t^*) = q_t^*$, and instead $G_v(v_t)$ is a martingale.*

Proof. See Appendix. ■

Proposition 6 is self-explanatory. Proposition 6(b) is intuitive, as a pension contribution mechanism can always replicate the allocation of laissez faire by doing nothing.

Proposition 6(a) means it is never optimal to retire with less pension than the one under laissez faire. Suppose not and agents retire with less pension under the optimal mechanism. At that level of pension, an agent under laissez faire can always choose to retire, but he does not. That implies agents find it is better to keep working and accumulate more saving for pension under laissez faire. But by Proposition 6(b), under an optimal mechanism, the continuation value of working is higher than the one under laissez faire, so it contradicts with the premise that agents prefer to retire with less pension.

We also find, maybe counterintuitively, the level of pension is higher under the optimal mechanism than laissez faire. To see the reason, we decompose pension accumulation along intensive margin, which is the level saving at each time, and extensive margin, which is the length of participating the mechanism. Because contribution is taxed, the intensive margin decreases as agents save less under the optimal mechanism. But the extensive margin increases as the insurance aspect of the optimal mechanism induces agents to stay longer in the mechanism and postpone their retirement. It turns out that the extensive margin dominates the intensive margin, thus agents retire with higher pension under the optimal mechanism.

The source of efficiency gain comes from insuring pension accumulation, which reflects by Proposition 6(c). In particular, recall in (41) that there is less volatility of pension under optimal mechanism. So there is less motivation to save for self-insurance, so consumption increases. A trade-off of providing insurance of pension is that it results in lower drift of pension balance than the one under laissez faire. Given the same balance, on average pension is accumulated slower under the optimal mechanism. It is because of the higher consumption thus less saving under the optimal mechanism. In sum, pension under optimal mechanism features lower drift and lower volatility.

One may note that, under the optimal pension contribution mechanism, there could be higher volatility in promised continuation utility than laissez faire. Recall under the optimal mechanism, the evolution of promised continuation utility is given by

$$\begin{aligned} dv_t^* &= r(v_t^* - \mathcal{U}^*(C(q^*(v_t^*)))) dt + r\sigma q_t^* dB_t. \\ dv_t &= r(v_t - u(C(a_t))) dt + \sigma dB_t. \end{aligned}$$

For comparison, consider the decentralization of optimal mechanism. Suppose there is wealth effect on consumption. Then by Proposition 5(d), we have $r\sigma q_t^*$ decreasing in a_t . So if $r\sigma q^*(G^{-1}(-a_{ret}^*)) < \sigma < r\sigma q^*(G^{-1}(0))$, then $r\sigma q_t^*$ will single cross σ from above when a_t is increasing. So for sufficiently high balance a_t , the optimal mechanism leads to less volatility in promised continuation utility than laissez faire, and vice versa. Then the optimal mechanism on average insures the continuation value of agents who are old or about to retire, but results in volatile continuation value of agents who are young or about to quit the mechanism, since the former is more likely to be with high balance a_t , and vice versa.

On average, agents take less time to quit under the optimal mechanism. It is because agents save less under the optimal mechanism, and volatility of pension accumulation is lower. On the other hand, one might expect we can establish that, on average agents take more time to retire under the optimal mechanism. This is not necessary. Though agents save less and also have higher level of pension target to hit under the optimal mechanism, which implies longer time to retire on average, there is an opposite force that volatility of pension accumulation is lower, which implies less time.

Proposition 6(e) states that "inverse-Euler equation" in Rogerson (1985) and Sannikov (2008) does not satisfy under optimal pension contribution mechanism. This is because of hidden consumption. Inverse- Euler equation states $-1/(ru_{c,t})$ is a martingale. In general, inverse-Euler equation does not necessarily hold, as in Williams (2010). Here under the optimal mechanism, from the first order condition of q , we have

$$G_{v,t} = \frac{-1}{ru_{c,t}} + \frac{r\sigma^2 G_{vv,t}}{\mathcal{C}_q(q_t^*)}.$$

Compared to Sannikov (2008), there is an extra term $(r\sigma)^2 G_{vv,t}/\mathcal{C}_q(q_t^*)$, which captures the cost of hidden consumption. Without this term, $-1/(ru_{c,t})$ is equal $G_{v,t}$, which is a martingale. The left hand side is the marginal revenue of promised continuation utility $G_{v,t}$, and the right hand side is its decomposition. The first term is the marginal consumption needed to fulfill an additional utility at time t . The second term is the marginal cost of inducing higher hidden consumption. The incentive compatibility of recommended consumption implies that the social planner has to reduce volatility of promised continuation utility q_t^* in order to increase hidden consumption. The marginal

revenue of q_t^* is captured by the second term.

3.8 Duality Between Optimal Mechanism and Principal-Agent Problem

A benevolent social planner might step back and ask whether the dual really optimizes agent's value. In the primal problem, the social planner maximizes agent's value, given incentive compatibility and feasibility constraints. In dual problem, the social planner maximizes revenue of pension contribution mechanism, given a level of promised lifetime utility. Usually problem is formulated in dual in principal-agent problem, as well as our context; in primal in optimal taxation problem. Though the design of optimal pension contribution mechanism is in the spirit of optimal taxation problem, we want to formulate in dual such as avoid the situation where promised continuation utility is both the objective and constraint. The following lemma provides the condition to establish duality.

Proposition 7 *Given assumptions, the dual maximizes agents' continuation value if and only if $G_v(v_t^*) < 0$.*

Proof. See Appendix. ■

An example with $G_v(v_t^*) > 0$ can be found in Sannikov (2008). There is no problem of $G_v(v_t^*) > 0$ in principal-agent problem, as principal only cares its own revenue maximization. There is problem of $G_v(v_t^*) > 0$ for a benevolent social planner. In the case $G_v(v_t^*) > 0$, the corresponding optimal public pension mechanism is not represented by a solution G to (34). In our model, $G_v(v_t^*) < 0$ is always satisfied.

