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Power and core-periphery networks

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Abstract

Social and industrial networks frequently demonstrate a core-periphery structure where the set of nodes can be partitioned into a clique ("core") and an empty network ("periphery"). A possible explanation for the formation of this structure is provided by introducing type heterogeneity into the basic connections model. A two-type society is "power based" if for any given path length, both types prefer to connect to the "superior" type over the "inferior" type. Core-periphery networks, in which the "superior" agents are in the core and the "inferior" agents are in the periphery, emerge as a dominant architecture (pairwise stable and strongly efficient) in a "power based" society as long as the linking costs are intermediate. Thus, in this framework, the network formation process converts an exogenous, intellectual advantage into a central position in the network.

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1 Introduction

A social or industrial network obtains a core-periphery structure if its set of agents can be partitioned into two subsets, the core and the periphery, such that each agent in the core is directly connected to all other core members, while each periphery member is directly connected to none of the other periphery agents[1]. Since the 1970's, the empirical literature identifies the core-periphery architecture as a dominant structure in many contexts. White et al. (1976) mention the "hangers-on" architecture, a core-periphery network that consists of some bidirectional relations between core and periphery agents, as one of two "substantively important" architectures that occurred frequently in their analysis of various social networks (p. 742). Similar findings were mentioned in relation to industrial networks (e.g. Mintz and Schwartz (1981) and Baker et al. (2008)), scientific networks (e.g. Mullins et al. (1977), Goyal et al. (2006) and van der Leij and Goyal (2009)) and drug users networks (e.g. Curtis et al. (1995)).

The normative literature finds no consensus as to the social favorability of core-periphery networks. On the one hand, it is agreed that this architecture is an efficient spreader of knowledge because of its low average path length (e.g. Borgatti (2005))[2]. On the other hand, most researchers disapprove of the ability of

[1]The core-periphery structure is not a well-defined concept in the literature. Generally, it describes a network in which there is one group of agents that is densely connected internally, while all the other agents are sparsely connected among themselves (Borgatti and Everett, 1999). Our definition is the one used by Bramoullé and Kranton (2005) and Bramoullé (2007). Other definitions use the term "core-periphery networks" for a subset of these networks by adding a restriction on the pattern of links between the core agents and the periphery agents (e.g. Galeotti and Goyal (2009) and Goyal (2007)). In addition, some architectures that are mentioned in the literature, could be considered as special cases of core-periphery networks (e.g. the "dominant group" in Goyal and Joshi (2003)).

[2]Dodds et al. (2003) claim that core-periphery networks are robust to congestion due to the decentralization among the core members.

the central agents to control messages and ideas introduced by peripheral agents[3].

A common characteristic to many real-life core-periphery networks, is that the core members possess some intrinsic advantage over the members of the periphery. However, being advantageous is not enough for a subset of agents to constitute the core. For a core-periphery architecture to evolve, the advantage of the core members must lie in an attribute that affects the linking behavior of all the agents in the society. This observation is articulated very clearly by Mullins et al. (1977) in relation to scientific networks:

> "Scientific specialties typically display a social structure that ... can be described as a "center-periphery" pattern ... . The center is a subset of members *who are recognized by others* as being central to the specialty. These scientists are *more productive* than their peers, and among them are those who made the original discoveries that are the intellectual basis of research in the specialty." (p. 556, emphasis added)[4].

Another observation is that in a core-periphery network the location of the core members is preferable to that of the periphery agents. Kadushin (2002), for example, refers to the positional advantage of the core members by stating that:

> "Not only are they able to take advantage of gaps in the connections within the less powerful periphery, but they are also mutually related to other powerful central actors." (p. 85)[5].

The combination of these two simple observations suggests that core members are "powerful". Not only do they possess an advantage in a highly valued attribute,

[3]See the discussion of Chubin (1976) and Borgatti (2005) on inefficiencies inherent to a core-periphery structure of scientific networks. Bramoullé and Kranton (2005) demonstrate one type of such inefficiency by showing that agents in the core experiment less than others since they have greater access to other agents' information. Other forms of inefficiencies are discussed in Krackhardt and Hanson (1993) and Dodds et al. (2003) in reference to the internal organization of a firm. Cummings and Cross (2003) find partial evidence for negative correlation between core-periphery structure and performance in small teams.

[4]A similar observation can be found in Kadushin (2004) (Proposition 10). Other examples to attributes that may affect the linking behavior of all the agents are demographics (e.g. Fershtman and Gneezy (2001)) and personal appearance.

[5]See also footnote 3 in Hojman and Szeidl (2008).

but they are able to translate this advantage into a positional advantage in the social network. Characterizing an environment that allows for such a "powerful" subgroup of agents to emerge, constitutes the motivation for this paper. Identifying conditions that enable the formation of an elite that includes the gifted agents bears important consequences on issues as inequality and mobility in the social context, diffusion of ideas in the academic context and power distribution in the industrial organization context.

We generalize the homogeneous connections model of Jackson and Wolinsky (1996) by introducing two types of agents into the framework. The benefit of an agent from a connection (direct or indirect) depends on the types of the two end agents and the minimal distance between them. Only direct connections are costly (the linking costs are homogeneous). Thus, in this general model, as opposed to the homogeneous model, two agents may enjoy different benefits from the same path if they are of different types. Moreover, two agents may induce different benefits on the same third agent, even if their connection to this agent is identical.

Assume there are two types of agents in the society. The society is "power-based" if, for any given path length, every agent prefers to connect to agents of one type over agents of the other type. We show that in a "power-based" society, core-periphery networks where the preferred agents are in the core while the other agents populate the periphery, are the dominant architectures both as stable networks and as efficient networks. This suggests that the existence of a single criterion with a unanimously accepted ranking that motivates the linking preferences, offers a possible explanation for the emergence of core-periphery networks.

To interpret the model, consider a set of agents that form a social network. Periodically, one of the agents comes up with an innovative idea. Other agents get the information regarding this new idea through the network, with a delay that increases with their distance to the source. Only upon arrival can they exploit the new idea. Assume that in this society there are two types of agents - “superior” and “inferior”, such that the probability that a “superior” agent will come up with a new idea is higher than the probability that an “inferior” agent will do so. Moreover, “superior” agents are more able than “inferior” agents in exploiting new ideas. In any other respect, such as information transmission speed or direct linking costs, the two types are identical.

The utility of an agent decreases with the delay of being informed about the innovation of her counterpart. Since all the links are assumed to convey information with the same speed, this delay can be represented by the geodesic distance between the two agents. The utility of an agent from being connected to a “superior” agent is higher than his utility from being connected to an “inferior” agent since the probability that this path will yield an idea is higher. In addition, the benefit from any given path is higher for “superior” agents than for “inferior” agents, since they have higher ability to exploit an idea, if one will be provided by this path. Thus, the value of a path for a given agent depends on his own execution skills and on the innovative qualities of the other end agent. Innovation and execution skills are indistinguishable and the “superior” type enjoys an advantage in both.

Our analysis suggests that (for intermediate linking costs), core-periphery networks where the “superior” agents form the core and the “inferior” agents form the periphery are both probable and favorable. Thus, the “superior” agents, in addition to their exogenous superiority, are also located in a position that enables them to get the innovative information faster than the “inferior” agents.

For a brief overview of the literature, consider an architecture to be a non-degenerate core-periphery network if it is a core-periphery network and it can accommodate simultaneously at least 4 agents in the core (to differentiate from a cyclic core) and at least 2 agents in the periphery (to differentiate from the complete network)[6]. While many architectures that are shown to be dominant in the strategic network formation literature are degenerate forms of the core-periphery architecture[7], non-degenerate core-periphery networks are rarely found to be either stable or efficient. One exception is Galeotti and Goyal (2009). In their model, an agent can either acquire information personally or gather information from agents that acquired it personally. However, in the version of their model in which core-periphery networks are dominant, there is no use of indirect connections, while in our model, these paths are crucial. In addition, in their model, agents are homogeneous and the partition to core and periphery is equilibrium-specific. In our model, on the other hand, the agents are heterogeneous and the partition of agents to core and periphery is equilibrium-independent.

This paper also contributes to the literature on the role of heterogeneity in the formation of social networks. One way of introducing heterogeneity is by conditioning the linking costs on the geographic distance between the agents[8]. Fewer models introduce type heterogeneity (sometimes called value heterogeneity). Hojman and Szeidl (2006) characterize the conditions under which a socially "gifted" agent becomes the center of a stable star architecture. Zeggelink (1995) introduces a dynamic network formation model with two types of agents. In this model, the agent's utility is maximized if she has an "ideal" number of friends, all of them are similar to her. The extended preferential attachment model of Bianconi and

[6]Galeotti (2006) have stable networks in which a subset of agents forms a wheel, while some of them serve also as centers of star structures ("wheel with local centered sponsored stars"). Similar architecture might be weakly stable in the insiders-outsiders model of Galeotti et al. (2006) without decay.

[7]One obvious example is the star network which is found to be pairwise stable and efficient both in undirected networks formation models (e.g. the symmetric connections model of Jackson and Wolinsky (1996)) and in directed formation models (e.g. Hojman and Szeidl (2008)). Another example is the interlinked stars architecture (e.g. the two-way flow model with decay in Bala and Goyal (2000)).

[8]See Johnson and Gilles (2000), Carayol and Roux (2003) and Jackson and Rogers (2005) for models that introduce geography into the connections model of Jackson and Wolinsky (1996). See Galeotti et al. (2006) and Hojman and Szeidl (2006) for models that introduce geography into the one-side two-way model Bala and Goyal (2000).

Barabási (2001) introduces individual fitness into the standard preferential attachment model such that higher fitness leads to faster links accumulation.
The crucial difference between these two approaches is that the linking costs heterogeneity is relevant only to direct connections, while the intrinsic values heterogeneity is carried through both direct and indirect connections. This property induces different preferences over network structures[9]. Galeotti (2006) and Galeotti et al. (2006) introduce heterogeneity both in the costs and in the benefits of the network formation model of Bala and Goyal (2000). Our results, however, are not directly comparable to theirs since we analyze an undirected network formation model with two types while they analyze a directed network formation model with infinite number of types.

2 The Model

Preliminaries

Let $N = \{1, 2, \cdots, n\}$ be a finite set of utility-maximizing agents. Denote the set of all subsets of N of size two by $g^N = \{\{i, j\} | i, j \in N, i \neq j\}$. The set of all possible networks on N is $\{g | g \subseteq g^N\}$. We will say that g^N is the *complete network* and that the empty set is the *empty network*.

