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Tapan Mitra
Cornell University

Santanu Roy
Southern Methodist University

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In this paper we investigate the roles of environmental uncertainty and risk aversion in the theory of optimal management of a renewable resource. We study in detail an example in which the growth rate of the resouce at low stock levels exceeds the discount rate for every realization of the random shock, so that optimally managed resource stocks would exhibit growth for low stocks in any deterministic version of the model. We then show that (i), if the Arrow-Pratt measure of relative risk aversion is sufficiently high, then the resource stock cannot be bounded away from zero in the long-run; and (ii) if relative risk aversion decreases sufficiently with increasing consumption, then there will exist a positive minimum safe standard of conservation when the best shock is sufficiently favorable for growth of the resource.

Could fit a session on resource economics, dynamic environmental economics, stochastic dynamic models, optimal stochastic growth or any other session on dynamics.

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On Risk Aversion and the Theory of Optimal Management of Renewable Resources

Tapan Mitra [*] and Santanu Roy [†]

Abstract

In this paper we investigate the roles of environmental uncertainty and risk aversion in the theory of optimal management of a renewable resource. We study in detail an example in which the growth rate of the resouce at low stock levels exceeds the discount rate for every realization of the random shock, so that optimally managed resource stocks would exhibit growth for low stocks in any deterministic version of the model. We then show that (i), if the Arrow-Pratt measure of relative risk aversion is sufficiently high, then the resource stock cannot be bounded away from zero in the long-run; and (ii) if relative risk aversion decreases sufficiently with increasing consumption, then there will exist a positive minimum safe standard of conservation when the best shock is sufficiently favorable for growth of the resource.

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[*]Department of Economics, Cornell University, Ithaca, NY 14853; E-mail: tm19@cornell.edu

[†]Department of Economics, Southern Methodist University, Dallas, TX 75275-0496; E-mail: sroy@smu.edu.

1 Introduction

Extinction of biological species is an important ecological concern of the current age. Extinction is likely whenever a renewable resource is harvested persistently at a rate exceeding the level required to sustain its current stock. The economics of extinction relates the depletion of resources to economic incentives that affect harvesting. Traditionally, economists have related overexploitation of resources to failures of markets and property rights. However, even if such failures are corrected and society "manages" its resources *optimally*, the nature of intertemporal trade-offs between current and future welfare that a society is willing to make can lead to eventual extinction. It is important to understand the way in which intertemporal preferences of society and the biological growth of resources interact to determine the possibility of extinction and conservation for an optimally managed resource.

One important factor here is the sensitivity of biological growth to random environmental fluctuations. The other factor is that such environmental uncertainty also affects the incentive to harvest a resource; this depends on the attitude to risk of the "manager" of the renewable resource. It is common to refer to this aspect as *risk aversion*.

Beginning with Clark (1973), much of the analysis of the economics of optimal extinction and conservation of renewable resources has been carried out in *deterministic* models. The conventional wisdom from this literature suggests that stocks of an optimally managed resource ought to be bounded away from zero as long as the resource has an intrinsic growth rate that exceeds the rate at which society discounts the future. However, when the natural growth of the resource is stochastic, comparing the natural growth rate of the resource to the discount rate is no longer sufficient to characterize the possibility of extinction.

In Mitra and Roy (2006), we studied the conditions under which resource stocks are bounded away from zero in the long run; in particular, we characterized conditions for global conservation and the existence of a safe standard of conservation. However, those characterizations were in the form of joint conditions on the production function and the utility function. They do not allow us to understand the explicit role of risk aversion.[1]

In this paper, we wish to examine in detail the relationship between the environmental uncertainty, the degree of risk aversion, and long run extinction and conservation of renewable resources in a model of optimal resource management. Unlike the optimal growth literature, where an assumption of infinite marginal product at zero is commonly made on the production function, we assume that

[1] There is a significant literature on characterization of extinction and non-extinction in terms of the transition law for a given Markov process (rather than the primitives of an economic model that generates such transition law). For the case of a multiplicative shock with a smooth density function whose support is the entire positive real line (so that from any current stock one may reach any interval of stocks, however high or low, with strictly positive probability), Nishimura, Rudnicki and Stachurski (2006) provide conditions on the transition function under which the stochastic process converges globally to a degenerate distribution at zero as well as conditions under which it converges globally to a unique distribution that assigns zero probability mass at zero.

the production function (also known in the literature as the stock-recruitment function) has finite slope at zero. The assumption of infinite marginal product at zero is not well suited to our purpose because the rate of natural growth for most biological species is rather small when the stock depletes to a level close enough to zero. On the other hand, to highlight the role of environmental uncertainty in the stochastic model of optimal resource management, we assume that the resource has an intrinsic growth rate that exceeds the rate at which society discounts the future, for every realization of the random shock, so that optimally managed resource stocks would exhibit growth for low stocks in any deterministic version of the model.

We first try to assess the role that risk aversion can play in determining the extinction of optimally managed renewable resources. In the context of a specific example of the production framework described in the previous paragraph, we show that with low relative risk aversion (RRA) the resource expands for low stock levels with probability one, but for high relative risk aversion, resource stocks can decline for *all* stock levels under the worst shock and this implies that resource stocks get arbitrarily close to zero with probability one, *independent of the initial resource stock*.

Second, we show (again in the context of a specific example of the production framework described above) that if the measure of relative risk aversion decreases sufficiently with increasing consumption, there will exist a *minimum safe standard of conservation* when the best shock is sufficiently favorable for growth of the resource. That is, there is a resource stock level $m > 0$, such that if the initial resource stock y is at least as large as m, then the optimally managed resource stock is always as large as the resource stock level m, independent of the random shock. On the other hand, optimally managed resource stocks starting from $y \in (0, m)$ always decline under the worst shock.

Our two results stand in stark contrast to the standard theory of optimal management of renewable resources under certainty. Consider a deterministic model in which the production possibilities are specified by our production function under the worst shock. This is very similar to the standard Ramsey-Cass-Koopmans model of optimal growth (but with bounded steepness of the production function), since we assume that the production and the utility functions are (strictly) concave. In this case, there would exist a positive modified golden-rule stock, y^*, such that optimally managed resource stocks would exhibit growth for *all* positive initial stocks below y^* (with asymptotic monotone convergence to y^*).