4 Extended Pension Provision: Accidental Retirement

To be completed. Consider there is probability that an agent is hit by a unobservable retirement shock as in Diamond and Mirrlees (1978,1982). We call it accidental retirement, which captures the event that the agent is no longer able to work. In particular, let π denote the Poisson rate of accidental retirement. Under laissez faire, the value

function of an agent given saving a_t is

$$rV(a_t) = \max_{c_t} \left\{ ru(c_t) + V_a(a_t)(ra_t + \theta - c_t) + \frac{\sigma^2}{2}V_{aa}(a_t) + \pi(V(a_t) - U(a_t)) \right\}. \quad (46)$$

The boundary conditions and smooth-pasting are

$$V(a_{ret}) = U(a_{ret}), V(0) = V_{\min}, V_a(a_{ret}) = U_a(a_{ret}). \quad (47)$$

Under pension contribution mechanism, the evolution of promised continuation utility under incentive compatible recommendations becomes

$$dv_t = [r(v_t - \mathcal{U}(q_t)) + \pi(V_t^A - v_t)] dt + r\sigma q_t dB_t, \text{ given } v_0 > V_{\min}, v_{T_L} = V_{\min}. \quad (48)$$

The continuation revenue given promised continuation utility v is the solution solving:

$$G(v) = \max_{q, V^A \leq v} \left\{ \begin{array}{l} \theta - \mathcal{C}(q) + G_v(v)[r(v - \mathcal{U}(q)) + \pi(V^A - v)] \\ + \frac{(r\sigma q)^2}{2}G_{vv}(v) + \pi(G(v) + U^{-1}(V^A)) \end{array} \right\}, \quad (49)$$

where the value-matching conditions and smooth-pasting condition are:

$$G(V_{ret}^*) = -U^{-1}(V_{ret}^*), \quad (50)$$

$$G(V_{\min}) = 0. \quad (51)$$

$$G_v(V_{ret}^*) = \frac{-1}{U' \circ U^{-1}(V_{ret}^*)}. \quad (52)$$

So all the previous results are maintained.

5 Conclusion

To be completed.

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6 Appendix

6.1 Proof of Proposition 1(a) to 1(d):

We now guess and verify agent continuation value V_t given by (5) would be a function of saving. Consider a solution $V(a) : [a_{\min}, a_{\text{ret}}] \rightarrow \mathbb{R}$ satisfying a HJB equation stated in the proposition 1:

$$rV = \max_{c \geq 0} \{ru(c, 1) + V_a(ra + \theta - c) + \frac{\sigma^2}{2}V_{aa}\}, \quad \forall a \in [0, a_{\text{ret}}], \quad (53)$$

where $V \equiv V(a)$ and, V_a and V_{aa} denote respectively the first and second derivative of V with respect to a . The smooth-matching conditions determines a_{ret} as

$$V(a_{ret}) = U(a_{ret}), V'(a_{ret}) = U'(a_{ret}). \quad (54)$$

Another boundary condition, the value matching condition, is given by

$$V(0) = V_{\min}. \quad (55)$$

A solution V to HJB (53) must be twice-differentiable, as (53) implies V_{aa} exists. The first order condition of c is:

$$u_c(c_t) = V_a(a_t), \quad (56)$$

and zero if no such c_t exists. Consider stopping times to retire and to quit:

$$T_R = \inf\{t : a_t = a_{ret}\}. \quad (57)$$

$$T_L = \inf\{t : a_t = 0\}. \quad (58)$$

The sketch of the proof is as follows. First, in Lemma 1, given $\beta \geq 0$, we show the solution $V(a)$ to the second order ODE (53) satisfying (55) and $V_a(0) = \beta$ exists and unique. Second, a crucial step in our proof, Lemma 2 shows the set of solutions V , which are different in β only, satisfies single-cross property, that is two solutions to (53) cross at most once. An implication of Lemma 3 is that, the solution to (53) with (55) and higher β always has higher value than the one with lower β , so we can rank the set of solutions by β . Third, Lemma 3 shows any solution $V(a)$ to (53) with (55) and $V_a(0) = \beta$ is concave. This also proves the second part of (b). Forth, use this implication to show there is β such that the solution is tangential to U . Such β is the greatest out of all solutions satisfying smooth-pasting condition (54). We also show such β is strictly positive, and V_a is strictly bounded above by zero, so the solution $V(a)$ to the second order ODE (53) satisfying (54) and (55) is strictly increasing. The proof of proposition 1(a) - 1(c) verifies the solution $V(a_0)$ is a continuation value following consumption plan according to (56) and stopping time (57), and the solution $V(a_0)$ attains the highest value.

Lemma 1 *Given $\beta \geq 0$, the solution $V(a)$ to the second order ODE (53) satisfying (55) and $V_a(0) = \beta$ exists and unique.*

Proof. The proof of existence and uniqueness is to apply Picard-Lindelof theorem, which states that, if $f_i(x, y_1 \dots y_n)$ for $i \in \{1, \dots, n\}$ are continuous in x and Lipschitz continuous in $y_1, \dots, y_n$, then the system of ordinary differential equation defined as

$$\frac{dy_i}{dx} = f_i(x, y_1 \dots y_n),$$

with the initial condition $y_i(x_0) = y_{i0}$ for $i \in \{1, \dots, n\}$ has a unique solution $y_i(x)$ for $i \in \{1, \dots, n\}$.

To apply, first we set

$$\begin{aligned} \frac{dy_1}{da} &= y_2, \\ \frac{dy_2}{da} &= f(a, y_1, y_2) \equiv \frac{2}{\sigma^2} \left\{ ry_1 - \max_{c \geq 0} \{ ru(c, 1) + y_2(ra + \theta - c) \} \right\}. \end{aligned}$$

So f is Lipschitz continuous if $y_2 \geq 0$. Then the solution $y_1(a) = V(a)$ and $y_2(a) = V_a(a)$ exist given $V(0) = V_{\min}$ and $V_a(0) \geq 0$.

Lemma 2 *Given $\beta_1 > \beta_2 \geq 0$, the solution $V^i(a)$, $i \in \{1, 2\}$, to the second order ODE (53) satisfying $V^i(0) = V_{\min}$. and $V_a^i(0) = \beta_i$ only intersect each other once at $a = 0$.*

■

Proof. Consider V^1 and V^2 such that $V^1(0) = V^2(0) = V_{\min}$ but $V_a^1(0) = \beta_1 > \beta_2 = V_a^2(0) \geq 0$. By continuity of V^1 and V^2 , there is ε such that $V^1(a) > V^2(a)$ for all $a \in (0, \varepsilon)$. Suppose V^1 crosses V^2 other than $a = 0$, then we must have $V_a^1 \leq V_a^2$ at the point of intersection, and by intermediate value theorem there is a' such that $V_a^1(a') = V_a^2(a')$, and $V^1(a) > V^2(a)$ and $V_a^1(a) > V_a^2(a)$ for all $a \in (0, a')$. Then we have

$$\begin{aligned} \frac{\sigma^2}{2} V_{aa}^1(a') &= rV^1(a') - V_a^1(a')(ra' + \theta) - \max_{c \geq 0} \{ ru(c, 1) - V_a^1(a')c \}, \\ &> rV^2(a') - V_a^2(a')(ra' + \theta) - \max_{c \geq 0} \{ ru(c, 1) - V_a^2(a')c \}, \\ &= \frac{\sigma^2}{2} V_{aa}^2(a'). \end{aligned}$$

So by continuity of V_a^1 and V_a^2 , there is $\varepsilon > 0$ such that $V_a^1(a' - \varepsilon) < V_a^2(a' - \varepsilon)$, which is contradiction. Thus V^1 and V^2 have no intersection other than $a = 0$. ■