[9]For example, in the connections model setting with two types of agents, consider the preferences of an agent of type a in a cost heterogeneity framework versus his preferences in a value heterogeneity framework. Assume that in both frameworks he will prefer a direct link with another type a agent over a direct link with a type b agent. Define an *a-star* to be a star in which the center is of type a while the leafs are of type b and define a *b-star* to be a star in which the center is of type b while the leafs are of type a. Further assume that the agent considers a direct connection to the center of the stars and that she has no other path to any of the agents in the stars. In the cost heterogeneity framework, she will prefer to connect to the a-star over the b-star for any size of network since his benefits are the same and his costs are lower. In the value heterogeneity framework, there is a critical size, above which the agent will prefer the b-star, since the costs are the same, but a large b-star guarantees a path of length 2 to many type a agents, while the a-star guarantees a path to only one such agent.

Denote by ij the element of g^N that contains i and j. If $ij \in g$ we say that agents i and j are *directly connected* in network g. Denote by $N(i,g) = \{j | ij \in g\}$ the set of agent i's *neighbors* in network g and denote its cardinality by n_i. Let $g + ij$ denote the network obtained by adding the link ij to the network g and let $g - ij$ denote the network obtained by severing the link ij from the network g.

A *path* p of *length* $L(p)$ between agents i and j exists in network g, if there is a set of distinct agents $\{i_1, i_2, \ldots, i_{L(p)}, i_{L(p)+1}\}$ such that $\{i_1i_2, i_2i_3, \ldots, i_{L(p)}i_{L(p)+1}\} \subseteq g$ and $i_1 = i, i_{L(p)+1} = j$.

p is a *shortest path* between agent i and agent j in network g, if there is no other path p' between them such that $L(p') < L(p)$. Denote the set of all shortest paths between agent i and agent j in network g by $S(i,j,g)$, denote its cardinality by s_{ij} and denote the shortest path's length by d_{ij}.

If a path between agent i and agent j exists in network g, we say that agent i and agent j are *connected* in network g. Otherwise, we say that agent i and agent j are *disconnected* in network g and we set d_{ij} to infinity. If agent i and agent j are connected but not directly connected in network g, we say that agent i and agent j are *indirectly connected* in network g. If for each pair of agents $i, j \in N$, agent i and agent j are connected in g, we say that g is *connected*. For a subset of agents, $N' \subseteq N$, define the *subnetwork* $g' = \{ij | i, j \in N', ij \in g\}$. A subnetwork g' is a *component* of network g if it is connected and there is no pair of agents $i \in N'$ and $k \in N \backslash N'$ such that $ik \in g$. Let $\tilde{N}(i,g)$ denote the set of agents that reside in the same component as agent i in network g.

A network g is a *star* network if g is connected and $\exists i \in N, \forall kj \in g : i \in \{k, j\}$. A network g is a *core-periphery* network if there is a partition of N into two subsets K, the *core*, and L, the *periphery* ($K \cup L = N, K \cap L = \emptyset$), such that $\forall i, j \in K : ij \in g$ while $\forall l, m \in L : lm \notin g$. Various classes of core-periphery networks can be characterized by the pattern of the direct connections between the core members and the periphery members. For every periphery member, $l \in L$, define his *local core* as $LC_l = \{k|kl \in g, k \in K\}$ and denote its size by lc_l. For every core member, $k \in K$, define his *local periphery* as $LP_k = \{l|kl \in g, l \in L\}$ and denote its size by lp_k.

A network g is a *disconnected core-periphery* network if g is a core-periphery network and $\forall l \in L : lc_l = 0$. A network g is a *maximally connected core-periphery* network if g is a core-periphery network and $\forall l \in L : LC_l = K$[10]. A network g is a *minimally connected core-periphery* network if g is a core-periphery network and $\forall l \in L : lc_l = 1$. A network g is a *one-gate minimally connected core-periphery* network if g is a minimally connected core-periphery network and $\exists k \in K : LP_k = L$. Agent k will be called the *gate*. Note that the disconnected core-periphery network, the maximally connected core-periphery network and the one-gate minimally connected core-periphery network are unique in the unlabeled set of networks (given the partition of N to K and L). Also note that this classification is not exhaustive, and there are many core-periphery networks that belong to none of these classes. See Figure 1 for a visual demonstration of the core-periphery networks' classification.

The heterogeneous symmetric connections model

The heterogeneous symmetric connections model with no side payments allows for two types of agents, n^a type a agents and n^b type b agents ($n^a + n^b = n$) such that $n^a \geq 1$ and $n^b \geq 1$.

[10]Every maximally connected core-periphery network with $|K|$ core members and $|L|$ periphery members, can be identified also as a core-periphery network with $|K| + 1$ core members and $|L| - 1$ periphery members where one of the core members is disconnected from all the periphery members. This ambiguity exists for every core-periphery network where at least one of the periphery members have direct connections to all the core members. Fortunately, this ambiguity bears no consequence on the following analysis.

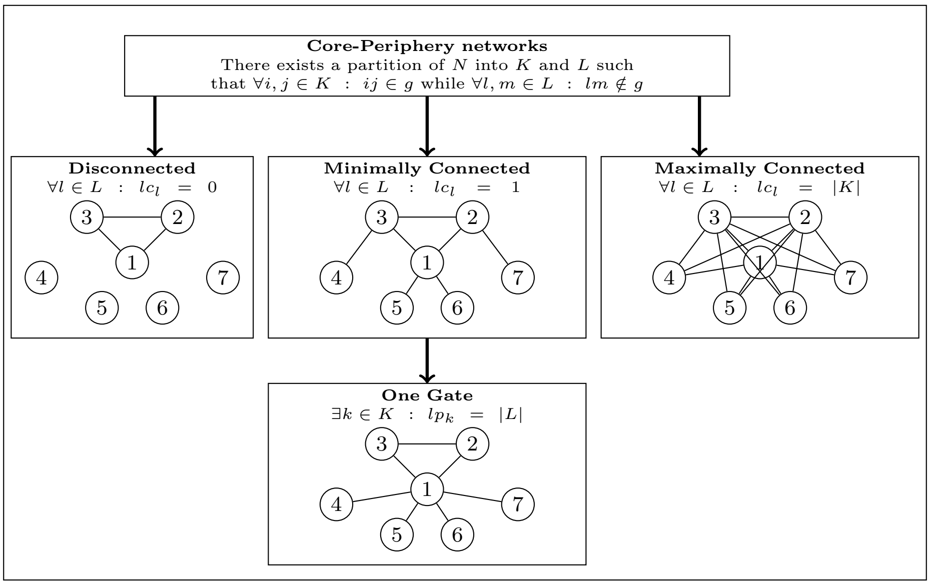

Figure 1: The classification of core-periphery networks. Agents $1, 2$ and 3 belong to the core while the others belong to the periphery. The network depicted in the Minimally Connected section is only a representative of this class. Also, note that agent 1 is the gate in One Gate network.

The utility of agent i, of type $t_i \in \{a, b\}$, from network g is

$$u_i(g) = \sum_{j \neq i} [\delta^{d_{ij}} \times f(t_i, t_j)] - n_i \times c \tag{1}$$

where $0 < \delta < 1$ captures the idea that the value that agent i derives from being connected to agent j is proportional to their proximity, $f(t_i, t_j)$ captures the *intrinsic value* that this agent provides her[11] and $c > 0$ is the universal direct connection costs. The intrinsic value is a discrete function of the types of both end agents:

$$f(t_i, t_j) = \begin{cases} w_1 & \text{if } \ t_i = t_j = a \\ w_2 & \text{if } \ t_i \neq t_j \\ w_3 & \text{if } \ t_i = t_j = b \end{cases} \tag{2}$$

[11] "Intrinsic value" is the term used by Jackson and Wolinsky (1996) while describing the general connections model.

In this paper we concentrate on the case where $w_1 > w_2 > w_3 > 0$ for all the agents. In this case, both types prefer a connection to an agent of type a over a connection of the same length to an agent of type b. We say that linking preferences with such values of the intrinsic value function are "power based" linking preferences[12]. We assume that the intrinsic values are the same for all the agents. Therefore, we say that the society is "power based".

Definition 1. *The linking preferences represented by Eq. 1 and Eq. 2 are called "power based" linking preferences if* $w_1 > w_2 > w_3 > 0$.

Definition 2. *A society is called "power based" society if all of its agents follow the "power based" linking preferences represented by* $w_1 > w_2 > w_3 > 0$.

Recall the interpretation of the model. The "superior" agents are more innovative than the "inferior" agents and therefore both types prefer linking to a "superior" agent over an "inferior" agent ($f(a, a) > f(a, b)$ and $f(b, a) > f(b, b)$). In addition, "superior" agents execute better than the "inferior" agents ($f(a, a) > f(b, a)$ and $f(a, b) > f(b, b)$).

Four remarks should be made regarding the preferences of the agents as reflected in Equations 1 and 2. First, type a is the preferred type for linking for exogenous reasons, and in particular, for reasons which are independent from the network structure. Second, a major simplification is achieved by ignoring the types of the mediators in the shortest paths. The complexity induced by taking account of the mediators, arises mainly from the need to redefine the "shortest path" concept, in a very similar manner to the redefinition required for the analysis of weighted networks. Third, the symmetry of the intrinsic function ($f(a, b) = f(b, a)$) is not essential in most cases. In the subsequent analysis it will turn out that, in most

[12] In the same spirit, one can call the linking preferences "homophilic" if $w_1, w_3 > w_2$ or "heterophilic" if $w_2 > w_1, w_3$. Jackson and Wolinsky (1996) use "homogeneous" linking preferences ($w_1 = w_2 = w_3 = 1$).

cases, the type a agents decide whether links will form between agents of different types. This is a direct result of the required mutual consent in link formation (that will be stated formally shortly). Therefore, the results hold if $f(b,a) > f(a,b)$. Things might change if $f(a,b) - f(b,a)$ is positive and high enough to reverse the dominance of the type a agents (the critical difference increases with n^a). We do not attend this case here. Last, for simplicity, we use $\delta^{d_{ij}}$ as the distance motive in the utility function. Any general "distance based" measure that qualifies for the analysis of "distance based" utility functions in Propositions 6.1 and 6.2 of Jackson (2008) fits in our model as well.