Our second result is also to be distinguished from the existence of a safe standard of conservation obtained in deterministic models of optimal resource management, in which the intrinsic growth rate is below the rate at which society discounts the future, but because the stock-recruitment function is S-shaped, the growth rate exceeds the discount rate for a range of high stock levels.[2]

[2] For this literature, see especially Lewis and Schmalensee (1979), Majumdar and Mitra (1982, 1983), Dechert and Nishimura (1983), Cropper (1988) and Clark (1990).

2 Preliminaries

2.1 The Framework

We consider an infinite horizon one-good representative agent economy. Time is discrete and is indexed by $t = 0, 1, 2, ...$ The initial stock of output $y_0 > 0$ is given. At each date $t \geq 0$, the representative agent observes the current output stock $y_t \in \mathbb{R}_+$ and chooses the level of consumption (harvest) level c_t and current input stock x_t, such that

$$c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t$$

This generates y_{t+1}, the output stock next period through the relation

$$y_{t+1} = f(x_t, r_{t+1})$$

where $f(.,.)$ is the production function (stock-recruitment function) and r_{t+1} is a random production shock realized at the beginning of period $(t+1)$. Given current output stock $y \geq 0$, the feasible set for consumption and input is denoted by $\Gamma(y)$; that is,

$$\Gamma(y) = \{(c, x) : c \geq 0, x \geq 0, c + x \leq y\}$$

The following assumption is made on the sequence of random shocks:

(A.1) $\{r_t\}_{t=1}^{\infty}$ *is an independent and identically distributed random process defined on a probability space* $(\Omega, \mathcal{F}, P)$, *where the marginal distribution function is denoted by* F. *The support of this distribution is a compact set* $A \subset \mathbb{R}$.

The production function $f : \mathbb{R}_+ \times A \to \mathbb{R}_+$ is assumed to satisfy the following:

(T.1) *For all* $r \in A, f(x, r)$ *is concave in* x *on* $\mathbb{R}_+$.

(T.2) *For all* $r \in A$, $f(0, r) = 0$.

(T.3) $f(x, r)$ *is continuous in* (x, r) *on* $\mathbb{R}_+ \times A$. *For each* $r \in [a, b]$, $f(x, r)$ *is differentiable in* x *on* $\mathbb{R}_+$ *and, further,* $f'(x, r) \equiv \frac{\partial f(x,r)}{\partial x} > 0$ *on* $\mathbb{R}_+ \times A$.

Assumptions (T.1)-(T.3) are standard monotonicity, concavity and smoothness restrictions on production. For any input level $x \geq 0$, let the upper and lower bound of the support of output next period be denoted by $\overline{f}(x)$ and $\underline{f}(x)$, respectively. In particular,

$$\overline{f}(x) = \max_{r \in A} f(x, r), \underline{f}(x) = \min_{r \in A} f(x, r).$$

It is easy to check that $\underline{f}(x)$ is continuous, concave and strictly increasing on $\mathbb{R}_+$.[3] Further, $\overline{f}(x)$ is continuous and strictly increasing on $\mathbb{R}_+$.

Let

$$\nu = \lim_{x \downarrow 0} \frac{\underline{f}(x)}{x}.$$

[3]Continuity follows from the maximum theorem. To see concavity observe that for any $x_1, x_2 \in \mathbb{R}_+, \lambda \in [0, 1]$, $\underline{f}(\lambda x_1 + (1-\lambda)x_2) = f(\lambda x_1 + (1-\lambda)x_2, \widehat{r})$ for some $\widehat{r} \in A$ which is $\geq \lambda f(x_1, \widehat{r}) + (1-\lambda) f(x_2, \widehat{r}) \geq \lambda \underline{f}(x_1) + (1-\lambda)\underline{f}(x_2)$.

We assume that

(T.4) $\nu > 1$ and $\limsup_{x\to\infty} \frac{\overline{f}(x)}{x} < 1$.

The first part of assumption (T.4) ensures that it is feasible for capital and output to grow with probability one in a neighborhood of zero; that is, even under the most adverse realization of the random shock. The second part of the assumption implies that the technology exhibits bounded growth.

Let $\delta \in (0,1)$ denote the utility discount factor. Given the initial stock $y_0 > 0$, the representative agent's objective is to maximize the expected value of the discounted sum of utilities from consumption:

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right]$$

where u is the one period utility function from consumption.

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The utility function $u : \mathbb{R} \to \bar{\mathbb{R}}$ satisfies the following restrictions:

(U.1) *u is strictly increasing, continuous and strictly concave on* $\mathbb{R}_+$ (on $\mathbb{R}_{++}$ if $u(0) = -\infty$); $u(c) \to u(0)$ as $c \to 0$.

(U.2) *u is continuously differentiable on* $\mathbb{R}_{++}$ with $u'(c) > 0$ for all $c > 0$.

(U.3) $\lim_{c\to 0} u'(c) = +\infty$.

Assumptions (U.1) and (U.2) are standard. Note that we allow the utility of zero consumption to be $(-\infty)$. (U.3) requires that the utility function satisfy the Uzawa-Inada condition at zero.

2.2 Value and Policy Functions

The *partial history* at date t is given by $h_t = (y_0, x_0, c_0, \ldots, y_{t-1}, x_{t-1}, c_{t-1}, y_t)$. A *policy* π is a sequence $\{\pi_0, \pi_1, \ldots\}$ where π_t is a conditional probability measure such that $\pi_t(\Gamma(y_t)|h_t) = 1$. A policy is *Markovian* if for each t, π_t depends only on y_t. A Markovian policy is *stationary* if π_t is independent of t. Associated with a policy π and an initial state y is an expected discounted sum of social welfare:

$$V_\pi(y) = E\sum_{t=0}^{\infty} \delta^t u(c_t),$$

where $\{c_t\}$ is generated by π, f in the obvious manner and the expectation is taken with respect to P.