Lemma 3 *Given $\beta \geq 0$, the solution $V(a)$ to the second order ODE (53) satisfying (55) and $V_a(0) = \beta$ is either linear, strictly concave or strictly convex.*

Proof. Applying envelope theorem on (53) implies

$$0 = V_{aa}(a) [ra + \theta - c] + \frac{\sigma^2}{2} V_{aaa}(a), \quad (59)$$

so $V_{aa} = 0$ implies $V_{aaa} = 0$. Repeating envelope theorems implies any derivative with order higher than the two is zero once $V_{aa} = 0$. Suppose there exists a' such that $V_{aa}(a') = 0$. Then applying Taylor expansion around a' leads to

$$V(a) = V(a') + V_a(a')(a - a'),$$

so if such a' exists then V is linear. So V is one of the following cases: V is linear, or $V_{aa} > 0$ for all a , or $V_{aa} < 0$ for all a . In other words, V can only be linear, strictly concave or strictly convex. ■

Proof. Proposition 1(a) to 1(d). Consider $V'(a) = V_{\min} + \beta a/r$, where β solves $rV_{\min} = \beta\theta + \max_{c \geq 0} \{ru(c) - \beta c\}$, is a solution to (53) given initial conditions $V'(0) = V_{\min}$ and $V'_a(0) = \beta$. According to assumption 1, if there is a_{ret} such that $\beta = U_a(a_{ret})$ then $a = a_{ret}$ must be the unique tangential point, so (54) and (55) satisfy and we establish V' as the unique solution to HJB (53) given (54) and (55). We are done with this case.

On the other hand, suppose $V'(a) > U(a)$ for all a . By Lemma 2 and Lemma 3, we known that for any solution V to (53) given initial conditions $V(0) = V_{\min}$ and $V_a(0) > \beta$, V is strictly convex. Similarly, for any solution V to (53) given initial conditions $V(0) = V_{\min}$ and $V_a(0) < \beta$, V is strictly concave. For those convex V , we know $V \geq V' > U$ for all $a > 0$, so there is no a_{ret} satisfies (54), thus V cannot be convex and hence must be concave. Also, suppose V is the solution to (53) given initial conditions $V(0) = V_{\min}$ and $V_a(0) = 0 < \beta$, then by strict concavity, V must be decreasing, thus V must cross U . Since the proof of Picard-Lindelof theorem implies the set of solution V is continuous in the initial values (omit to proof here), then by continuity and we decrease $V_a(0)$ from β to zero, here must be a value of β' such that the corresponding solution V to (53) given initial conditions $V(0) = V_{\min}$ and

$V_a(0) = \beta'$ satisfies (54) and (55). Then β' is the maximal $V_a(0)$ over the set of solution V to (53) given initial conditions $V(0) = V_{\min}$ that satisfies (54) and (55).

Finally, to verify solution V given initial conditions $V(0) = V_{\min}$ with largest $V_a(0)$ that satisfies (54) and (55) attains the highest value, consider a derivation $c' = \{c'_t, 0 \leq t < \infty\}$ not satisfying (56) until t such that the value becomes

$$W_t \equiv r \int_0^{t \wedge T_R \wedge T_L} e^{-rs} (u(c'_s) - a) ds + e^{-rt} 1_{t < \min(T_R, T_L)} V(a_t) + e^{-rT_R} 1_{T_R \leq \min(t, T_L)} U(a_{T_R}) + e^{-rT_L} 1_{T_L < \min(t, T_R)} V_{\min},$$

where the stopping time T_R and T_L are defined in (57) and (58). By Ito's lemma, the drift of W_t is

$$re^{-rt} \left[u(c'_t) - a - V(a_t) + V_a(a_t) [ra_t + \theta - c'_t] + \frac{\sigma^2}{2} V_{aa}(a_t) \right], \text{ if } t < \min(T_R, T_L),$$

and zero otherwise. In both cases, the drift term is non-positive, since the former is:

$$\begin{aligned} & u(c'_t) - a - V(a_t) + V_a(a_t) [ra_t + \theta - c'_t] + \frac{\sigma^2}{2} V_{aa}(a_t), \\ & \leq -V(a_t) + \max_{c \geq 0} \left\{ u(c'_t) - a + V_a(a_t) [ra_t + \theta - c'_t] + \frac{\sigma^2}{2} V_{aa}(a_t) \right\}, \\ & = 0. \end{aligned}$$

Thus W_t is a bounded supermartingale. Then we have

$$\mathbb{E}W_t \leq W_0 = V(a_t).$$

Therefore $V(a_t)$ is the maximal continuation value. To verify the stopping time T_R defined in (57) is optimal, suppose not, then there is t where it is optimal the agent retires when $a_t < a_{ret}$, or the agent does not retire when $a_t = a_{ret}$. The second case can be ruled out, since $V(a_t)$ is the maximal continuation value, then if the agent retire at $a_t = a_{ret}$, he at most gets continuation value $V(a_t) = V(a_{ret}) = U(a_{ret})$, so retire at t is not dominated. On the other hand, if the agent retires when $a_t < a_{ret}$, then he would get continuation value $U(a_t)$, which is strictly less than the maximal value $V(a_t)$, as from above we know a_{ret} is the tangential point so $V(a_t) > U(a_t)$ for all $a_t \in [0, a_{ret})$. So the agent retires when $a_t < a_{ret}$ is not optimal as well. Similar argument to establish T_L defined in (58) is optimal. ■

6.2 Proof of Proposition 1(e)

Proof. Proposition 1(e). Applying Ito lemma on $V_a(a)$, then the drift of $V_a(a_t)$ is $V_{aa}(a_t)[ra_t + \theta - C(a_t)] + \frac{\sigma^2}{2}V_{aaa}(a_t)$, which is zero by applying envelope theorem on (53). So V_a is a martingale. Since a_t is always bounded between zero and a_{ret} , then $V_a(a_t)$ is also bounded from $V_a(0) > 0$ and $V_a(a_{ret}) < \bar{\beta}$. Arranging (59), the drift of saving is

$$ra_t + \theta - C(a_t) = \frac{r\sigma^2 V_{aaa}(a_t)}{-2V_{aa}(a_t)},$$

where the sign follows $V_{aaa}(a_t)$ as $V_{aa}(a_t) < 0$ by proposition 1(b). By Ito lemma, the drift of $V(a_t)$ is

$$V_a(a_t)[ra_t + \theta - C(a_t)],$$

since we have $V_a(a_t) > 0$ from proposition 1(b), the sign also follows $V_{aaa}(a_t)$. From the first order condition of consumption (56), the drift of consumption is

$$\begin{aligned} & \frac{V_{aa}(a_t)}{u_{cc}(C(a_t))}[ra_t + \theta - C(a_t)] + \frac{\sigma^2}{2} \left(\frac{-u_{ccc}(C(a_t))V_{aa}^2(a_t)}{u_{cc}^3(C(a_t))} + \frac{V_{aaa}(a_t)}{u_{cc}(C(a_t))} \right) \\ &= -\frac{\sigma^2}{2} \frac{V_{aa}^2(a_t)}{u_{cc}^3(C(a_t))} u_{ccc}(C(a_t)), \end{aligned}$$

since u is strictly concave, the sign of consumption drift follows the sign of u_{ccc} . ■