Additional assumptions

The linking preferences described above are bidimensional. The first dimension is the distance. If an agent (i) with "power based" linking preferences agrees to connect to another agent, she will surely agree to connect to a more distant agent, all else is equal (including the other agent's type). The second dimension is the type. If an agent with a "power based" linking preferences agrees to connect to a type b agent she will surely agree to connect to a type a agent, all else is equal (including the distance of the other agent). Thus, if agent i agrees to connect directly to a type b agent who is otherwise d_b links away, she surely agrees to connect directly to a type a agent who is otherwise $d_a \geq d_b$ links away, all else is equal. However, her preferences are not clear if $d_b > d_a$.

The following two assumptions (one for each type) expand the "power based" linking preferences so that in some cases, linking to a distant low type could be compared to linking to a close high type.

Assumption 1. $(\delta - \delta^2)w_1 > \delta w_2$

Assumption 2. $(\delta - \delta^2)w_2 > (\delta - \delta^3)w_3$

Assumption 1 states that if a type a agent agrees to connect to a type b agent to whom she has no alternative path, she will surely agree to connect to a type a agent to whom she has an alternative path of length two, other things being equal. Assumption 2 is weaker and it states that if a type b agent agrees to connect to a type b agent to whom she has an alternative path of length three, she will surely agree to connect to a type a agent to whom she has an alternative path of length two, other things being equal[13]. It is straight forward to see that the additional structure provided by these two assumption is not sufficient to turn the "power based" linking preference into a complete preference relation. Note that these assumptions can be viewed also as restrictions on either the intrinsic values function[14] or the depreciation rate[15].

We will use these assumptions to define two new sets of linking preferences.

Definition 3. *"Power based" linking preferences are called "strong power based" linking preferences if they satisfy Assumptions 1 and 2. A society where all the agents hold such preferences is called "strong power based" society.*

Definition 4. *"Power based" linking preferences are called "partially strong power based" linking preferences if they satisfy Assumption 1. A society where all the agents hold such preferences is called "partially strong power based" society.*

Last, let us introduce a stronger version of Assumption 2, which is symmetric to Assumption 1,

[13]An alternative interpretation of these assumption can be driven from writing Assumption 1 as $\delta w_1 > \sum_{k=1}^{\infty} \delta^k w_2$ and Assumption 2 as $\delta w_2 > \sum_{k=1}^{2} \delta^k w_3$. Thus, Assumption 1 states that if a type a agent agrees to connect to an infinite line of type b agents to whom she has no alternative path, she will surely agree to connect to a single type a agent to whom she has no alternative path, other things being equal. Assumption 2 states that if a type b agent agrees to connect to a connected pair of type b agents to whom she has no alternative path, she will surely agree to connect to a single type a agent to whom she has no alternative path, other things being equal. This interpretation is specific to the distance function chosen here ($\delta^{d_{ij}}$).

[14]Assumption 1 argues that $\frac{w_1}{w_2} > \frac{1}{1-\delta}$ while Assumption 2 states that $\frac{w_2}{w_3} > 1 + \delta$.

[15]For both assumptions to hold simultaneously, the depreciation rate parameter should satisfy $0 < \delta < \min\{1 - \frac{w_2}{w_1}, \frac{w_2}{w_3} - 1\}$.

Assumption 2*. $(\delta - \delta^2)w_2 > \delta w_3$

This assumption states that if a type b agent agrees to connect to a type b agent to whom she has no alternative path, she will surely agree to connect to a type a agent to whom she has an alternative path of length two, other things being equal. We will use this assumption for the following definition.

Definition 5. *"Power based" linking preferences are called "ultra strong power based" linking preferences if they satisfy Assumptions 1 and 2*. A society where all the agents hold such preferences is called "ultra strong power based" society.*

Stability and Efficiency

A network g is *pairwise stable* if the following conditions hold:

$$\forall ij \in g : u_i(g) \geq u_i(g - ij), u_j(g) \geq u_j(g - ij) \quad (3)$$

$$\forall ij \notin g : u_i(g + ij) > u_i(g) \Rightarrow u_j(g + ij) < u_j(g) \quad (4)$$

Thus, in a pairwise stable network, for every existing link, both agents would not gain by severing it (Condition 3) and for every missing link, either at least one of its agents strictly loses from forming it or both agents do not gain from forming it (Condition 4)[16].

The *value* of a network g is the sum of the utilities of its agents, $v(g) = \sum_{i \in N} u_i(g)$.

Network g is *strongly efficient* if $\forall g' \subseteq g^N : v(g) \geq v(g')$.

In the following analysis we will show that core-periphery networks where the core includes all the type a agents and the periphery includes all the type b agents, are

[16]Myerson (1991) proposes a normal form game of network formation in which agents simultaneously announce all the links they wish to form. The resulting network is formed by the mutually announced links. A Nash equilibrium outcome where no mutually beneficial links are left aside is called a pairwise Nash network. A simple generalization of Claim 1 in Calvó-Armengol and İlkılıç (2009) shows that the pairwise stability solution concept and the pairwise Nash solution concept coincide in our model.

the dominant architecture in the heterogeneous connections model with "power-based" linking preferences. Therefore, a useful notation is:

Notation. *An "AB core-periphery" network is a core-periphery network in which all the core agents are of type a and all the periphery agents are of type b (similar notations are used for the various classes of the core-periphery structure mentioned above).*

3 Results

This section starts with a brief description of the pairwise stable and strongly efficient architectures in the homogeneous symmetric connections model as analyzed by Jackson and Wolinsky (1996). To analyze our heterogeneous model (i.e. characterizing the stable and efficient networks), we fix the depreciation rate (δ) and the intrinsic value function (w_1, w_2 and w_3) and gradually increase the linking costs (c). The results of the homogeneous model are used as a baseline that assists in clarifying the role of heterogeneity in our model.

The homogeneous symmetric connections model

Propositions 1 and 2 in Jackson and Wolinsky (1996) characterize the pairwise stable and strongly efficient architectures in the homogeneous symmetric connections model. When costs are very low ($c < \delta - \delta^2$, Area A in Figure 2), the unique pairwise stable network and the unique strongly efficient network is the complete network. When the costs are intermediate ($\delta - \delta^2 < c < \delta$, Area B in Figure 2), the star network is pairwise stable (but not unique) and the unique strongly efficient network. When the costs are high ($\delta < c < \delta + \frac{n-2}{2}\delta^2$, Area C in Figure 2), the empty network is pairwise stable (any pairwise stable network has "no loose ends"), while the star network is the unique strongly efficient network (but, obviously, not

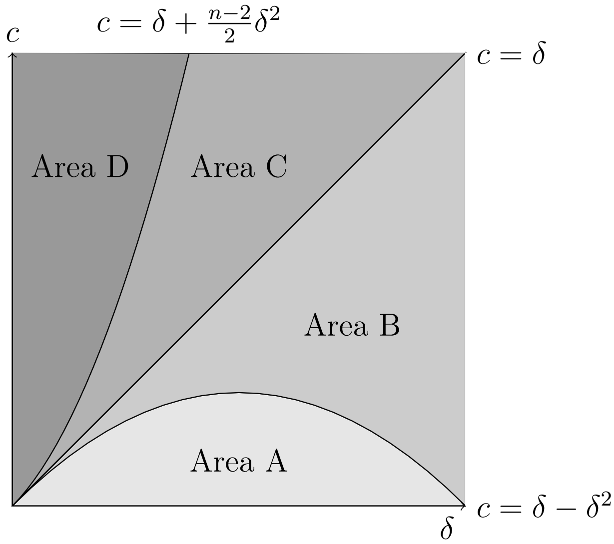

Figure 2: Stability and efficiency in the homogeneous symmetric connections model of Jackson and Wolinsky (1996). The depreciation rate values (δ) are on the horizontal axis while the linking costs are presented on the vertical axis (bounded by $c = 1$). Note that the only line that depends on the number of agents is the one that divides areas C and D (here $n = 10$).

pairwise stable). Last, when the costs are extremely high ($\delta + \frac{n-2}{2}\delta^2 < c$, Area D in Figure2), the empty network is the unique pairwise stable network and the unique strongly efficient network[17].

Extremely low linking costs

Proposition 1 shows that when the linking costs are extremely low, the complete network will emerge both as the predicted outcome and as the favorable outcome.

Proposition 1. *Let $(\delta - \delta^2)w_3 > c$. In a "power based" society, the complete network is the unique pairwise stable network and the unique efficient network.*

The proof is trivial and, as all other proofs, is relegated to the appendix. The

[17] The pairwise stability uniqueness of the empty network in the extremely high linking costs range is not stated by Jackson and Wolinsky (1996). However, it could be easily shown by Lemma 6 in the Appendix. It can also be shown by Claim 1 in Calvó-Armengol and İlkılıç (2009) or alternatively by Theorem 1 in Buechel and Hellmann (2009).

complete network is stable since the linking costs are low enough for every pair of agents to prefer a costly direct connection over a free indirect connection. Since the model is of positive externalities[18], the complete network is also uniquely efficient. Proposition 1 serves as a baseline for the following results by showing that when the linking costs are very low, the social structure does not reflect any social heterogeneity[19].

Low linking costs

Next, we analyze the case where the linking costs are low, but high enough for a direct connection between type b agents not to be worthwhile, provided that the pair have an alternative path of length two between them and that this link does not shorten any of their other paths. These costs are still low in the sense that direct connections that involve at least one type a agent are worthwhile even if the pair shares an alternative path of length two. Proposition 2 characterizes the predicted and the socially favorable outcomes in this range of linking costs.

Proposition 2. *Let $(\delta - \delta^2)w_2 > c > (\delta - \delta^2)w_3$. In a "power based" society, the AB maximally connected core-periphery network is the unique pairwise stable network and the unique efficient network.*

The AB maximally connected core-periphery network (see Figure 1) is pairwise stable and efficient since the linking costs are low enough to preserve every link that yields a sufficient level of intrinsic utility, namely, every link that involves at least one type a agent. A pair of type b agents, on the other hand, have an alternative path of length two (through a type a agent). Under these linking costs, and since type a agents connect to all agents, such an indirect path is more beneficial

[18]See Lemma 1 in the Appendix. We say that a model exhibits positive externalities if $\forall g, \forall ij \notin g, \forall k \in N\backslash\{i,j\} : u_k(g + ij) \geq u_k(g)$.