The *value function* $V(y)$ is defined on $\mathbb{R}_{++}$ by:

$$V(y) = sup\{V_\pi(y) : \pi \textit{ is a policy}\}.$$

Under assumption **(T.4)**, it is easy to check that

$$-\infty < V(y) < +\infty \text{ for all } y > 0.$$

A policy, π^*, is *optimal* if $V_{\pi^*}(y) = V(y)$ for all $y > 0$. Standard dynamic programming arguments[4] imply that there exists a unique optimal policy, that

[4]See, for instance, Schäl (1975), Majumdar, Mitra and Nyarko (1989).

this policy is stationary and that the value function satisfies the functional equation:

$$V(y) = \sup_{x \in \Gamma(y)} [u(y - x) + \delta E[V(f(x, r))]]. \tag{FE}$$

It can be shown that $V(y)$ is continuous, strictly increasing and strictly concave on $\mathbb{R}_{++}$. Further, the maximization problem on the right hand side of (FE) has a unique solution, denoted by $x(y)$. The stationary policy generated by the function $x(y)$ is the optimal policy and we refer to $x(y)$ as the *optimal input function* and $c(y) = y - x(y)$ as the *optimal consumption function*.

2.3 Basic Properties of Optimal Policies

Using standard arguments[5] in the literature, (U.3) can be used to show that:

Lemma 1 *(i) For all* $y > 0, x(y) > 0$ *and* $c(y) > 0$.
(ii) $x(y)$ *and* $c(y)$ *are continuous and strictly increasing in* y *on* $\mathbb{R}_+$.

Given initial stock $y > 0$, the stochastic process of optimal output stocks $\{y_t(y, \omega)\}$ evolves over time according to the transition rule:

$$y_t(y, \omega) = f(x(y_{t-1}(y, \omega)), \omega_t) \text{ for } t \geq 1 \tag{SP}$$

and $y_0(y, \omega) = y$.

Next, we note that the stochastic Ramsey-Euler equation holds:

Lemma 2 *For all* $y > 0$,

$$u'(c(y)) = \delta E[u'(c(f(x(y), r)))f'(x(y), r)]. \tag{RE}$$

2.4 An Example

In what follows, we will confine our analysis to an example of the above production framework. While aspects of the example can be generalized, it helps us to focus on some key elements of the model and to simplify some of the analysis involved in describing the optimal input function.

Assume that production is subject to a random *multiplicative* shock, so that:

$$f(x, r) = rh(x) \text{ for all } x \in \mathbb{R}_+ \text{ and } r \in A$$

where $A = [\alpha, \beta]$, with $0 < \alpha < \beta < \infty$, and:

$$h(x) = Bx/(1 + x) \text{ for all } x \in \mathbb{R}_+ \tag{DPF}$$

with $B > 1$. Without loss of generality, we choose $\alpha = 1$, and so $\beta \in (1, \infty)$. Then, (T.1)-(T.4) are clearly satisfied.

[5] See especially Brock and Mirman (1972) and Mirman and Zilcha (1975).

The sequence of random shocks is iid, with the common *distribution* given by:

$$F(r) = \begin{cases} 0 & \text{for } r < 1 \\ (r-1)/(\beta-1) & \text{for } 1 \le r \le \beta \\ 1 & \text{for } r > \beta \end{cases}$$

The extent of *risk* can be captured by the magnitude of $(\beta - 1)$, so that the parameter β itself can be used as a proxy for it. We denote by G the density associated with F; that is,

$$G(r) = \frac{1}{(\beta - 1)} \text{ for all } r \in A = [1, \beta]$$

A *deterministic version* of the production framework[6] is obtained when the distribution of r is degenerate at the value $r = \alpha = 1$, so that the deterministic production function (DPF) completely describes the production possibilities.

3 On the Concept of Resource Conservation

The central focus of this paper is the transition possibilities of an optimally managed resource, and in particular the possibility of conservation of the resource stock under optimal management. In order to proceed with our analysis, we need to be precise about the concepts of resource conservation. Several concepts are discussed and analyzed in Mitra and Roy (2006, 2010); we describe below the concepts that are relevant for the purpose of this paper.

Following Mitra-Roy (2010), we can refer to an economy (u, δ, f, F) as *nowhere bounded away from zero* (NBZ) if its optimal input policy $x(y)$ satisfies:

$$h(x(y)) < y \text{ for all } y > 0 \tag{DWS}$$

The property (DWS) of the input policy function means that *no matter what the current state of the economy*, the resource stock necessarily declines under the worst realization of the production shock. Using Proposition 2 in Mitra and Roy (2007), one can check that (DWS) implies that for every initial stock $y_0 = y > 0$, the stochastic process of optimal output $\{y_t(y, \omega)\}$ defined by (SP) must satisfy the property:

$$\Pr\{\lim_{t\to\infty} \inf y_t(y, \omega) = 0\} = 1 \tag{NSS}$$

That is, the optimal resource stock is not bounded away from zero in the long run and there is no invariant distribution whose support is bounded away from zero. (DWS). In this case, it is legitimate to say that there is *no safe standard of conservation*; that is, there is no initial stock, however large, which guarantees

[6] There are other deterministic versions of the stochastic model, which are also of interest. For example, one might consider a deterministic model, with a production function given by $f(x) = (Er)h(x)$, where Er is the mean value of the random shock in the stochastic model.

that stocks under optimal management will be bounded away from zero in the long run.

For this paper, a weaker condition than (DWS) will also be of interest to us, namely where the condition indicated in (DWS) holds for "low" levels of the resource stock:

$$\text{There exists } \tilde{y} > 0, \text{ such that } h(x(y)) < y \text{ for all } y \in (0, \tilde{y}) \qquad \text{(DWS')}$$

Condition (DWS') allows for the possibility that:

$$\text{There exists } y' > 0, \text{ such that } h(x(y')) \geq y' \qquad \text{(GWS)}$$

In this case, for all $y \geq y'$, we have by Lemma 1,

$$rh(x(y)) \geq rh(x(y')) \geq ry' \geq y'$$

for all $r \in A$, so that the stochastic process of optimal output $\{y_t(y, \omega)\}$ defined by (SP) must satisfy the property:

$$\Pr\{\omega : y_t(y, \omega) \geq y' \text{ for all } t \geq 0\} = 1 \text{ for all } y \geq y'$$

That is, the optimal resource stock never goes below y', if one starts from an initial stock $y \geq y'$. Thus when (GWS) holds we can call y' a *safe standard of conservation*.