6.3 Proof of Proposition 1(f)

Proof. Proposition 1(f). Given the stopping time T_R by (57), the expected remaining time to retire with saving a_t is

$$S^R(a_t) \equiv \mathbb{E} \left\{ \int_t^{T_R} 1 ds \middle| \mathcal{F}_t^B \right\}.$$

Since agents retire once a_t hits a_{ret} , and agents never retire if a_t hits zero, the boundary conditions are

$$S^R(a_{ret}) = 0, S^R(0) = \infty.$$

The associated HJB with a as state variable is

$$0 = 1 + S_a^R(a)[ra + \theta - C(a)] + \frac{\sigma^2}{2} S_{aa}^R(a).$$

Similarly, the expected remaining time to quit with saving a_t , and boundary conditions are

$$S^L(a_t) \equiv \mathbb{E} \left\{ \int_t^{T_L} 1 ds \middle| \mathcal{F}_t^B \right\}, S^L(a_{ret}) = \infty, S^L(0) = 0.$$

The associated HJB with a as state variable is the same as

$$0 = 1 + S_a^L(a) [ra + \theta - C(a)] + \frac{\sigma^2}{2} S_{aa}^L(a).$$

■

6.4 Necessity of Smooth-Pasting Condition

Note that we always have $V_a(a_{ret}) \leq U_a(a_{ret})$, otherwise suppose $V_a(a_{ret}) > U_a(a_{ret})$, by continuity of V_a there is $\varepsilon > 0$ such that $V_a(a) > U_a(a)$ for any a within the ε -open ball centered at a_{ret} . So for any $a \in (a_{ret} - \varepsilon, a_{ret})$, we have $V(a) = V(a_{ret}) - \int_a^{a_{ret}} V_a(a) da < U(a_{ret}) - \int_a^{a_{ret}} U_a(a) da = U(a)$, which contradicts the fact that a_{ret} reflects the optimal stopping time. By similar argument we have $V_a(a_{ret}) \geq U_a(a_{ret})$, combining both cases we establish the continuously differentiable pasting condition.

6.5 Proof of Proposition 2

Proof. Proposition 2. Note we have $V^{FB}(a_0) \geq V(a_0)$. Suppose $a_{ret}^{FB} < a_{ret}$, so agents do not retire when $a_t = a_{ret}^{FB}$ under the benchmark economy, that implies $V(a_{ret}^{FB}) > U(a_{ret}^{FB})$. Since $V^{FB}(a)$ is tangential to $U(a)$, we have $V^{FB}(a) > U(a)$ for all $a > a_{ret}^{FB}$, thus $V(a_{ret}) = U(a_{ret}) < V^{FB}(a_{ret})$. So there must be $a' \in (a_{ret}^{FB}, a_{ret})$ such that $V(a') = V^{FB}(a')$. Note both $V(a)$ and $V^{FB}(a)$ satisfy (53), then by Lemma 2 there is at most one intersection, which is $a = a'$. But since $V^{FB}(a_0) \geq V(a_0)$, then there must be another $a'' \in [a_0, a_{ret}^{FB})$ such that $V(a'') = V^{FB}(a'')$, which leads to contradiction. So we always have $a_{ret}^{FB} \geq a_{ret}$.

Note consumptions are given by

$$c^{FB} = u_c^{-1} \left(\frac{U'(a_{ret}^{FB})}{r} \right) \geq u_c^{-1} \left(\frac{U'(a_{ret})}{r} \right) \geq u_c^{-1} \left(\frac{U'(a_t)}{r} \right) = C(a_t).$$

So consumption is always higher under first best. ■

6.6 Proof of Proposition 3(a)

Proof. Proposition 3(a). To recover $\mathcal{D}(A)$ under original space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$, we can apply Girsanov theorem once more on $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$. Given $(\mathcal{C}(q), Y, T_R(Y), T_L(Y)) \in \mathcal{D}(A)$ under risk-neutral space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_Y)$ with Brownian motion Y , construct the implied Brownian motion B^0 on $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_0)$ such that

$$dB_t^0 = dY_t - \left(\frac{\theta - \mathcal{C}(q_t)}{\sigma} \right) dt. \quad (60)$$

Note the effect of Y on T_R and T_L can be summarized by v' . After making substitutions, we can rewrite (23) under measure $\mathbb{P}_0$ as

$$\begin{cases} dv'_t = r[v'_t - \mathcal{U}(q_t)]dt + r\sigma q_t dB_t^0(Y), \\ v'_{T_R} = U(A(Y, T_R)), v'_{T_L} = V_{\min} \end{cases} \quad (61)$$

Then we have $(\mathcal{C}(q), Y, T_R(Y), T_L(Y)) \in \mathcal{D}(A)$ under the space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P}_0)$, where Y is given by (60), $T_R(v')$ by (24) and $T_L(v')$ by (??). Since $(\mathcal{C}(q), Y, T_R(Y), T_L(Y)) \in \mathcal{D}(A)$, we have $B^0 = B$ and $\mathbb{P}_0 = \mathbb{P}$, so $(\mathcal{C}(q), Y, T_R(Y), T_L(Y)) \in \mathcal{D}(A)$ under the original space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$ as well. Hence $(\mathcal{C}(q), T_R(v'), T_L(v'))$ is incentive compatible with respect to $A(Y, T)$ under $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$. ■

6.7 Proof of Proposition 3(b)

One can follow the elegant trick in Sannikov (2008), which uses Martingale Representation Theorem to represent continuation utility v as a solution to a stochastic differential equation, and then verify $v' = v$. As he aptly points out, generally Martingale Representation Theorem (Theorem 3.4.15 KS) may fail as it requires the filtration is big enough to include the Brownian motion and the solution v to the stochastic differential equation. So v may not exist. Such problem does not happen here as all the results are found on augmented filtration $\{\mathcal{F}_t^Y\}_{t=0}^\infty$. Since we only need to verify $v' = v$, where v is given by (15) rather than a solution to stochastic differential equation, we can provide a shorter proof instead.