[19]This observation stems from the positivity of the intrinsic value function. Therefore, this observation holds also for other cases, specifically the ones described in footnote 12.

than maintaining a costly direct link. Proposition 2 reflects a transformation of the advantage of the type a agents, from an exogenous advantage to a positional advantage. The advantageous social position of these agents stems both from their multiple connections to other advantageous agents and from serving as bridges for the type b agents (see above for the quote from Kadushin (2002)).

Medium linking costs

The next case exhibits a range of linking costs in which the worthiness of a direct connection between a type a agent and a type b agent depends on their alternative indirect path. In this range, if the agents share a path of length two, a direct connection between them is not worthwhile, provided that the link does not shorten any of their paths to other agents. On the other hand, if they do not share any alternative path a direct link is worthwhile. Proposition 3 uses the additional structure on the linking preferences of the agents in order to obtain a characterization of the predicted and the socially favorable outcomes in this range of linking costs.

Proposition 3. *Let $\delta w_2 > c > (\delta - \delta^2)w_2$. In a "strong power based" society,*

1. *Every AB minimally connected core-periphery network is pairwise stable.*
2. *Other members of the set of pairwise stable networks are connected networks in which all type a agents are directly connected to each other and no type b agent is directly linked to more than one type a agent.*
3. *The AB one-gate minimally connected core-periphery network is uniquely efficient.*

The dominance of the AB minimally connected core-periphery networks (see Figure 1) in this linking costs range is a result of three motives coming together.

First, Assumption 1 guarantees that type a agents have strong enough preferences towards their own type to ensure that they are not satisfied with indirect paths to other type a agents, even very short ones. Note that the attractiveness of a type a agent is an increasing function of the size of her local periphery[20]. Assumption 1 is strong enough to turn even the type a agents who have an empty local periphery to be attractive enough for their fellow type a agents.

Second, Assumption 2 guarantees that type b agents have weak enough preferences towards their own type to ensure that they are satisfied with an indirect path of length three to other type b agents. Thus, there is no need for type b agents to be directly connected among themselves. It should be mentioned, however, that Assumption 2 is not required to obtain the efficiency result. The reason is that the social interest leads to indirect paths of length two between type b agents. These paths are better than direct links, due to the linking costs range alone. In addition, it is easy to see that in a "partially strong power based" the AB one-gate minimally connected core-periphery network (see Figure 1) would be the uniquely efficient network and the only pairwise stable AB core-periphery network.

Third, the linking costs range guarantees that, in a pairwise stable network, no type b agent is left isolated. However, none of them maintains more than one direct link to a type a agent. When there are no links between type b agents, type a agents are the ones who decide whether a link between them and a type b agent will be formed[21]. This results from the inability of type b agents to supply additional short paths besides the one to themselves. Since type a agents are com-

[20]This sort of attractiveness in a minimally connected core-periphery network, could also be measured by most of the centrality measures (e.g. degree centrality, closeness centrality and betweeness centrality). The higher the centrality, the more attractive is the agent.

[21]This may be viewed as a specific demonstration of the "Principle of Least Interest" that is usually attributed to Waller's study of the dating scene among college students in the 1930's (For an overview see Sprecher et al. (2006) and for the original see Waller (1938)). The principal states that the person that dictates the conditions of association is the person whose interest in this association is the least.

pletely connected among themselves and they are satisfied with a path of length two to type b agents, they are interested in connecting directly only to otherwise isolated type b agents.

The efficiency result resembles the efficiency of the star in the homogeneous symmetric connections model of Jackson and Wolinsky (1996). The efficiency stems from the direct connections between the type a agents, the minimal costs that the society bears to keep the type b agents connected and the short paths, of length two at most, to and within type b agents.

Note that the AB one-gate minimally connected core-periphery network is, obviously, Pareto efficient, but there are many other Pareto efficient networks. For example, any AB minimally connected core-periphery network is Pareto efficient since type a agents always prefer an indirect connection to type b agents over a direct connection, but they are willing to suffer the costs of a direct connection to a type b agent over having no path to her at all.

Proposition 3 exhibits the first case of tension between probable and favorable networks. Although this tension can be mitigated by a central planner, since the favorable network is also probable, it demonstrates clearly two distinct types of inefficiency. One type of inefficiency is mis-coordination where the agents fail to nominate a "gate". The other type is non optimal connectivity, which is demonstrated by the over-connectivity of the pairwise stable non core-periphery networks that are characterized in the following remark[22]:

Remark 1. *Let $\delta w_2 > c > (\delta - \delta^2)w_2$. Let g be a pairwise stable network in an "ultra strong power based" society. Any type b agent who is not directly linked to any type a agent must maintain at least two links (must not be a "loose end").*

[22]Buechel and Hellmann (2009) do not regard these networks as over-connected since no improvement to social welfare could result from severing any subset of their links (Lemma 1 and Theorem 1). Their notion may be viewed as local over-connectivity, while we use the global over-connectivity notion of Jackson and Wolinsky (1996) which relates to the number of links in the network.

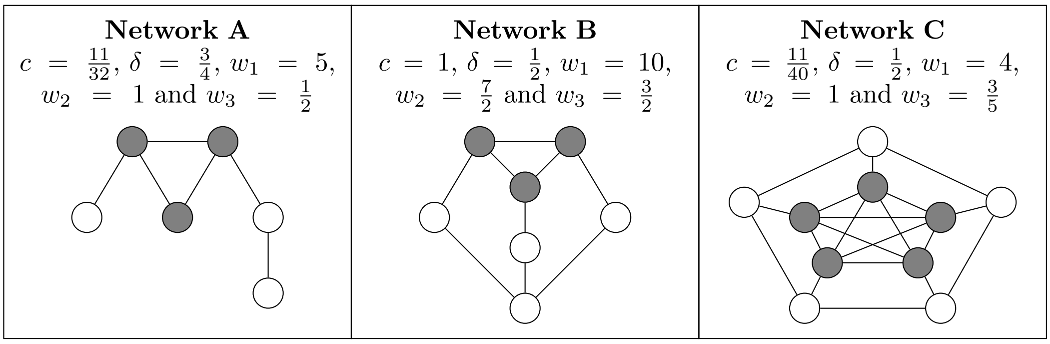

Figure 3: Three examples of pairwise stable non core-periphery networks (black circles stand for type a agents and white circles stand for type b agents). In each example, below the title, a set of parameters for which the network is pairwise stable is specified.

To conclude the analysis of this linking costs range, Figure 3 demonstrates three examples of non core-periphery networks which are pairwise stable. Network A is pairwise stable under the "strong power based" linking preferences but not under "ultra strong power based" linking preferences since, by Remark 1 it has a type b agent who is not connected to a type a agent and maintains only one link. Note that Network A is inefficient but has the same number of links as the efficient network. Thus, the source of its inefficiency is similar to that of the AB minimally connected core-periphery networks, a coordination failure. Network B is pairwise stable under "ultra strong power based" linking preferences, and, indeed, it has no "loose ends". Network B is inefficient due to over-connectivity in addition to a coordination failure. Last, Network C is pairwise stable under the "strong power based" linking preferences but not under "ultra strong power based" linking preferences. Note that Network C demonstrates an extreme case of over-connectedness since each type b agent maintains three links in comparison to the single link he maintains in the efficient network. The spread of the type b agents between the various type a agents (extreme coordination failure) causes the internal links between type b agents (over-connectivity) to be worthwhile, since it makes the indirect paths between type b agents (through the type a agents) too long. Thus, the type b agents compensate by creating an independent circle of links[23].

High linking costs

Next we analyze the case where a direct connection between a type a agent and a type b agent is not worthwhile even if this link is the only path between them, provided that they do not supply each other with additional shorter paths. On the other hand, a direct connection between a pair of type a agents is worthwhile even if otherwise they have a path of length two between them and they do not supply each other with additional benefits. Such a range of linking costs exists if and only if Assumption 1 is satisfied.

Proposition 4 requires an additional notation. Let g be the AB one-gate minimally connected core-periphery network and let g' be the AB disconnected core-periphery network (see Figure 1).

Notation. *Let Q denote the additional total utility from an AB one-gate minimally connected core-periphery network relative to AB disconnected core-periphery network per additional payment.*

$$Q = \frac{v(g) - v(g')}{2n^b} = \delta w_2 + (n^a - 1)\delta^2 w_2 + \frac{n^b - 1}{2}\delta^2 w_3 - c \tag{5}$$

Intuitively, Q is the net social return from connecting all the type b agents into the central component of the network. If Q is positive it is beneficial for the whole society to incorporate the "weak" agents into the central component and otherwise it is not[24].

Note that, if $n \geq 3$, the high linking costs range surely contains an interval in

[23]This compensation is not socially damaging since by Theorem 1 (and Lemma 1) in Buechel and Hellmann (2009) the total value of Network C is at least as high as the total value of the AB minimally connected core-periphery network that results from severing this circle.

[24]$Q \geq (\delta w_2 - c)$ and therefore Q is always positive in the medium and low linking costs ranges.

which Q is positive. An interval where Q is negative may also be contained in this linking costs range. Since Q decreases in the linking costs such an interval will always lie on the higher segment of the linking costs range. Proposition 4 will attend both the case of positive Q and the case of negative Q.

Proposition 4. *Let $(\delta - \delta^2)w_1 > c > \delta w_2$. Given "partially strong power based" linking preferences,*

1. *if $Q > 0$*
   - *(a) The AB disconnected core-periphery network is pairwise stable.*
   - *(b) Other members of the set of pairwise stable networks are non core-periphery networks in which all type a agents are directly connected to each other and every type b agent is either isolated or maintains at least two links*[25]*.*
   - *(c) The AB one-gate minimally connected core-periphery network is uniquely efficient.*
2. *if $Q < 0$, the AB disconnected core-periphery network is the unique pairwise stable network and the unique efficient network.*

The proposition is a consequence of the reluctance of all agents to connect to type b agents that do not supply additional value, and of the willingness of type a agents to connect to their own type even if such additional value is not provided. For the efficiency results we show that either AB one-gate minimally connected

[25]By Theorem 1 and Lemma 1 of Buechel and Hellmann (2009) no pairwise stable network is a super-network of the AB one-gate minimally connected core-periphery network (network g is a super-network of network g' if $g \supseteq g'$). In addition, we conjecture that in a setting of "strong power based" linking preferences, the AB disconnected core-periphery network is the unique pairwise stable network.

core-periphery network or the AB disconnected core-periphery network are the strongly efficient networks. The sign of Q determines between them. Note that the possibility that disconnected networks will achieve efficiency and stability is introduced here for the first time, due to the reluctance of all agents to connect directly to isolated type b agents.