When (DWS') and (GWS) both hold, we can define:

$$m = \inf\{y' > 0 : h(x(y')) \geq y'\} \qquad \text{(MSS)}$$

Any element in the set $\{y' > 0 : h(x(y')) \geq y'\}$ must be at least as large as $\tilde{y}$ by (DWS'), and so $m \geq \tilde{y}$. Further, by Lemma 1, $h(x(m)) \geq m$. Thus, m itself satisfies (GWS) and so m is a safe standard of conservation. Further, for all $y < m$, it must be the case that $h(x(y)) < y$. We call m the *minimum safe standard of conservation* in the sense that m satisfies property (GWS) and no lower positive y satisfies property (GWS).

This terminology differs from that in Mitra-Roy (2006); we adopt it here because the asymptotic properties of the stochastic process (SP) are not altogether clear when the initial resource stock y is less than m. That is, it might be possible that for positive initial stocks y lower than m, even though the resource declines under the worst shock, it recovers under good shocks sufficiently to eventually get larger than m. This is a topic we are investigating, but we only have partial results to report at present.

4 On the Absence of a Safe Standard of Conservation

To begin our discussion on the absence of a safe standard of conservation, consider a *deterministic* version of the stochastic growth model outlined in Section

2.4. This is the well known Ramsey-Cass-Koopmans discounted classical optimal growth model, described by (h, u, δ), where h is given by (DPF). As is well-known, if

$$h'(0) > \frac{1}{\delta} \tag{DP}$$

that is, the net marginal productivity at zero exceeds the discount rate (the technology is "delta-productive" at zero), the sequence of optimal output stocks from every strictly positive initial output stock, converges monotonically to a unique strictly positive limit, the "modified golden rule" output stock y^* defined by

$$h'(x^*) = \frac{1}{\delta} \tag{MGR}$$

where $h(x^*) = y^*$.In other words, under (DP), the optimal resource stock is always bounded away from zero and indeed, if $0 < y_0 < y^*$, the optimal resource stock exhibits growth over time. Beyond the general assumptions (U.1)-(U.3), the actual form of the utility function plays no role in this deterministic setting.

The situation may, however, be qualitatively different in the stochastic model where the probability distribution of the production function $f(x, r)$ is non-degenerate.[7] Suppose the utility function satisfies constant relative risk aversion (CRRA) with relative risk aversion parameter ρ; that is, $u : \mathbb{R}_{++} \to \mathbb{R}$ is given by:

$$u(c) = \begin{cases} \frac{c^{1-\rho}}{1-\rho} & \text{if } \rho \neq 1 \\ \ln c & \text{otherwise} \end{cases}$$

with $u(0) = \lim_{c \to 0} u(c) = 0$ when $\rho \in (0, 1)$, and $u(0) = -\infty$ otherwise.

Let the production framework be as described in Section 2.4. Further, assume that $\delta \in (0, 1)$ satisfies

$$\delta h'(0) = \delta B > 1$$

so that the deterministic version of the stochastic growth model (obtained by selecting the deterministic production function to be the stochastic production function under the worst shock) is "delta-productive". Then, if $\delta(Er)B$ is close to 1, we show in this section that one can explicitly specify ρ' such that for all $\rho > \rho'$,

$$h(x(y)) < y \text{ for all } y > 0 \tag{DWS}$$

so that the no safe standard (NSS) holds.[8]

[7] Our analysis is based on the phenomenon described in Mirman-Zilcha (1976). In the example of Mirman-Zilcha, the production function satisfies the Uzawa-Inada condition at zero, a feature we wanted to avoid. In addition, the degree of relative risk aversion in their example must go to infinity as consumption goes to zero (as demonstrated in Mitra-Roy (2010)), a feature which appears unnecessary to make the point we wish to make in the current context.

[8] Kamihigashi (2006) shows that if the marginal product at zero is finite, then every feasible path (including, therefore, any optimal path) converges almost surely to zero, provided the random shocks are "sufficiently volatile". His result is not driven by any property of the utility function, and appears to be different from our results described below, which emphasize the role of risk aversion.

Thus, in the stochastic growth model (and in contrast to the deterministic model), the presence of environmental uncertainty and the degree of risk aversion play important roles in determining the long run destiny of the economy.

4.1 Parametric Restrictions

We will impose one parametric restriction before studying the extinction issue. This restriction involves the parameters (B, δ), and does not involve the parameter ρ of the utility function, or the magnitude of the most favorable shock, captured by the parameter β.

We will restrict the corresponding deterministic economy to be δ-productive; that is,

$$\delta h'(0) = \delta B > 1 \tag{1}$$

Note that when (1) holds, we also have:

$$\delta(Er)h'(0) = \delta(Er)B > 1 \tag{2}$$

Define:

$$\eta \equiv [\delta(Er)B]^{\frac{1}{2}} - 1 \tag{3}$$

Using (2), we have $\eta > 0$. We can then define:

$$\mu \equiv \frac{(1+\beta)\eta B}{(\eta+1)(B-1)} > 0 \tag{4}$$

Clearly, if η is small enough, then $\mu < 1$.

To take a numerical example, choose $B = 4, \beta = 2$ and $\delta = (25/96)$. Then, $(Er) = (\beta + 1)/2 = (3/2)$, and:

$$\delta B = (25/24) > 1$$

so that (1) is satisfied. Also,

$$\delta(Er)B = (25/24)(3/2) = (5/4)^2 > 1$$

so that we have $\eta = (5/4) - 1 = (1/4)$, and $\eta/(\eta+1) = (1/5)$. Also $[B/(B-1)] = (4/3)$. Thus,

$$\mu = \frac{(1+\beta)\eta B}{(\eta+1)(B-1)} = (3)(4/3)(1/5) = (4/5) < 1$$

For future use, we also define:

$$\theta \equiv \frac{1}{\{1 + [(1-\mu)/\mu]2B\}} \tag{5}$$

Clearly, $\theta \in (0, 1)$ when $\mu \in (0, 1)$.