Proof. Proposition 3(b). For any bounded q_t , all the relevant functions in (26) are Lipschitz continuous and satisfy linear growth condition, so the strong solution v'_t

exists. Note that (26) implies

$$\begin{aligned}
& e^{-rt} dv'_t - r e^{-rt} v'_t dt = -r e^{-rt} \mathcal{U}(q_t) dt + r \sigma e^{-rt} q_t dB_t^0, \\
\Rightarrow & \int_t^T d e^{-rs} v'_s = -r \int_t^T e^{-rs} \mathcal{U}(q_s) ds + r \sigma \int_t^T e^{-rs} q_s dB_s^0, \text{ for any } T. \\
\Rightarrow & \mathbb{E} \left\{ \begin{array}{l} -e^{-rt} v_t + e^{-rT_R} 1_{T_R \leq T_L} v'_{T_R} \\ + e^{-rT_L} 1_{T_R > T_L} v'_{T_L} \end{array} \middle| \mathcal{F}_t \right\} = -\mathbb{E} \left\{ r \int_t^{T_R \wedge T_L} e^{-rs} \mathcal{U}(q_s) ds \middle| \mathcal{F}_t \right\}, \\
\Leftrightarrow & v'_t = \mathbb{E} \left\{ \begin{array}{l} r \int_t^{T_R \wedge T_L} e \mathcal{U}(q_s) ds + e^{-r(T_R-t)} 1_{T_R \leq T_L} U(a_{T_R}) \\ + e^{-r(T_L-t)} 1_{T_R > T_L} V_{\min} \end{array} \middle| \mathcal{F}_t^Y \right\} = v_t.
\end{aligned}$$

■

After the showing v'_t is the promised continuation utility under original space $(C[0, \infty), \{\mathcal{F}_t^Y\}_{t=0}^\infty, \mathbb{P})$. We can also verify that v'_0 attains the maximum, although stochastic Pontryagin principle has guaranteed so. Suppose there is an action profile $(c'', Y'', T_R'', T_L'') \neq (\mathcal{C}(q), Y, T_R(Y), T_L(Y))$ which implies continuation value v''_0 and attains maximal instead. So we have $v''_0 > v'_0$. As in Sannikov (2008), by Martingale representation theorem, there is process q'' such that

$$\begin{aligned}
dv''_t &= (rv''_t - ru(c''_t)) dt + r \sigma q''_t dB_t, \\
&= r(v''_t - u(c''_t) - q''_t(\theta - c''_t)) dt + r \sigma q''_t dY''_t.
\end{aligned}$$

Suppose the agent follows the recommendations, then given such q'' , the evolution of promised continuation utility is

$$\begin{aligned}
dv'_t &= r[v'_t - \mathcal{U}(q''_t) - q''_t(\theta - \mathcal{C}(q''_t))] dt + r \sigma q''_t dY''_t, \\
&= \min_{c''_t} r[v'_t - u(c''_t) - q''_t(\theta - c''_t)] dt + r \sigma q''_t dY''_t, \\
&\leq r(v'_t - u(c''_t) - q''_t(\theta - c''_t)) dt + r \sigma q''_t dY''_t.
\end{aligned}$$

So the difference of v'_t and v''_t is given by

$$\begin{aligned}
d(v'_t - v''_t) &= r \left(\left[(v'_t - v''_t) + u(c''_t) - q''_t(\theta - c''_t) - \max_{c''_t} [u(c''_t) - q''_t(\theta - c''_t)] \right] \right) dt, \\
de^{-rt}(v'_t - v''_t) &= re^{-rt} \left[u(c''_t) - q''_t(\theta - c''_t) - \max_{c''_t} [u(c''_t) - q''_t(\theta - c''_t)] \right] dt \leq 0.
\end{aligned}$$

So $e^{-rt}(v'_t - v''_t)$ is a supermartingale. Since the promised continuation utility is bounded, then we have $(v'_0 - v''_0) \geq 0$, which is contradiction.

6.8 Proof of Corollary 1

Proof. Corollary 1. Note $U(0) < V_{\min}$, so v_t must pass through $V_{\min}$ first then $U(0)$. But by above the social planner recommends the agent to quit when v_t hits $V_{\min}$, so the above implies the social planner never recommends agent to retire with zero pension. Thus the social planner always release $U^{-1}(v_T(Y))$ unit of pension when the agent follow recommended stopping time to retire at $T_R(Y)$. Then it is straightforward to verify that the social planner recommendation described above, $(\mathcal{C}(q), T_R(Y), T_L(Y))$, satisfies (23) to (25), thus $(\mathcal{C}(q(Y)), T_R(Y), T_L(Y))$ is incentive compatible. ■

6.9 Proof of Proposition 4

We sketch how to modify the proof of Proposition 1(a) to 1(d) to prove Proposition 4(a) to 4(c).

Lemma 4 *Given $\beta \leq 0$, the solution $G(v)$ to the second order ODE (34) satisfying $G_v(V_{\min}) = 0$ and $G_v(V_{\min}) = \beta$ exists and unique.*

Proof. Note the solution to the following second-order ODE also solves HJB equation (34):

$$G_{vv}(v) = \min_q \left\{ \frac{rG(v) - \theta + \mathcal{C}(q) - rG_v(v)[v - \mathcal{U}(q)]}{(r\sigma q)^2/2} \right\}. \quad (62)$$

To see, let q^* denote the minimizer

$$\Leftrightarrow G_{vv}(v) \geq \theta - \mathcal{C}(q) + rG_v(v)[v - \mathcal{U}(q)] + \frac{(r\sigma q)^2}{2} G_{vv}(v), \forall q \neq q^*.$$

where the equality holds when $q = q^*$, which coincides with the HJB (34). Then use (62) and proof and Lemma 1 we can show the existence and uniqueness of solution to (62) given $G(V_{\min}) = 0$ and $G_v(V_{\min}) = \beta$. ■

Lemma 5 *Given $\beta_1 < \beta_2 \leq 0$, the solution $G^i(v)$, $i \in \{1, 2\}$, to the second order ODE (62) satisfying $G^i(V_{\min}) = 0$. and $G_v^i(V_{\min}) = \beta_i$ only intersect each other once at $v = V_{\min}$.*

Proof. Suppose not. Consider G^1 and G^2 such that $G^1(V_{\min}) = G^2(V_{\min}) = 0$ but $G_v^1(V_{\min}) = \beta_1 < \beta_2 = G_v^2(V_{\min}) \leq 0$. Then there is $v' > V_{\min}$ such that

$G_v^1(v') = G_v^2(v')$, and $G^1(v) < G^2(v)$ and $G_v^1(v) < G_v^2(v)$ for all $v \in (V_{\min}, v')$. Then we have

$$\begin{aligned} G_{vv}^1(v') &= \min_q \left\{ \frac{rG^1(v') - \theta + \mathcal{C}(q) - rG_v^1(v')[v' - \mathcal{U}(q)]}{(r\sigma q)^2/2} \right\}, \\ &< \min_q \left\{ \frac{rG^2(v') - \theta + \mathcal{C}(q) - rG_v^1(v')[v' - \mathcal{U}(q)]}{(r\sigma q)^2/2} \right\}, \\ &= \min_q \left\{ \frac{rG^2(v') - \theta + \mathcal{C}(q) - rG_v^2(v')[v' - \mathcal{U}(q)]}{(r\sigma q)^2/2} \right\}, \\ &= \frac{\sigma^2}{2} G_{vv}^2(v'). \end{aligned}$$