The stability uniqueness of the AB disconnected core-periphery network when Q is negative, follows from the following. It is unstable for a pair of type a agents not to be directly connected. In addition, in a pairwise stable network all agents have non-negative utility (see Lemma 6). Thus, every pairwise network must have at least as high total value as the AB disconnected core-periphery network. As a consequence of the unique efficiency of the AB disconnected core-periphery network when $Q < 0$, it is also uniquely pairwise stable in this case.

While it is natural, by definition, that the efficiency result depends on the sign of Q, it is quiet surprising that the stability uniqueness of the AB disconnected core-periphery network depends also on this magnitude. Thus, the sign of Q serves as an indicator for the existence of tension between the favorable and probable networks. If Q is negative such tension does not exist, while if Q is positive, severe tension exists since the socially favorable network is not pairwise stable. This tension results from the failure of agents to internalize the positive externalities embedded in the model[26].

An important feature of Proposition 4 is its dependence on the community size and composition. The reason is that Q is an increasing function of both the number of type a agents and the number of type b agents. The greater the

[26]The heterogeneity that is introduced in this specific version of the connections model is not the "force" behind the Q. A similar Q can be defined in the homogeneous model of Jackson and Wolinsky (1996) as the additional total utility per payment from the star network relative to the empty payment, $Q^{JW} = \delta + \frac{n-2}{2}\delta^2 - c$. If Q^{JW} is negative the empty network is the unique pairwise stable and unique efficient network, while if it is positive the empty network is inefficient and it is a non-unique pairwise stable network.

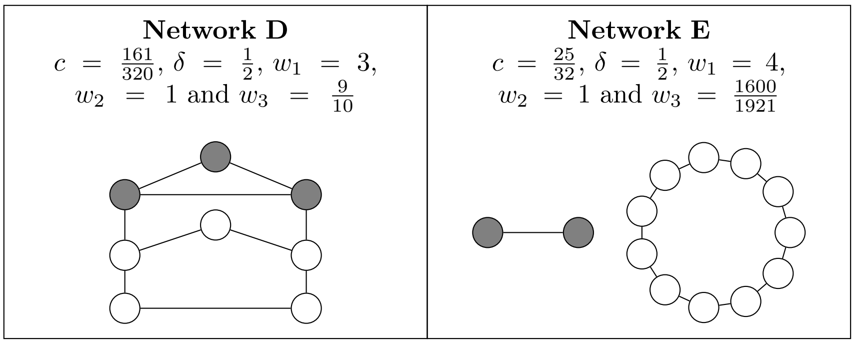

Figure 4: Two examples of pairwise stable non core-periphery networks (black circles stand for type a agents and white circles stand for type b agents). In each example, below the title, a set of parameters for which the network is pairwise stable is specified.

population, the higher is the social loss from isolated type b agents. As a result, the larger the population the higher are the linking costs required to prevent efficiency-stability tension. This feature appears also in the homogeneous model (see Figure 2). However, heterogeneity introduces a new dimension to this issue since Q is more sensitive to the number of type a agents than to the number of type b agents ($\frac{\partial Q}{\partial n^a} > \frac{\partial Q}{\partial n^b}$).

Figure 4 demonstrates two examples of non core-periphery networks which are pairwise stable for the case of a positive Q. Network D is over-connected in the sense that it includes two links more than the efficient network. Two type a agents are willing, in this range of parameters, to connect directly to type b agents since each such connection supplies them with short paths to additional two agents. Moreover, the type b agents that are not directly connected to the type a agents satisfy also the "no loose ends" requirement, since otherwise their fellow type b agents would refuse to maintain a link with them. Although inefficient, Network D satisfies the social preferences (at least those reflected by Q) by its connectivity. Network E, on the other hand, is a pairwise stable and disconnected network. Thus, Network E reflects a very unfavorable social equilibrium - not only it is inefficient due to maintaining more links than necessary, it also does not provide a path for conveying information between the two types. In this set of parameters, we have a rare example of type b agents who refuse to connect to type a agents while the type a agents wish to establish such links. The reason is that while the linking costs are very high, the return to a type b agent from linking directly to a type a agent is just one direct connection and one indirect connection. The return to a type a agent from a direct link to a type b agent is, on the other hand, one direct connection and 10 indirect paths.

This example also demonstrates that segregation must not always stem from homophilic linking preferences. By our interpretation of the model, the "inferior" types are not willing to connect to the "superior" types. The probability that the very few gifted agents will come up with a new idea is low. Thus, by taking into account the "inferiors" poor "execution" ability, the costs of maintaining those links are too high, and segregation emerges.

Extremely high linking costs

To conclude, we will analyze briefly the linking costs range in which a direct connection between a pair of type a agents is not worthwhile if they have an alternative path of length two between them and by linking directly they do not shorten each other's paths to other agents.

Proposition 5. *Let $c > (\delta - \delta^2)w_1$. Given "partially strong power based" linking preferences and $n^a \geq 3$, AB core-periphery networks are neither pairwise stable nor strongly efficient.*

Proposition 5 establishes that AB core-periphery networks are not dominant when the linking costs are very high. The main motivation for this result is that the high linking costs drive type a agents to be satisfied with indirect paths to other type a agents that do not supply additional short paths. The combination of the linking costs range and Assumption 1 guarantees that type a agents decline any direct link to an otherwise isolated type b agent that does not provide additional benefits. Therefore, in any AB core-periphery network type a agents are disconnected from type b agents and offer no additional benefits to their direct neighbors. From the social point of view, the inefficiency of core-periphery networks is a result of the favorability of the star architecture over the complete network architecture for type a agents in a setting with very high linking costs.

The heterogeneous symmetric connections model - summary

Propositions 1 to 5 are summarized in Figure 5. When costs are extremely low (Area A in Figure 5), the unique pairwise stable network and the unique strongly efficient network is the complete network. When the costs are low (Area B in Figure 5), the AB maximally connected core-periphery network is the unique pairwise stable network and the unique strongly efficient network. When the costs are intermediate (Area C in Figure 5), in a "strong power based" society, any AB minimally connected core-periphery network is pairwise stable (but not unique) and the AB one-gate minimally connected core-periphery network is the unique strongly efficient network. When the costs are high, in a "partially strong power based" society, the results depend on the sign of Q. If Q is positive (Area D in Figure 5), a major tension between the probable and the favorable arises. The AB disconnected core-periphery network is the only pairwise stable AB core-periphery network while the AB one-gate minimally connected core-periphery network is the unique strongly efficient network. Thus, it is efficient for the society to be connected while if an AB core-periphery architecture emerges the type b agents are left out. On the other hand, If Q is negative (Area E in Figure 5), no such tension arises, since it is efficient for the society to be disconnected. The AB disconnected core-periphery network is the unique pairwise stable network and the

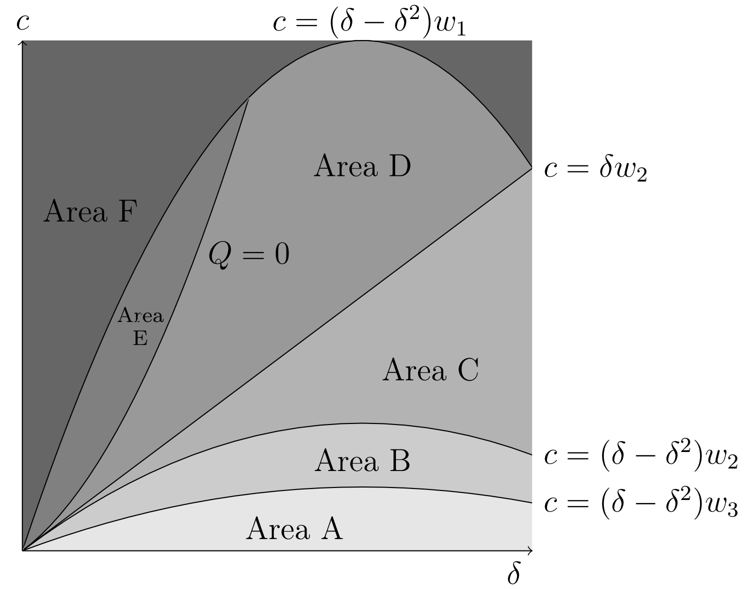

Figure 5: Stability and efficiency in the heterogeneous symmetric connections model. The intrinsic values are $w_1 = 4$, $w_2 = 1$ and $w_3 = \frac{1}{2}$. For these values, "strong power based" linking preferences are satisfied for $\delta \in (0, \frac{3}{4})$. The depreciation rate values lie on the horizontal axis (bounded by $\delta = \frac{3}{4}$) while the linking costs are presented on the vertical axis (bounded by $c = 1$). Note that the only line that depends on the number of agents is $Q = 0$ (in this figure $n^a = 5$, $n^b = 5$).

unique strongly efficient network. When the costs are extremely high (Area F in Figure 5), in a "partially strong power based" society, AB core-periphery networks are neither pairwise stable nor strongly efficient.

4 Conclusions

The motivation for this paper is derived from the combination of two observations regarding core-periphery networks. First, core members possess an intrinsic advantage over the members of the periphery. Second, the location of the core members is preferable to that of the periphery agents. Therefore, this architecture grants the core members with a mechanism to translate their intrinsic advantage into a positional advantage in the social network.

We provide a detailed characterization of an environment in which these networks may arise. If a set of agents includes "superior" and "inferior" individuals, and if their linking preferences depends on this classification then core-periphery networks in which all the "superior" agents are in the core and all the "inferior" agents are in the periphery may arise if the linking costs are intermediate. Thus, these linking preferences reinforce the advantage of the "superior" agents by placing them in the central positions of the social network which may provide them with the ability to dominate the flow of information. We view the characterization provided here as a clarification of one aspect of power in networks.