4.2 A Sufficient Condition for Stocks to be Arbitrarily Close to Zero

We now provide a sufficient condition on the parameters of the model (u, δ, f, F) for condition (DWS) to hold, and therefore for the absence of a safe standard (NSS).

Proposition 1 *Suppose (B, δ) are such that:*

$$\delta h'(0) = \delta B > 1 \tag{6}$$

and β is such that:

$$\mu \equiv \frac{(1+\beta)\eta B}{(\eta+1)(B-1)} < 1 \tag{7}$$

Define:

$$\rho' \equiv \frac{\delta B \beta^2}{(\beta-1)(1-\theta)} - 1 \tag{8}$$

and note that ρ' is clearly positive. If the RRA (ρ) of the utility function satisfies $\rho > \rho'$ then the optimal consumption policy function, $c(y)$, satisfies:

$$h(y - c(y)) < y \quad \textit{for all } y > 0 \tag{9}$$

The proposition provides a restriction on the RRA for which stocks are arbitrarily close to zero (infinitely often) with probability one. The lower bound ρ' is in terms of the basic parameters of the model, and in terms of θ, which is defined in terms of μ. A more convenient lower bound ρ'' can be written in terms of the basic parameters of the model, and in terms of μ, by noting that:

$$(1-\theta) \geq (1-\mu)$$

Corollary 1 *Suppose (B, δ) are such that:*

$$\delta h'(0) = \delta B > 1$$

and β is such that:

$$\mu \equiv \frac{(1+\beta)\eta B}{(\eta+1)(B-1)} < 1$$

Define:

$$\rho'' \equiv \frac{\delta B \beta^2}{(\beta-1)(1-\mu)} - 1 \tag{10}$$

and note that ρ'' is clearly positive. If the RRA (ρ) of the utility function satisfies $\rho > \rho''$ then the optimal consumption policy function, $c(y)$, satisfies:

$$h(y - c(y)) < y \quad \textit{for all } y > 0$$

Proof. Using the definition of θ, we have:

$$\theta = \frac{\mu}{\mu + (1-\mu)2B}$$

and so:

$$(1-\theta) = \frac{(1-\mu)2B}{\mu + (1-\mu)2B} \geq (1-\mu)$$

Thus $\rho'' > \rho'$, and the result follows from Proposition 1. ■

4.3 A Necessary Condition for Stocks to be Arbitrarily Close to Zero

Note that condition (1) ensures delta-productivity in the deterministic case and therefore for the least favorable shock in the stochastic case. Thus, by Chatterjee and Shukaev (2008), if $\rho \in (0,1)$, then since the utility function is bounded below, we must have that the stocks are bounded away from zero. [The concept of the optimal stock being bounded away from zero (BAZ) is discussed more fully in Mitra and Roy (2010)].

Along the same lines, we might also note the following *necessary condition* for (DWS) to be satisfied. That is, if this (necessary) condition is violated, then there will be some $y' > 0$ for which (GWS) will be satisfied, and therefore starting from any initial stock $y \geq y'$, optimal stocks must be bounded away from zero.

Proposition 2 *Suppose* (B, δ) *are such that:*

$$\delta h'(0) = \delta B > 1$$

and β *is such that:*

$$\mu \equiv \frac{(1+\beta)\eta B}{(\eta+1)(B-1)} < 1$$

If the RRA (ρ) *of the utility function is such that the corresponding optimal consumption policy function,* $c(y)$*, satisfies:*

$$h(y - c(y)) < y \quad \text{for all } y > 0$$

then:

$$\frac{(\beta-1)}{\delta B} \geq \begin{cases} \left[\frac{(B-1)}{B}\right]^{\rho} \left[\frac{\beta^{2-\rho}-1}{(2-\rho)}\right] & \text{for } \rho \neq 2 \\ \left[\frac{(B-1)}{B}\right]^{\rho} \ln \beta & \text{for } \rho = 2 \end{cases} \tag{11}$$

4.4 A Bifurcation Result

We can see how good the restriction in Proposition 1 is by doing the following exercise. Suppose we are free to choose (B, β, δ) in any way we like provided the delta-productivity condition (1) is satisfied. What is the minimal RRA (call this ρ^*) for which the stocks are arbitrarily close to zero (infinitely often) almost surely?

It turns out that, at least for the example we have been discussing, one can provide a definitive answer to this question. When $\rho > 1$, then one can choose (B, β, δ) appropriately, satisfying restriction (1), so that optimal resource stocks will be nowhere bounded away from zero. And, if $\rho < 1$, then there is *no* choice of (B, β, δ), satisfying restriction (1), for which optimal resource stocks will be nowhere bounded away from zero. Thus $\rho^* = 1$ is a bifurcation value for the NBZ phenomenon to arise. We report this result in two propositions below.

Proposition 3 *Let $\rho > 1$. Then there exists $E \equiv (B, \beta, \delta)$ satisfying the restriction (1), such that the optimal consumption policy function, c_E, corresponding to E, satisfies:*

$$h_E(y - c_E(y)) < y \text{ for all } y > 0$$

Proposition 4 *Let $\rho \in (0, 1)$. Then there is no $E \equiv (B, \beta, \delta)$ satisfying the restriction (1), such that the optimal consumption policy function, c_E, corresponding to E, satisfies:*

$$h_E(y - c_E(y)) < y \text{ for all } y > 0$$

It is useful to keep in mind that the bifurcation result is obtained under the strong assumption that the production function is delta-productive even for the least favorable shock; that is (1) holds. If instead we assumed that the production function is delta-productive on the average, that is if (2) holds (but not necessarily (1)) then our bifurcation result could be different.