So by continuity of G_v^1 and G_v^2 , there is $\varepsilon > 0$ such that $G_v^1(v' - \varepsilon) > G_v^2(v' - \varepsilon)$, which is contradiction. ■

Lemma 6 *Given $\beta \leq 0$, the solution $G(v)$ to the second order ODE (62) satisfying $G_v(V_{\min}) = 0$ and $G_v(V_{\min}) = \beta$ is either linear, strictly concave or strictly convex.*

Proof. Applying envelope theorem on (62), then if $G_{vv}(v') = 0$ for some v' implies $G(v)$ is linear. So $G(v)$ is either linear, strictly concave or strictly convex. ■

Proof. Proposition 4. To show 4(a), consider $G'(v) \equiv -(v - V_{\min})/\bar{\beta}$, where $\bar{\beta}$ solves $V_{\min} = \bar{\beta}\theta + \max_{c \geq 0} \{ru(c, 1) - \bar{\beta}c\}$. Then we can verify that $G'(v)$ solves (62) with $G'(V_{\min}) = 0$. By the assumption $V_{\min} + \bar{\beta}a \geq U(a)$, we have

$$V_{\min} - \bar{\beta}G'(v) \geq U(-G'(v)) \Rightarrow G'(v) \geq -U^{-1}(G'(v)),$$

Suppose $G'(v) \geq -U^{-1}(G'(v))$ for all v . Then we decrease $G_a(V_{\min})$ from $-1/\bar{\beta}$ to negative infinite, here must be a value of β' such that the corresponding solution G to (62) given initial conditions $G(V_{\min}) = 0$ and $G_v(V_{\min}) = \beta'$ satisfies (33) and (35). Then β' is the maximal $G_v(V_{\min})$ over the set of solutions G to (62) given initial conditions $G(V_{\min}) = 0$ that satisfies (33) and (35). So we have $G_v(V_{\min}) \in (-\infty, -1/\bar{\beta})$ and $G(v)$ is strictly concave and decreasing. We can verify $G(v)$ is maximal as in the proof of Proposition 1(a) to 1(d).

To show 4(b) and 4(c), note the first order condition of q is

$$q_t^* = \mathcal{C}_q^*(q_t^*) \frac{rG_v(v_t^*)q_t^* + 1}{(r\sigma)^2 G_{vv}(v_t^*)} = u_c(\mathcal{C}_t^*), \quad (63)$$

where the last equality comes from (25). By Proposition 3 we know v is promised continuation utility, where the stopping times are given by (24).

To show 4(d), apply envelope theorem on (34):

$$v_t^* - \mathcal{U}(q_t^*) = -\frac{r(\sigma q_t^*)^2}{2} \frac{G_{vvv}(v_t^*)}{G_{vv}(v_t^*)},$$

so the drift of v_t^* points in the direction in which $G_{vv}(v_t^*)$ is increasing.

To show 4(e), note from (63), since $q_t^* = u_c(\mathcal{C}_t^*) > 0$, then we have $rG_v(v_t^*)q_t^* + 1 > 0$ as $\mathcal{C}_q^*(q_t^*) < 0$.

To show 4(f), since $G_v(v)$ is strictly decreasing in v , so if $G_v(V_{\min}) \leq -1$, then $-G_v(v_t^*) \geq 1$ for all v_t^* and hence from 4(e) we have $r q^*(v_t^*) < 1$ for all v_t^* .

To show 4(g), total differentiating both side of (63):

$$\frac{dq_t^*(v_t^*)}{dv_t^*} = \frac{r q_t^* [\mathcal{C}_{q,t} G_{vv,t} - r \sigma^2 G_{vvv,t}]}{-\mathcal{C}_{qq,t} (1 + r G_{v,t} u_{c,t}) - \mathcal{C}_{q,t}^2 r G_{v,t} u_{cc,t} + (r \sigma)^2 G_{vv,t}},$$

where the denominator is negative by the second order condition. Since G_{vv} and $\mathcal{C}_{q,t}$ are negative, the numerator is positive if and only if G_{vvv} is sufficiently positive.

It is straight forward to write HJB equation for the expected remaining time to retire and to quit, which is similar to Proposition 1(f). That establishes 4(h). ■

6.10 Proof of Proposition 5

Proof. Proposition 5. Apply Ito Lemma on $a_t = -G(v_t)$:

$$\begin{aligned} da_t &= \left[-r G_{v,t}(v_t - \mathcal{U}_t) - \frac{(r \sigma q_t)^2}{2} G_{vv,t} \right] dt - r \sigma q_t G_{v,t} dB_t. \\ &= \left[-r G_{v,t}(v_t - \mathcal{U}_t) - \frac{(r \sigma q_t)^2}{2} G_{vv,t} \right] dt - r q_t G_{v,t} [\sigma dY_t - (\theta - \mathcal{C}_t) dt]. \end{aligned}$$

Matching with the decentralization taxation:

$$da_t = r(1 - \tau_{int,t}) a_t dt + (1 - \tau_{sav,t}) \sigma dY_t.$$

Then we show 5(b)

$$\tau_{sav,t} = 1 + r q_t G_{v,t} > 0.$$

$$\begin{aligned}
ra - \tau_t &= -rG_{v,t}(v_t - \mathcal{U}_t + q_t(\theta - \mathcal{C}_t)) - \frac{(r\sigma q_t)^2}{2}G_{vv,t} \\
&= -rG_{v,t}q_t(\theta - \mathcal{C}_t) - rG_t + \theta - \mathcal{C}_t \\
\Leftrightarrow \tau_t &= (1 - \tau_{sav,t})(\theta - \mathcal{C}_t).
\end{aligned}$$

Substituting τ_t , the evolution of saving can be reduced to

$$da_t = (ra_t + \theta - \tau_{sav,t}\mathcal{C}_t - (1 - \tau_{sav,t})c_t)dt + (1 - \tau_{sav,t})\sigma dB_t.$$

The ex ante saving is

$$\mathbb{E}da_t = \left[-rG_{v,t}(v_t - \mathcal{U}_t) - \frac{(r\sigma q_t)^2}{2}G_{vv,t} \right] dt = [ra_t + \theta - \mathcal{C}_t] dt,$$

which proves 5(c)