In studying real-life societies many researchers point out the existence of complex network structures that resemble the core-periphery architecture (see Borgatti and Everett (1999) and Holme (2005)). Extending the current model by introducing a slightly richer heterogeneous setting may provide a similar explanation to their emergence. One such case is a three-layer core-periphery networks in which there is a mediating layer between the core and the periphery, usually referred to as "semi periphery"[27]. The agents in this mediating layer act as hubs for peripheral agents and thus supply the periphery agents with an access to the core. The core agents regard these mediators as attractive because a direct link to them provides many indirect paths to otherwise disconnected peripheral agents. One strategy to model the emergence of these networks is by adding a third type of agents which are ranked, by all the agents, as being less attractive than the "superior" agents but more attractive than the "inferior" ones. Another strategy may be to introduce some geography to the linking costs, possibly in the flavor of the "islands model" of Jackson and Rogers (2005). A third strategy may be weakening Assumption 1 so that the type a agents would not necessarily be completely connected. In this case it might be that type a agents will form the core, type b will form the periphery

[27] See, for example, Mintz and Schwartz (1981), Baker et al. (2008), Goyal et al. (2006), Mahutga (2006) and Cattani and Ferriani (2008).

and agents of both types will form the “semi periphery”.
Another such case of a complex structure that resembles the core-periphery architecture, may be the emergence of networks with one core and multiple peripheries. In these networks the core is internally completely connected and also connected to several distinct groups. These groups are disconnected among themselves and have various patterns of internal connectivity. To consider a model that may produce such networks as stable and efficient, recall our interpretation of the current model. In this story, “superior” agents are better than the “inferior” agents, both in innovation and in execution of new ideas. Now, suppose that innovative ability and execution ability are separate mental abilities. Each agent may be high or low type in her innovative ability and high or low type in her execution ability. Plausible linking preferences should reflect the greater demand for innovation by better “executors”. In this framework, it seems that there should be a linking costs range where, while better “executors” who are also better “innovators” connect to all other agents, other mixed links do not form. However, the mixed types (high in one characteristic and low in the other) should be well connected among themselves while the “weak” types should be isolated.
Exploring these, and other cases of complex core-periphery-like architectures, will strengthen the main motivation of this paper, which is that certain linking preferences serve as a natural tool for deepening inequality, by granting an additional positional advantage to the already exogenously privileged.

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Appendix

Lemma 1. $\forall g, \forall ij \notin g : [u_i(g+ij) - u_i(g)] + [u_j(g+ij) - u_j(g)] \leq v(g+ij) - v(g)$

Proof. By Equation 1 the heterogeneous symmetric connections model exhibits positive externalities[28], meaning, $\forall g, \forall ij \notin g, \forall k \in N \backslash \{i,j\} : u_k(g+ij) \geq u_k(g)$. Therefore,

$$\sum_{k \in N \backslash \{i,j\}} u_k(g+ij) \geq \sum_{k \in N \backslash \{i,j\}} u_k(g)$$

By the definition of the value of a network:

$$v(g) = u_i(g) + u_j(g) + \sum_{k \in N \backslash \{i,j\}} u_k(g)$$
$$v(g+ij) = u_i(g+ij) + u_j(g+ij) + \sum_{k \in N \backslash \{i,j\}} u_k(g+ij)$$

By subtracting the first from the second and introducing the inequality, the lemma is proved. ∎

Lemma 2. *Let $c > (\delta - \delta^2)w_2$. In a "power based" society, the AB one-gate minimally connected core-periphery network has higher total utility than any other connected network in which each type a agent is connected to all other type a agents.*

Proof. In every connected network with n^a type a agents and n^b type b agents, there are $\frac{n^a(n^a-1)}{2}$ paths between type a agents, $n^a n^b$ paths between type a and type b agents and $\frac{n^b(n^b-1)}{2}$ paths between type b agents.
The total value of each network is the sum of the net values of its shortest paths. Since each type a agent is connected to all other type a agents, the total value of

[28] Similar definition and intuition can be found in Buechel and Hellmann (2009).

the network with the highest total value includes $\frac{n^a(n^a-1)}{2}(2\delta w_1 - 2c)$. Denote the number of direct links between a type a and a type b agents by M_1. Denote the number of direct links between type b agents by M_2. By connectivity it must be that $M_1 + M_2 \geq n^b$. Thus, there are $n^a n^b - M_1$ indirect paths between a type a agent and a type b agent, and $\frac{n^b(n^b-1)}{2} - M_2$ indirect paths between pairs of type b agents.

Now, we can write the maximal total value of a network that belongs to the set of connected networks where each type a agent is connected to all other type a agents. The maximal total value is achieved if all the indirect paths are of length two. Thus, the maximal total value is:

$$\begin{aligned} V_{MX} = \quad & \frac{n^a(n^a-1)}{2}(2\delta w_1 - 2c) \quad + \\ & M_1(2\delta w_2 - 2c) \quad + \quad 2\delta^2 w_2(n^a n^b - M_1) \quad + \\ & M_2(2\delta w_3 - 2c) \quad + \quad 2\delta^2 w_3(\frac{n^b(n^b-1)}{2} - M_2) \end{aligned} \tag{6}$$

The value of the AB one-gate minimally connected core-periphery network is:

$$\begin{aligned} V_{OG} = \quad & \frac{n^a(n^a-1)}{2}(2\delta w_1 - 2c) \quad + \\ & n^b(2\delta w_2 - 2c) \quad + \quad 2\delta^2 w_2(n^a - 1)n^b \quad + \\ & 2\delta^2 w_3(\frac{n^b(n^b-1)}{2}) \end{aligned} \tag{7}$$

The difference between the maximal value and the AB one-gate minimally connected core-periphery network value is:

$$V_{MX} - V_{OG} = 2(M_1 - n^b)(\delta w_2 - \delta^2 w_2 - c) + 2M_2(\delta w_3 - \delta^2 w_3 - c) \tag{8}$$

This difference has to be non-negative. Therefore, it must be that:

$$(n^b - M_1)(c + \delta^2 w_2 - \delta w_2) \geq M_2(c + \delta^2 w_3 - \delta w_3) \tag{9}$$

By the linking costs condition ($c > (\delta - \delta^2)w_2$) and by the "power based" linking preferences we get $(c + \delta^2 w_3 - \delta w_3) > (c + \delta^2 w_2 - \delta w_2)$. By the non-negativity of M_2 both sides of the inequality are positive. Therefore, it must be that either $n^b - M_1 > M_2$ or $M_1 = n^b$ and $M_2 = 0$. However, by the connectivity condition it must be that $M_1 + M_2 \geq n^b$. Thus, for the highest value network among the connected networks in which each type a agent is connected to all other type a agents, to have the same total value as the AB one-gate minimally connected core-periphery network, there must be no direct connections between type b agents and each type b agent must have a single direct connection to a type a agent. Moreover, since in Equation 6 we assume that all indirect paths are of length two, all the type b agents must be linked to the same type a agent. Therefore, the AB one-gate minimally connected core-periphery network is the only network that achieves the maximal value among the set of connected networks in which each type a agent is connected to all other type a agents. ∎

Lemma 3. *Let $c > (\delta - \delta^2)w_2$. In a "power based" society, the highest total utility in the set of networks where each type a agent is connected to all other type a agents is either the AB one-gate minimally connected core-periphery network or the AB disconnected core-periphery network.*

Proof. First, we characterize the network that has the highest total utility in the set of disconnected networks where each type a agent is connected to all other type a agents. Note that these networks include one component that contains all the type a agents completely connected among themselves (the a-component). By Lemma 2 the a-component is an AB one-gate minimally connected core-periphery network.

Further, note that since all the agents that do not belong to the a-component are of type b and since $c > (\delta - \delta^2)w_3$, by Proposition 1 of Jackson and Wolinsky (1996)[29], they are organized either as a star (b-star) or as isolates. Denote by g_h the network in which there are $n^b \geq h \geq 1$ type b agents outside the a-component and they constitute a star. The total value of g_h is:

$$\begin{aligned} v(g_h) = \quad & \frac{n^a(n^a-1)}{2}(2\delta w_1 - 2c) \quad + \\ & (n^b - h)(2\delta w_2 - 2c) \quad + \quad 2\delta^2 w_2 (n^a - 1)(n^b - h) \quad + \\ & 2\delta^2 w_3 \frac{(n^b - h)(n^b - h - 1)}{2} \quad + \\ & (h-1)(2\delta w_3 - 2c) \quad + \quad 2\delta^2 w_3 \frac{(h-1)(h-2)}{2} \end{aligned} \tag{10}$$

The total value of g_1 is:

$$\begin{aligned} v(g_1) = \quad & \frac{n^a(n^a-1)}{2}(2\delta w_1 - 2c) \quad + \\ & (n^b - 1)(2\delta w_2 - 2c) \quad + \quad 2\delta^2 w_2 (n^a - 1)(n^b - 1) \quad + \\ & 2\delta^2 w_3 \frac{(n^b - 1)(n^b - 2)}{2} \end{aligned} \tag{11}$$

The difference between the total value of g_1 and the total value of g_h is:

$$v(g_1) - v(g_h) = 2(h-1)[\delta(w_2 - w_3) + (n^a - 1)\delta^2 w_2 + (n^b - h)\delta^2 w_3] \tag{12}$$

[29]Note that by the general version of this proposition to distance based utility functions (Proposition 6.1 in Jackson (2008)), the result applies for the case in which $\delta^{d_{ij}}$ is multiplied by a positive constant.

Due to the "power based" linking preferences, $\forall h > 1 : v(g_h) < v(g_1)$.

Thus, the highest total utility in the set of disconnected networks where each type a agent is connected to all other type a agents, is achieved by a network with an a-component that is organized as an AB one-gate minimally connected core-periphery network while all the type b agents who are not in this component are isolated. Moreover, by Lemma 2 the AB one-gate minimally connected core-periphery network has higher total utility than any other connected network where each type a agent is connected to all other type a agents. Thus, we can conclude that the highest total utility in the set of networks where each type a agent is connected to all other type a agents is achieved by a network where all the type b agents are either isolated or connected to the same type a agent (the "gate").

Denote each such network by g^m where $0 \leq m \leq n^b$ denotes the number of non-isolated type b agents in the network. The total value of g^m is:

$$\begin{aligned} v(g^m) = \quad & \frac{n^a(n^a-1)}{2}(2\delta w_1 - 2c) \quad + \\ & m(2\delta w_2 - 2c) \quad + \quad 2\delta^2 w_2 m(n^a-1) \quad + \\ & 2\delta^2 w_3 \frac{m(m-1)}{2} \end{aligned} \tag{13}$$

The coefficient of m^2 is $\delta^2 w_3 > 0$ and therefore $v(g^m)$ is an upward parabola in m. Thus, its maximum is achieved on one of the edges - either in $m = n^b$ (the AB one-gate minimally connected core-periphery network) or in $m = 0$ (the AB disconnected core-periphery network). ∎

Additional definitions

The *position* of agent k in path p, $T(k,p)$, equals l if $i_l = k$ in path p (the function T is not defined for $k \notin p$). Let $S(i,j,k,l,g) = \{p | p \in S(i,j,g), T(k,p) = l\}$ be the set of all shortest paths between agent i and agent j in network g that have agent k in position l. Denote its cardinality by $s^k_{ij}(l)$.