5 On a Safe Standard of Conservation

This section presents the main finding of the paper with regard to conservation of a resource: the existence of a positive minimum safe standard of conservation. We establish the existence result in the context of the example described in Section 2.4, and with a particular specification of the utility function, which captures (in crude form) the feature that the degree of risk aversion is relatively high for low consumption levels and relatively low for high consumption levels. A more satisfactory existence result, using a well-known family of decreasing relative risk aversion (DRRA) utility functions, would be useful.[9]

Since our demonstration of the existence result is somewhat lengthy, it is useful to break it up into several steps, which we describe in separate subsections below.

[9] Conditions for conservation in a model where the utility function depends on both consumption as well as resource stock have been explored by Olson and Roy (2000).

5.1 Specification of the Example

We can confine our analysis to the state space $X = [0, K]$, where K is defined to be the maximum sustainable stock under the best realization of the stock, so that:

$$\frac{\beta BK}{(1+K)} = K$$

and:

$$K = \beta B - 1 \tag{12}$$

If $y \in X$, then the stochastic process of output $\{y_t(y, \omega)\}$ generated by any policy will satisfy:

$$\Pr\{\omega : y_t(y, \omega) \in X \text{ for all } t \geq 0\} = 1 \tag{13}$$

We describe the utility function u, using (12), and keeping in mind that (13) holds. Define:

$$\theta \geq 2, \sigma \in (0,1), L = \frac{K^{1-\sigma}}{(1-\sigma)}, M = L - \frac{1}{(1-\sigma)} - \frac{1}{\theta - 1} \tag{14}$$

and:

$$u(c) = \begin{cases} [c^{1-\theta}/(1-\theta)] - M & \text{for } 0 < c < 1 \\ [c^{1-\sigma}/(1-\sigma)] - L & \text{for } c \geq 1 \end{cases} \tag{15}$$

We restrict the parameters[10] in (14) so that:

$$L > \frac{1}{(1-\sigma)} + \frac{1}{\theta - 1}$$

Thus, we have $M > 0$, and so $u(c) < 0$ for all $c \in (0, 1)$ by (15), since $\theta > 1$. For $c \in [1, K]$, we have $u(c) \leq 0$ by (15) and the definition of L in (14). Thus, we may note for future reference:

$$u(c) \leq 0 \ \text{ for all } c \in X \tag{16}$$

To check the continuity of u, note that by (14),

$$\lim_{c \to 1-} u(c) = \frac{1}{(1-\theta)} - M = \frac{1}{(1-\sigma)} - L = u(1)$$

To verify the continuous differentiability of u, note that:

$$\lim_{c \to 1-} u'(c) = \lim_{c \to 1-} \frac{1}{c^{\theta}} = 1 = u'(1) \tag{17}$$

Since u' is decreasing on $(0, 1)$ and on $[1, \infty)$, (17) implies that it is decreasing on $(0, \infty)$, and so u is strictly concave on $(0, \infty)$. Further, $u(0) = -\infty$ and

[10] For numerical values satisfying the restrictions, choose $B = 4, \theta = 2, \sigma = (1/2)$ and $\beta > 1$. Then $K > 3$, $L > 3$, and so M defined in (14) is positive.

$u'(c) \to \infty$ as $c \to 0$ by (15). Using (15) and (17), we have $u'(c) > 0$ for all $c > 0$, so u is increasing on $(0, \infty)$. Thus, assumptions (U.1)-(U.3) are satisfied.

The discount factor $\delta \in (0,1)$ is specified to be such that:

$$\delta B > 1 \tag{18}$$

Thus, we continue to assume (as in Section 4) that the production function is "delta-productive" even under the worst shock. In particular, given that $B > 1$, δ can be arbitrarily close to 1, so the phenomenon to be described in this example has nothing to do with strong discounting of future utilities.

Apart from the specifications of the example detailed above, we will be choosing the most favorable shock β to be sufficiently large, where "sufficiently large" will be defined more precisely in terms of the other parameters as we proceed to the next subsections.

5.2 Growth for Some Resource Stock

In this subsection we will show that there is a resource stock for which the optimal input leads to a growth in the resource stock even under the worst shock; that is, we demonstrate that (GWS) holds.

This demonstration consists of two parts. First, we show that there is a resource stock $y \in (0, K)$ such that the optimal input choice $x(y)$ corresponding to it is equal to 1. Second, we show that $h(x(y)) = h(1) > y$.

For the first part, we will use the condition that β is large enough to satisfy:

$$\beta h(1) - 1 \geq 1 \text{ and } (\beta + 1) > (8/\delta B) \tag{19}$$

Under the restriction (19), we have $(K-1) = (\beta B - 2) \geq 2$.

We claim that there is some $y \in (0, K)$ such that $x(y) = 1$. If this is not true, then by Lemma 1, it must be the case that $x(K) \leq 1$. Then, we have:

$$c(K) = K - x(K) \geq K - 1 \tag{20}$$

Using the Ramsey-Euler equation for the initial stock $y = K$, we get:

$$u'(c(K)) = \delta \int_1^\beta u'(c(rh(x(K))))rh'(x(K))G(r)dr \tag{21}$$

Since $x(K) \leq 1$, we have $c(rh(x(K))) \leq rh(x(K)) \leq rh(1) \leq \beta h(1)$, so that:

$$u'(c(rh(x(K)))) \geq u'(\beta h(1)) \tag{22}$$

Note that $c(K) \geq (K-1) > 1$ and $\beta h(1) = (\beta B/2) \geq 2 > 1$. Using (22) in (21), and noting the definition of u in (15), we obtain:

$$\begin{aligned}
\frac{1}{(K-1)^\sigma} &\geq u'(c(K)) \geq \frac{\delta u'(\beta h(1))B}{(1+x(K))^2} \int_1^\beta rG(r)dr \\
&\geq \frac{\delta B}{(\beta B/2)^\sigma (1+x(K))^2} \int_1^\beta rG(r)dr \\
&\geq \frac{\delta B(\beta+1)}{8(\beta B/2)^\sigma}
\end{aligned} \tag{23}$$

where the last line of (23) used the facts that $x(K) \leq 1$, and $\int_1^\beta rG(r)dr = (\beta+1)/2$. Using the definition of K in (12), the inequality in (23) yields:

$$\frac{(\beta B/2)^\sigma}{(\beta B - 2)^\sigma} \geq \frac{\delta B(\beta+1)}{8} \tag{24}$$

Since $\beta B \geq 4$ by (19), we have $(\beta B/2) \leq (\beta B - 2)$. Using this in (24), we obtain:

$$(8/\delta B) \geq (\beta + 1)$$

which contradicts the choice of β made in (19). This establishes our claim that there is some $y \in (0, K)$, such that $x(y) = 1$.