To show 5(a) and verify $V(a_t) = G^{-1}(-a_t)$ under the saving tax $\tau_{sav,t}$ and lump-sum tax $\tau_{int,t}$, consider the HJB of continuation value:

$$\begin{aligned}
rV(a) &= \max_c \left\{ ru(c) + V_a(a)[ra_t + \theta - (1 - \tau_{sav,t})c - \tau_{sav,t}\mathcal{C}_t] + \frac{(1 - \tau_{sav,t})^2\sigma^2}{2}V_{aa}(a) \right\}, \\
&= V_a(a)[ra_t + \theta + (rq_tG_{v,t} - 1)\mathcal{C}_t] \\
&\quad + \frac{(rq_tG_{v,t}\sigma)^2}{2}V_{aa}(a) + r \max_c \{u(c) - q_tV_a(a)G_{v,t}c\}.
\end{aligned} \tag{64}$$

Substitute

$$V_a = \frac{-1}{G_v}, V_{aa} = \frac{G_{vv}}{(G_v)^3}, a = -G$$

then the right hand side of (64) becomes

$$\frac{-1}{G_{v,t}} \left[-rG_t + \theta + (rq_tG_{v,t} - 1)\mathcal{C}_t + \frac{(rq_t\sigma)^2}{2}G_{vv,t} + rG_{v,t} \max_c \{u(c) + q_tc\} \right].$$

Since we have $q_t = u_c(\mathcal{C}_t)$, so we have

$$c_t = \mathcal{C}_t.$$

Hence the above can be further reduced to

$$\frac{-1}{G_{v,t}} \left[-rG_t + \theta - \mathcal{C}_t - \frac{(rq_t\sigma)^2}{2}G_{vv,t} + rG_{v,t}u(\mathcal{C}_t) \right] = rv,$$

where the equality follows the definition of HJB (34). So the right hand side (64) matches the left hand side under $V(a_t) = G^{-1}(-a_t)$, thus the saving tax $\tau_{sav,t}$ and interest tax $\tau_{int,t}$ decentralize the allocation of the optimal mechanism.

To show 5(d), since $\mathcal{C}(G^{-1}(-a_t))$ is increasing in a_t , then there exists a' such that for all $a_t > a'$ we have $C(G^{-1}(-a_t)) > \theta$ and vice versa. This implies $\tau_t = (1 - \tau_{sav,t})(\theta - C(G^{-1}(-a_t))) < 0$ if $a_t > a'$ and vice versa. On the other hand, $\mathcal{C}(G^{-1}(-a_t))$ is increasing in a_t implies $q^*(G^{-1}(-a_t))$ is decreasing in a_t , by (25), so we have

$$\frac{d\tau_{sav,t}}{da_t} = -r \left[q^*(G^{-1}(-a_t)) \frac{G_{vv}(G^{-1}(-a_t))}{G_v(G^{-1}(-a_t))} + G_v(G^{-1}(-a_t)) \frac{dq^*(G^{-1}(-a_t))}{da_t} \right] < 0.$$

■

6.11 Proof of Proposition 6:

Proposition 6 is the various implications of the following Lemma, which allows us to compare $G(v)$ and $-V^{-1}(v)$. This is straight forward to verify so we just state without showing the steps

Lemma 7 $G'(v) \equiv -V^{-1}(v)$ solves

$$rG'(v) = \theta - \mathcal{C}(q) + rG'_v(v)[v - \mathcal{U}(q)] + \frac{(r\sigma q)^2}{2}G'_{vv}(v), \quad (65)$$

for $q = V_a(a)/r$ and given $-V^{-1}(V_{\min}) = 0$.

Proof. Denote $v = V(-G'(v))$, and $a = G'(v)$, then the left hand side of (65) is $-ra$, and the right hand side is

$$\begin{aligned} & \frac{1}{V_a(a)} \left[V_a(a) [\theta - \mathcal{C}(q)] - r[v - \mathcal{U}(q)] + \frac{V_{aa}(a)}{2} \left(\frac{r\sigma q}{V_a(a)} \right)^2 \right], \\ &= -ra + \frac{1}{V_a(a)} \left[-rv + \max_c \left\{ ru(c) + V_a(a)[ra + \theta - c] + \frac{\sigma^2}{2} V_{aa}(a) \right\} \right. \\ & \quad \left. - \frac{\sigma^2}{2} V_{aa}(a) + \frac{V_{aa}(a)}{2} \left(\frac{r\sigma q}{V_a(a)} \right)^2 \right], \\ &= -ra + \frac{\sigma^2}{2} V_{aa}(a) \left[\left(\frac{rq}{V_a(a)} \right)^2 - 1 \right], \\ &= -ra. \end{aligned}$$

■

Consider the formulation (62) of (34). The following lemma establishes the single-crossing of $G(v)$ and $-V^{-1}(v)$.

Lemma 8 $G(v) \geq G'(v) \equiv -V^{-1}(v)$ and only intersect each other once at $v = V_{\min}$.

Proof. Suppose not, and $G_v(V_{\min}) \leq G'_v(V_{\min})$. Then there is $v' > V_{\min}$ such that there exists $\varepsilon > 0$ where $G(v') = G'(v')$, $G(v) < G'(v)$ and $G_v(v) > G'_v(v)$ for all $v \in (v' - \varepsilon, v')$. Then we have

$$\begin{aligned} G_{vv}(v') &= \min_q \left\{ \frac{rG(v') - \theta + \mathcal{C}(q) - rG_v(v')[v' - \mathcal{U}(q)]}{(r\sigma q)^2/2} \right\}, \\ &< \frac{rG'_v(v') - \theta + \mathcal{C}(V_a(G'(v'))/r) - rG'_v(v')[v' - \mathcal{U}(V_a(G'(v'))/r)]}{(\sigma V_a(G'(v')))^2/2}, \\ &= G'_{vv}(v'). \end{aligned}$$

So by continuity of G_v and G'_v , there is $\varepsilon' > 0$ such that $G_v(v' - \varepsilon') > G'_v(v' - \varepsilon')$, which is contradiction. So either we have $G_v(V_{\min}) > G'_v(V_{\min})$, or $G_v(V_{\min}) \leq G'_v(V_{\min})$ and there is no intersection other than $v = V_{\min}$.

Then we want to show the case $G_v(V_{\min}) \leq G'_v(V_{\min})$ is impossible. Suppose that, since there is no intersection between $G(v)$ and $G'(v)$, so we have $G(v_0) < G'(v_0)$, where v_0 is given by Proposition 4(b). But that implies G is not optimal as it is dominated by G' . So we must have $G_v(V_{\min}) > G'_v(V_{\min})$.

Finally, we want to show there is no intersection given $G_v(V_{\min}) > G'_v(V_{\min})$. Suppose not, then there is $v' > V_{\min}$ such that $G(v') = G'(v')$, and $G_v(v') < G'_v(v')$, as the order of slopes must be alternating for each intersection. By repeating the above proof but substitute $V_{\min}$ by v' , then we show such v' does not exist. So there must be no intersection. ■

The above lemma implies Proposition 6(b). To show 6(a), note that at $v_t = V_{ret}$, agents retire under laissez faire, we have $-U^{-1}(V_{ret}) = -a_{ret} = -V^{-1}(V_{ret})$. Since by the above lemma, we have $-V^{-1}(V_{ret}) < G(V_{ret})$, so agents under optimal mechanism do not retire at $v_t = V_{ret}$, hence $V_{ret}^* > V_{ret}$ and $-U^{-1}(V_{ret}^*) < -U^{-1}(V_{ret})$, so $a_{ret}^* > a_{ret}$.