Define the *relative contribution* of neighbor $k \in N(i,g)$ to the connection between agent i and agent j in g by $RC(i,j,k,g) = \frac{s_{ij}^k(2)}{s_{ij}}[\delta^{d_{ij}} \times f(t_i,t_j)]$.
Define the *total relative contribution* of neighbor $k \in N(i,g)$ to agent i in g by[30]:

$$TRC(i,k,g) = \sum_{j \in \tilde{N}(i,g)} RC(i,j,k,g) - c = \sum_{j \in \tilde{N}(i,g)} \frac{s_{ij}^k(2)}{s_{ij}}[\delta^{d_{ij}} \times f(t_i,t_j)] - c. \quad (14)$$

Lemma 4. $u_i(g) = \sum_{k \in N(i,g)} TRC(i,k,g)$

Proof.

$$\begin{aligned}
\sum_{k \in N(i,g)} TRC(i,k,g) &= \sum_{k \in N(i,g)} \Big[\sum_{j \in \tilde{N}(i,g)} \frac{s_{ij}^k(2)}{s_{ij}}[\delta^{d_{ij}} \times f(t_i,t_j)] - c \Big] \\
&= \sum_{j \in \tilde{N}(i,g)} \frac{\sum_{k \in N(i,g)} s_{ij}^k(2)}{s_{ij}}[\delta^{d_{ij}} \times f(t_i,t_j)] - \sum_{k \in N(i,g)} c \\
&= \sum_{j \neq i} [\delta^{d_{ij}} \times f(t_i,t_j)] - n_i \times c = u_i(g)
\end{aligned}$$

∎

Lemma 5. *Let g be a pairwise stable network. If $ij \in g$ then $TRC(i,j,g) \geq 0$.*

Proof. Assume $ij \in g$ and $TRC(i,j,g) < 0$. Let $l \in \tilde{N}(i,g)$.
If $s_{il}^j(2) = 0$ then none of the shortest paths between agents i and l go through agent j. Thus, the removal of ij has no effect on the set of shortest paths between agents i and l, $S(i,l,g) = S(i,l,g-ij)$. As a consequence, the removal of ij has no effect on the relative contribution of the other neighbors of i to his connection with agent l, $\forall k \in N(i,g-ij) : RC(i,l,k,g) = RC(i,l,k,g-ij)$.
If $s_{il}^j(2) = s_{il}$ then all the shortest paths between agents i and l go through agent

[30]The notion of total relative contribution is closely related to the notion of the marginal utility of links defined and used by Bloch and Jackson (2007), Calvó-Armengol and İlkılıç (2009) and Buechel and Hellmann (2009).

j and none go through i's other neighbors, $\forall k \in N(i, g - ij) : RC(i, l, k, g) = 0$. Hence, the removal of ij can not decrease the relative contribution of the other neighbors to i's path to agent l, $\forall k \in N(i, g - ij) : RC(i, l, k, g{-}ij) \geq RC(i, l, k, g)$. If $s_{il} > s^j_{il}(2) > 0$ then some of the shortest paths between agents i and l go through agent j and some go through i's other neighbors. The removal of ij shrinks the set of shortest path between agents i and l, $S(i, l, g) \supset S(i, l, g - ij)$. Therefore, also in this case, the relative contribution of the other neighbors of i to his connection with agent l can not decrease, $\forall k \in N(i, g - ij) : RC(i, l, k, g{-}ij) \geq RC(i, l, k, g)$. Thus if ij is removed from g, the relative contribution of the other neighbors of i to his connection with agent l can not decrease:

$$\forall k \in N(i, g - ij) : RC(i, l, k, g) \leq RC(i, l, k, g - ij) \tag{15}$$

Thus, if ij is removed from g, the total relative contribution of the other neighbors of i can not decrease:

$$\forall k \in N(i, g - ij) : TRC(i, k, g) \leq TRC(i, k, g - ij) \tag{16}$$

Therefore,

$$\sum_{k \in N(i,g), k \neq j} TRC(i, k, g) \leq \sum_{k \in N(i,g-ij)} TRC(i, k, g - ij) \tag{17}$$

Now, since $TRC(i, j, g) < 0$, we get:

$$\sum_{k \in N(i,g)} TRC(i, k, g) < \sum_{k \in N(i,g-ij)} TRC(i, k, g - ij) \tag{18}$$

However, this means, by Lemma 4 that $u_i(g) < u_i(g - ij)$ and therefore, by condition 3, g is not pairwise stable. Contradiction. ∎

Lemma 6. *Let g be a pairwise stable network. $\forall i \in N : u_i(g) \geq 0$.*

Proof. By Lemma 5 if g is pairwise stable then $\forall ij \in g : TRC(i, j, g) \geq 0$.
By Lemma 4 $u_i(g) = \sum_{k \in N(i,g)} TRC(i, k, g)$.
Thus, if g be a pairwise stable network then $\forall i \in N : u_i(g) \geq 0$. ∎

Proposition 1

Proof. The complete network is pairwise stable since no agent wishes to severe any of his links. In particular, if the intrinsic value of the link is w_i ($i \in \{1, 2, 3\}$) then in the case where $n > 2$ the net gains from severing the link are $\delta^2 w_i - (\delta w_i - c)$ while in the case where $n = 2$, the net gains from severing the link are $0 - (\delta w_i - c)$. Since the linking preferences are "power based" and since $(\delta - \delta^2)w_3 > c$, those net gains are negative.

Let g' be an incomplete pairwise stable network. Thus, g' includes at least one pair of agents (j and k) that are not directly linked. Denote the geodesic distance between j and k by $d \geq 2$ and let $f(t_j, t_k) = w_i$. For both agents, the net gains from adding the link are at least $(\delta w_i - c) - \delta^d w_i$. By the linking costs range and the "power based" linking preferences, these net gains are positive. Thus, the unique pairwise stable network is the complete network.

To show that the complete network is the unique efficient network[31], let g'' be an incomplete network. Thus, g'' includes at least one pair of agents (j and k) that are not directly linked. As we showed earlier, both agents wish to add the link jk since their net gains from adding the link are positive. By Lemma 1, $v(g'' + jk) > v(g'')$. Therefore, for any incomplete network there is a network with strictly higher total utility. Thus, the complete network is the unique efficient network. ∎

[31] Using Lemma 1 and Theorem 1 of Buechel and Hellmann (2009) it is straightforward to show that the complete network is efficient. However, to show uniqueness the rest of the proof is needed.

Proposition 2

Proof. First we show that the AB maximally connected core-periphery network is pairwise stable. To show that no member of a pair that consists of at least one type a agent wishes to severe this link, note that if $n^a > 1$ the net gains for a type a agent from severing a link are either $\delta^2 w_2 - (\delta w_2 - c)$ or $\delta^2 w_1 - (\delta w_1 - c)$. The net gains from severing such links for a type b agent are at most $\delta^2 w_2 - (\delta w_2 - c)$. If $n^a = 1$ the net gains for a type a agent from severing a link are at most $0 - (\delta w_2 - c)$ while the net gains for a type b agent are $0 - (\delta w_2 - c)$. Since the linking preferences are "power based" and since $(\delta - \delta^2) w_2 > c$, those net gains are negative. To show that no pair of type b agents wish to establish a direct link note that their net gains from this link are $(\delta w_3 - c) - \delta^2 w_3$. Since $c > (\delta - \delta^2) w_3$, those net gains are negative.

To show uniqueness assume first that there is a pairwise stable network, g', in which there is a pair of a type a agent (i) and another agent (j) who are not directly connected ($ij \notin g'$). Establishing the link ij supplies both agents with a net gain of either at least $(\delta w_1 - c) - \delta^2 w_1$ or at least $(\delta w_2 - c) - \delta^2 w_2$[32]. Since the linking preferences are "power based" and since $(\delta - \delta^2) w_2 > c$, those net gains are positive. Thus, g' is not pairwise stable and in every pairwise stable network the type a agents are completely connected to all other agents. Next, let g'' be a network where the type a agents are completely connected to all other agents and there is at least one pair of directly connected type b agents (i and j). The net gains of agent i from severing ij are $\delta^2 w_3 - (\delta w_3 - c)$ which are positive since $c > (\delta - \delta^2) w_3$. Therefore, g'' is not pairwise stable and the unique pairwise stable network is the AB maximally connected core-periphery network.

To prove unique efficiency, let g''' be a network in which there exist a pair

[32] The net gains are higher if the distance between these agents in g' is more than two and/or if establishing this link shortens their paths to other agents as well.

of agents i and j, at least one of them of type a, which are not directly linked. Consider the network $g''' + ij$. As we showed earlier, both agents wish to add the link ij since their net gains from adding the link are positive. By Lemma 1, $v(g''' + ij) > v(g''')$. Therefore, every efficient network must belong to the set of networks in which type a agents are directly connected to all other agents. let g'''' be a member of this class such that there exists a pair of directly linked type b agents (i and j, $ij \in g''''$). Since, as we showed earlier, severing ij increases the utility of both i and j and since severing ij harms the utility of none of the other agents, we can conclude that $v(g'''' - ij) > v(g'''')$. Thus, the unique efficient network is the AB maximally connected core-periphery network. ∎

Proposition 3

Proof. For an AB minimally connected core-periphery network to be pairwise stable four conditions should be met. Let us check them one-by-one given "strong power based" linking preferences and the linking costs range ($\delta w_2 > c > (\delta - \delta^2)w_2$). First, no type a agent wants to severe his link to another type a agent. If $n^a > 2$, then the net gains from dropping such a link are at most $\delta^2 w_1 - (\delta w_1 - c)$. If $n^a = 2$, the net gains from dropping such a link are at most $0 - (\delta w_1 - c)$. By "strong power based" linking preferences (Assumption 1) and by the linking costs range ($\delta w_2 > c$), both net gains are negative. Second, no pair of type b agents wish to connect directly. The net gains from such a link are at most $(\delta w_3 - c) - \delta^3 w_3$. By "strong power based" linking preferences (Assumption 2) and by the linking costs range ($c > (\delta - \delta^2)w_2$), those net gains are negative[33]. Third, no pair of type a agent (i) and type b agent who are directly connected wish to severe their link. Note that the net gains from deleting the link are higher for agent i since none of his paths to other agents are harmed. Thus, it suffices to show that the net

[33] The net gains remain negative if we replace Assumption 2 with Assumption 2*.

gains of agent i from deleting the link are negative. the net gains of agent i from dropping such a link are $0 - (\delta w_2 - c)$. By the linking costs range ($\delta w_2 > c$), these net gains are negative. Last, no pair of type a agent (i) and type b agent who are not directly connected wish to establish this link. It suffices to show that agent i refrains from establishing this link. The net gains of agent i form establishing this link are $(\delta w_2 - c) - \delta^2 w_2$. by the linking costs range ($c > (\delta - \delta^2)w_2$), these net gains are negative.