We now proceed to the second part of our demonstration. For this purpose we assume that, in addition to (19), β is sufficiently large to satisfy:

$$\beta^{1-\sigma} > 8(2^\sigma)/(\delta B) \tag{25}$$

We claim that for the $y \in (0, K)$ for which $x(y) = 1$, we must have:

$$h(x(y)) > y$$

Suppose, on the contrary that we have $h(x(y)) \leq y$. We use the Ramsey-Euler equation to obtain:

$$u'(h(x(y)) - x(y)) \geq u'(y - x(y)) = u'(c(y)) = \delta \int_1^\beta u'(c(rh(x(y))))rh'(x(y))G(r)dr \tag{26}$$

Note that we have $c(rh(x(y))) \leq rh(x(y)) = rh(1) \leq \beta h(1)$, so (26) yields:

$$u'(h(1) - 1) \geq \delta u'(\beta h(1))h'(1) \int_1^\beta rG(r)dr = \delta u'(\beta h(1))h'(1)[(\beta+1)/2] \tag{27}$$

Note that $h(1) - 1 = (B/2) - 1 > 1$, and $\beta h(1) > h(1) = (B/2) > 1$, so by the definition of u in (15), (27) can be rewritten as:

$$\frac{(h(1)-1)^\sigma}{(\beta h(1))^\sigma} \leq \frac{2}{\delta h'(1)(\beta+1)} = \frac{8}{\delta B(\beta+1)} \tag{28}$$

We observe that $\beta h(1) = \beta[h(1)-1] + \beta \leq \beta[h(1)-1] + \beta[h(1)-1]$, since $[h(1)-1] = (B/2) - 1 \geq 1$. Thus, $\beta h(1) \leq 2\beta[h(1)-1]$, and so (28) yields:

$$\frac{1}{(2\beta)^\sigma} \leq \frac{8}{\delta B(\beta+1)} \tag{29}$$

That is, $(2\beta)^\sigma \geq \delta B(\beta+1)/8 \geq [\delta B\beta/8]$, so that:

$$\beta^{1-\sigma} \leq 8(2^\sigma)/(\delta B)$$

But this contradicts the choice of β in (25), and establishes our claim.

5.3 A Lower Bound on the Propensity to Consume

In this subsection, we want to obtain a positive lower bound on the propensity to consume for low resource stocks. This is an intermediate step to getting the result (in the next subsection) that the resource stock declines under the worst shock when the initial resource stock is low.

Since $B > 1$, we can find $\lambda \in (0, 1)$ such that $B\lambda > 1$. Define

$$\eta = \left[\frac{(1-\delta)}{2}\right]^{1/(\theta-1)} (1-\lambda), \hat{y} = \min\{1, B\lambda - 1, \frac{1}{(1-\lambda)\{\delta M(\theta-1)\}^{1/(\theta-1)}}\} \tag{30}$$

We claim now that:

$$[c(y)/y] \geq \eta \text{ for all } y \in (0, \hat{y}) \tag{31}$$

Note by (FE) that for all $y \in (0, \hat{y})$, we have:

$$\begin{aligned} V(y) &\geq u((1-\lambda)y) + \delta \int_1^\beta V(rh(\lambda y))G(r)dr \\ &\geq u((1-\lambda)y) + \delta V(h(\lambda y)) \int_1^\beta G(r)dr \\ &\geq u((1-\lambda)y) + \delta V(y) \end{aligned} \tag{32}$$

The last line of (32) follows from the facts that V is increasing and that:

$$h(\lambda y) = \frac{B\lambda y}{1+\lambda y} > y \tag{33}$$

where we have used the fact that $y < \hat{y} \leq B\lambda - 1$. Thus, (32) yields the inequality:

$$V(y) \geq \frac{u((1-\lambda)y)}{(1-\delta)} \text{ for all } y \in (0, \hat{y}) \tag{34}$$

Suppose, contrary to (31), there is some $y \in (0, \hat{y})$, such that $[c(y)/y] < \eta$. Then, by (FE),

$$\begin{aligned} V(y) &= u(c(y)) + \delta \int_1^\beta V(rh(y - c(y)))G(r)dr \\ &\leq u(c(y)) \\ &< u(\eta y) \end{aligned} \tag{35}$$

the second line of (35) following from the fact (recall (16)) that $u(c) \leq 0$ for all $c \in X = [0, K]$, and $0 < y < \hat{y} \leq 1 < K$. Combining (34) and (35),

$$\frac{u((1-\lambda)y)}{(1-\delta)} < u(\eta y) \text{ for some } y \in (0, \hat{y}) \tag{36}$$

Note that $(1-\lambda)y < 1$, and $\eta y < 1$, so by (15) and (36),

$$-\frac{[(1-\lambda)y]^{1-\theta}}{(\theta-1)(1-\delta)} - \frac{M}{(1-\delta)} < -\frac{[\eta y]^{1-\theta}}{(\theta-1)} - M$$

This can be simplified to obtain:

$$\frac{[\eta y]^{1-\theta}}{(\theta-1)} - \frac{[(1-\lambda)y]^{1-\theta}}{(\theta-1)(1-\delta)} < \frac{\delta M}{(1-\delta)} \tag{37}$$

Since $\eta^{1-\theta} = [2/(1-\delta)](1-\lambda)^{1-\theta}$, we can use (37) to get:

$$\frac{[(1-\lambda)y]^{1-\theta}}{(\theta-1)(1-\delta)} < \frac{\delta M}{(1-\delta)} \tag{38}$$

Then, we obtain $y^{\theta-1} > 1/[(1-\lambda)^{\theta-1}\delta M(\theta-1)]$, which implies that $\hat{y} > 1/[(1-\lambda)\{\delta M(\theta-1)\}^{1/(\theta-1)}]$, contradicting the definition of $\hat{y}$ in (30). This establishes our claim (31).