To show 6(c), we compare agent's first order conditions under laissez faire and optimal mechanism respectively. Under laissez faire, the first order condition is

$u_c(C_t) = V_a(-G(v_t)) = -1/rG_v(v_t)$; under optimal mechanism, the first order condition is $u_c(C(v_t)) = q_t^*$. Since from Proposition 4(e) we know $q_t^* < -1/rG_v(v_t)$, so by the concavity of u we establish consumption is higher under optimal mechanism.

To show 6(d), note that the expected remaining time to quit under laissez faire $S^L(v) = S^L(-G(v))$, with $-G(v) = a$, solves the following HJB:

$$\begin{aligned} -1 &= S_v^L(v) \left[V_a(a) [\theta - C(a)] + \frac{\sigma^2}{2} V_{aa}(a) \right] + \frac{(\sigma V_a(a))^2}{2} S_{vv}^L(v) \\ &= r S_v^L(v) (v - u(C(a))) + \frac{(\sigma V_a(a))^2}{2} S_{vv}^L(v), \end{aligned}$$

where the boundary conditions are $S^L(0) = 0, S^L(V_{ret}) = \infty$. Rearrange the above and substituting $-G(v) = a$ and $V_a(-G(v)) = -1/rG_v(v)$, we have

$$\frac{S_{vv}^R(v)}{2} = (G_v(v))^2 \frac{r S_v^R(v) (u(C(-G(v))) - v) + 1}{\sigma^2}.$$

Recall the expected remaining time to quit under the optimal mechanism is

$$\frac{S_{vv}^{L*}(v)}{2} = (G_v(v))^2 \frac{r S_v^{L*}(v) (\mathcal{U}(q^*(v)) - v) + 1}{(\sigma r q^*(v) G_v(v))^2}.$$

We want to show a stronger result. Consider $0 < S_v^{L*}(v) < S_v^L(v)$ for all $v \in [V_{\min}, V_{ret}]$. Both $S_v^{L*}(v)$ and $S_v^L(v)$ are positive as it takes more time to quit if v is closer to V_{ret}^* , which is the maximal promised continuation utility as long as agents remain in the mechanism or the saving market. Suppose not, and consider $S_v^{L*}(V_{\min}) < S_v^L(V_{\min})$. By continuity of $S_v^{L*}(v)$ and $S_v^L(v)$ imply there is v' such that $S_v^{L*}(v') = S_v^L(v')$ and $S_v^{L*}(v) > S_v^L(v)$ for all $v \in (V_{\min}, v']$. So we have

$$\begin{aligned} \frac{S_{vv}^{L*}(v')}{2} &= (G_v(v))^2 \frac{r S_v^{L*}(v') (\mathcal{U}(q^*(v)) - v) + 1}{(\sigma r q^*(v) G_v(v))^2}, \\ &> (G_v(v))^2 \frac{r S_v^L(v') (u(C(-G(v))) - v) + 1}{\sigma^2}, \\ &= \frac{S_{vv}^L(v)}{2}. \end{aligned}$$

since from Proposition 4(e) we have $r q^*(v) G_v(v) < 1$, and from Proposition 6(c) we have $\mathcal{U}(q^*(v)) > u(C(-G(v)))$. So by continuity of $S_v^{L*}(v)$ and $S_v^L(v)$, there is $\varepsilon' > 0$ such that $S_v^{L*}(v' - \varepsilon') < S_v^L(v' - \varepsilon')$, which is contradiction. So if $S_v^{L*}(V_{\min}) < S_v^L(V_{\min})$, then we have $S_v^{L*}(v) < S_v^L(v)$ for all $v \in [V_{\min}, V_{ret}]$.

Suppose $S_v^{L*}(V_{\min}) \geq S_v^L(V_{\min})$. Since $S_v^L(V_{ret}) = \infty > S_v^{L*}(V_{ret})$, there is a v' such that $S_v^{L*}(v) < S_v^L(v)$ for all $v \in (v', V_{ret}]$ and $S_v^{L*}(v') = S_v^L(v')$. We want to show there is no intersection of $S_v^{L*}(v)$ and $S_v^L(v)$ other than $v = v'$. Suppose not, then there is $v'' < v'$ such that $S_v^{L*}(v) > S_v^L(v)$ for all $v \in (v'', v']$ and $S_v^{L*}(v'') = S_v^L(v'')$. Repeating the above proof we can show such v'' leads to contradiction, thus does not exist. Then we must have $v' = V_{\min}$, hence $0 < S_v^{L*}(v) < S_v^L(v)$ for all $v \in [V_{\min}, V_{ret}]$. Therefor we have $S^{L*}(v) < S^L(v)$ for all $v \in [V_{\min}, V_{ret}]$.

The comparison between optimal mechanism and the first best follows similar proof, so we omit here.

6.12 Proof of Proposition 7:

Proof. With abuse of notation, let denote the promised continuation utility supported by a level of revenue, ie $V(a_t) \equiv G^{-1}(-a_t)$. Suppose $G_v(V(a_t)) < 0$ for some a_t . Note dividing both side of (34) by $-G_v(V(a_t))$ implies:

$$\begin{aligned} -\frac{G(V(a_t))}{G_v(V(a_t))} &= \max_q \left\{ -\frac{\theta - \mathcal{C}(q)}{G_v(V(a_t))} - V(a_t) + \mathcal{U}(q) - \frac{r\sigma^2}{2} \frac{G_{vv}(V(a_t))}{G_v(V(a_t))} q^2 \right\}, \\ \Leftrightarrow V(a_t) &= \max_q \left\{ \mathcal{U}(q) + \frac{-G(V(a_t)) + \theta - \mathcal{C}(q)}{-G_v(V(a_t))} - \frac{r\sigma^2}{2} \frac{G_{vv}(V(a_t))}{(G_v(V(a_t)))^3} (-G_v(V(a_t)) q)^2 \right\}, \\ \Leftrightarrow V(a_t) &= \max_q \left\{ \mathcal{U}(q) + V_a(a_t) [-G(V(a_t)) + \theta - \mathcal{C}(q)] + \frac{r\sigma^2}{2} V_{aa}(a_t) (-G_v(V(a_t)) q)^2 \right\}, \end{aligned}$$

This is a value maximization problem of (64) given $a_0 = -G_t$. Note the direction from value maximization problem to revenue maximization problem is always established as $V_a > 0$. ■