However, there are networks that are pairwise stable and do not belong to the set of AB minimally connected core-periphery networks. To characterize those networks let g be a network in which there is a pair of type a agents (i and j) who are not directly connected. The net gains of agent i from establishing this link are at least $(\delta w_1 - c) - \delta^2 w_1$. By "strong power based" (Assumption 1) and by the linking costs range ($\delta w_2 > c$), those net gains are positive. Thus, the set of pairwise stable networks is a subset of the set of all networks in which each pair of type a agents is directly connected. Next, let g' be a network in which each pair of type a agents are directly connected and there is at least one pair of agents with no path between them. Thus, one component of this network includes at least all the type a agents while all the other components include only type b agents. There is at least one pair of a type a agent and a type b agent who have no path between them in g'. The net gains for both agents are at least $(\delta w_2 - c) - 0$. by the linking costs range ($\delta w_2 > c$), these net gains are positive. Thus, the set of pairwise stable networks is a subset of the set of all connected networks in which each pair of type a agents is directly connected. Third, let g'' be a connected network in which each pair of type a agents is directly connected and $\exists i \in \{k | t_k = b\} : lc_i > 1$. Let us denote such a type b agent by j and the type a agents to whom she maintains direct links by $j_1, j_2, \ldots, j_{lc_j}$. The net gains of agent j_1 from dropping the link to agent j are $\delta^2 w_2 - (\delta w_2 - c)$. By the linking costs range ($c > (\delta - \delta^2)w_2$), these net gains are positive. Therefore any pairwise stable network must be a connected network where each pair of type a agents is directly connected and no type b agent is directly linked to more than one type a agent.

Now, let us show that the AB one-gate minimally connected core-periphery network is uniquely efficient. First, Let g be a network in which there exists a pair of type a agents (i and j) which are not directly linked. As we showed above, by "strong power based" (Assumption 1) and by the linking costs range ($\delta w_2 > c$), their net gains from establishing the link are positive. By Lemma 1 we get that $v(g+ij) > v(g)$, meaning, the efficient network is a member of the set of networks where each type a agent is connected to all other type a agents. Second, Let g' be a network in which each pair of type a agents are directly connected and there is at least one pair of agents with no path between them. As we showed above, there is at least one pair of a type a agent (i) and a type b agent (j) who have no path between them in g'. By the linking costs range ($\delta w_2 > c$), the net gains of these agents from establishing a link are positive. By Lemma 1 we get that $v(g' + ij) > v(g')$, meaning, the efficient network is a member of the set of connected networks where each type a agent is connected to all other type a agents. By Lemma 2 the AB one-gate minimally connected core-periphery network is the unique efficient network. ∎

Remark 1

Proof. Let g be a a connected network where each pair of type a agents is directly connected and there is at least one type b agent (i) who is not directly connected to any type a agent and who maintains only one link (to agent j). Agent j is of type b and his net gains from severing ij are $0 - (\delta w_3 - c)$. by the "ultra strong power based" linking preferences (Assumption 2*) and by the linking costs range ($c > (\delta - \delta^2)w_2$), those net gains are positive and g is not pairwise stable. Therefore any type b agent who is not directly linked to any type a agent must maintain at least two links. ∎

Proposition 4

Proof. For the AB disconnected core-periphery network to be pairwise stable three conditions should be met. Let us check them one-by-one given "partially strong power based" linking preferences and the linking costs range ($(\delta-\delta^2)w_1 > c > \delta w_2$). First, no type a agent wants to severe his link to another type a agent. If $n^a > 2$, then the net gains from dropping such a link are $\delta^2 w_1 - (\delta w_1 - c)$. If $n^a = 2$, then the net gains from dropping such a link are $0 - (\delta w_1 - c)$. By the linking costs range ($(\delta - \delta^2)w_1 > c$), both net gains are negative. Second, no pair of type b agents wish to connect directly. The net gains from such a link are $(\delta w_3 - c) - 0$. By the linking costs range ($c > \delta w_2$) and by "power based" linking preferences those net gains are negative. Last, no pair of a type a agent (i) and a type b agent wish to establish a link. It suffices to show that agent i refrains from establishing this link. The net gains of agent i form establishing this link are $(\delta w_2 - c) - 0$. by the linking costs range ($c > \delta w_2$), these net gains are negative.

Next we show that the unique efficient network is the AB one-gate minimally connected core-periphery network if $Q > 0$ and the AB disconnected core-periphery network in case $Q < 0$. First, Let g be a network in which there exists a pair of type a agents (i and j) which are not directly linked. If $n^a > 2$, their net gains from establishing ij are at least $(\delta w_1 - c) - \delta^2 w_1$. If $n^a = 2$, then their net gains from forming the link are at least $(\delta w_1 - c) - 0$. By the linking costs range ($(\delta-\delta^2)w_1 > c$), both net gains are positive. By Lemma 1 we get that $v(g + ij) > v(g)$, meaning, the efficient network is a member of the set of networks where each type a agent is connected to all other type a agents. By Lemma 3 the efficient network is either the AB one-gate minimally connected core-periphery network or the disconnected core-periphery network. By the definition of Q the efficiency results of Proposition 4 are proven.

Next, we prove that when $Q < 0$ there are no pairwise stable networks other than the AB disconnected core-periphery network. Let g be a pairwise stable network in which there is a pair of type a agents who are not directly connected. As we showed above, by the linking costs range ($(\delta - \delta^2)w_1 > c$), their net gains are positive. Let g' be a pairwise stable network in which each pair of type a agents are directly connected and some of the type a agents are directly connected to type b agents. Let us compare the utilities of the agents in network g' to their utilities in the AB disconnected core-periphery network. Type a agents who are not connected directly to type b agents in g' surely have higher utility in g' than in the AB disconnected core-periphery network, since they benefit from the indirect connections to type b agents without changing their costs. In addition, type a agents that have direct connections to type b agents in g' have the utility they had in the AB disconnected core-periphery network, the utility they gain from their direct connections to type b agents and possibly a positive amount of additional utility from indirect connections to other type b agents. By Lemma 5, since g' is pairwise stable, the total relative contribution of each type b agent to his type a neighbor is non-negative. Therefore, the total utility of each type a agent is at least as high in g' as it is in the AB disconnected core-periphery network. Also, the utility of type b agents in the AB disconnected core-periphery network is zero. By Lemma 6 since g' is pairwise stable the utility of type b agents is at least zero. In conclusion, the total value of g' is strictly higher than that of the AB disconnected core-periphery network, in contradiction to the efficiency result we got earlier. Thus, in a pairwise stable network when $Q < 0$, type a agents are completely connected among themselves and completely disconnected from the type b agents. Let g'' be a pairwise stable network in which each pair of type a agents are directly connected, there are no direct links between type a and type b agents and there is at least one pair of type b agents that are directly connected. Note that the sum of utilities of type a agents in g'' is equal to the sum of utilities of type a agents in the AB disconnected core-periphery network. Thus, it must be that the sum of utilities of type b agents in g'' is negative, since the AB disconnected core-periphery network is uniquely efficient and the sum of utilities of type b agents in this network is zero. Therefore, there is at least one type b agent in g'' that have negative utility. This contradicts Lemma 6, to complete the uniqueness proof for the case where $Q < 0$.

However, when $Q > 0$ there are networks that are pairwise stable other then the AB disconnected core-periphery network. To characterize those networks let g be a pairwise stable network in which there is a pair of type a agents who are not directly connected. As shown above, by the linking costs range $((\delta - \delta^2)w_1 > c)$, their net gains from establishing a link are positive. Let g' be a network in which all the type a agents are completely connected among themselves and there is a type b agent (i), who has exactly one direct connection (to j). The net gains of agent j from severing ij are at least $0 - (\delta w_2 - c)$. By the linking costs range $(c > \delta w_2)$ these net gains are positive. We conclude the proof by showing that the disconnected core-periphery network is the only pairwise stable core-periphery network. Let g'' be an AB non-disconnected core-periphery network where no type b agent has exactly one link. Hence, there is at least one type b agent (i) who maintains direct links to at least two type a agents. The net gains of each of them from severing the link to agent i are $\delta^2 w_2 - (\delta w_2 - c)$. By the linking costs range $(c > \delta w_2)$ these net gains are positive. ∎

Proposition 5

Proof. First, let us prove the instability of core-periphery networks. Let g be an AB core-periphery network other than the AB disconnected core-periphery network. Thus, there is at least one type a agent (j) with a direct link to a type b agent. The net gains of agent j from severing such a link are at least $0-(\delta w_2-c)$. By the linking costs range and by "partially strong power based" linking preferences those net gains are positive. In addition, the AB disconnected core-periphery network is not pairwise stable since the net gains for a type a agent from severing her direct link to a fellow type a agent are $\delta^2 w_1 - (\delta w_1 - c)$ which are positive by the linking costs range (using $n^a \geq 3$).

Second, let g be the efficient network. By Lemma 3 the highest total utility in the set of networks where each type a agent is connected to all other type a agents is either the AB one-gate minimally connected core-periphery network (g^{OG}) or the AB disconnected core-periphery network. In particular, one of these two networks maximize the total value among the set of core-periphery networks. Let g' be a star network encompassing all the agents, where the center of the star is a type a agent. It is easy to see that $v(g') - v(g^{OG}) = (n^a k - 1)(n^a - 2)[\delta^2 w_1 - (\delta w_1 - c)]$. By the linking costs range and $n^a \geq 3$, $v(g') > v(g^{OG})$. Therefore the AB one-gate minimally connected core-periphery network is not efficient. To complete the proof, note that by the linking costs range and Proposition 1 of Jackson and Wolinsky (1996), the AB disconnected core-periphery network has lower total value than either the empty network or the network where all the type a agents are organized as a star and all the type b agents are isolates. ∎