5.4 Decline for Low Resource Stocks

In this final subsection, we establish that when the initial resource stock y is low (the term "low" will be defined precisely below), optimal resource stocks will decline under the worst shock. That is, we demonstrate that (DWS') holds.

In addition to the restrictions on β given in (19) and (25), we now also assume that β is large enough so that:

$$\frac{\ln\beta}{(\beta-1)} < \frac{\eta^{\theta}}{\delta B} \tag{39}$$

where η is given in (30). Note that since $[\ln\beta/(\beta-1)] \to 0$ as $\beta \to \infty$, (39) can always be satisfied by taking β to be large enough.

We define:

$$y'' = (\hat{y}/\beta) \tag{40}$$

where $\hat{y}$ is defined in (30).

We now claim that:

$$h(x(y)) < y \text{ for all } y \in (0, y'') \tag{41}$$

where y'' is defined in (40). Suppose, contrary to (41), there is some $y \in (0, y'')$, such that $h(x(y)) \geq y$. Using the Ramsey-Euler equation, we have:

$$u'(c(y)) = \delta \int_1^{\beta} u'(c(rh(x(y))))rh'(x(y))G(r)dr \tag{42}$$

Since $h(x(y)) \geq y$, we have $rh(x(y)) \geq ry$ for all $r \in A$, and by Lemma 1, $c(rh(x(y))) \geq c(ry)$ for all $r \in A$. Using this in (42), we obtain:

$$u'(c(y)) \leq \delta \int_1^{\beta} u'(c(ry)rh'(x(y))G(r)dr \tag{43}$$

which can be rewritten as:

$$\frac{(\beta-1)}{\delta h'(x(y))} \leq \int_1^{\beta} \frac{u'(c(ry))}{u'(c(y))} rdr \tag{44}$$

Note that $c(y) \leq y < y'' < \hat{y} < 1$ and $c(ry) \leq ry \leq \beta y < \beta y'' = \hat{y} \leq 1$, so (15) can be used with (44) to obtain:

$$\frac{(\beta - 1)}{\delta h'(x(y))} \leq \int_1^\beta \frac{c(y)^\theta}{c(ry)^\theta} r dr \tag{45}$$

Note that $ry \leq \beta y < \beta y'' = \hat{y}$, so that $c(ry) \geq \eta ry$ for all $r \in A$ by (31). Since $c(y) \leq y$, we can now obtain from (45):

$$\frac{(\beta - 1)}{\delta h'(x(y))} \leq \int_1^\beta \frac{y^\theta}{\eta^\theta (ry)^\theta} r dr = \frac{1}{\eta^\theta} \int_1^\beta \frac{1}{r^{\theta - 1}} dr \leq \frac{1}{\eta^\theta} \int_1^\beta \frac{1}{r} dr = \frac{\ln \beta}{\eta^\theta} \tag{46}$$

where the second inequality in (46) follows from $\theta \geq 2$. Since $h'(x(y)) \leq B$, we then obtain the inequality:

$$\frac{\eta^\theta}{\delta B} \leq \frac{\ln \beta}{(\beta - 1)}$$

This contradicts the choice of β in (39) and establishes our claim (41).

6 Concluding Remarks

The existence result presented in Section 5 leads to several open questions, even if we keep basically within the confines of the example described in Section 5.1. First, as mentioned at the outset of Section 5, instead of a (smooth) pasting together of two CRRA functions, it would be useful to see how the intuition obtained from this crude approximation carries over to a family of decreasing relative risk aversion utility functions.

Second, the existence result is non-constructive. It would be appropriate to obtain a minimum safe standard of conservation explicitly in terms of the parameters of the model.

Third, the asymptotic properties of the stochastic process (SP), when both (DWS') and (GWS) hold, are unknown to us. In particular, we do not know whether $\Pr\{\omega : \lim_{t \to \infty} \inf y_t(y, \omega) = 0\}$ is positive when y is positive but below the minimum safe standard of conservation.

7 Appendix

Proof of Proposition 1. Pick an arbitrary $\rho \in (\rho', \infty)$ and fix it in what follows. Consider now the stochastic optimal growth model. It has an optimal consumption policy function, which we will denote by $c(y)$ in what follows (since the stochastic optimal growth model has now been fixed).

Suppose, contrary to (9), that there is some $y > 0$, such that:

$$h(y - c(y)) \geq y$$

Note that for $y > K \equiv B - 1$, we have:

$$h(y - c(y)) \leq h(y) < y$$

Thus, we have:

$$z \equiv \sup\{y > 0 : h(y - c(y)) \geq y\} \tag{LFP}$$

to be well-defined, and by continuity of h and c, we have:

$$h(z - c(z)) \geq z$$

On the other hand, since the definition of z in (LFP) entails that:

$$h(y - c(y)) < y \text{ for all } y > z$$

continuity of h and c also imply that:

$$h(z - c(z)) \geq z$$

Thus, we must have:

$$h(z - c(z)) = z \tag{A1}$$

That is, z is a fixed point of the map $H(y, 1) \equiv h(y - c(y))$, and indeed is the largest fixed point of $H(y, 1)$.

We now break up the proof into several steps.

Step 1: (Ramsey -Euler equation with z as initial stock)

With initial stock z, the Ramsey-Euler equation is:

$$u'(c(z)) = \delta \int_1^\beta rh'(z - c(z))u'(c(rh(z - c(z)))G(r)dr$$

where G denotes the density of F on $[1, \beta]$. Using (A1), this can be rewritten as:

$$\frac{1}{\delta h'(z - c(z))} = \int_1^\beta r\frac{u'(c(rz))}{u'(c(z))}G(r)dr \tag{A2}$$

Most of the subsequent steps will concentrate on this form of the Ramsey-Euler equation, because it helps us to focus on the MRS between present and future consumption (inside the integral) on the right-hand side.

Step 2: [An upper bound on the largest fixed point of $H(\cdot, 1)$]

Using the Ramsey-Euler equation (A2), and noting that $c(rz) \geq c(z)$ for all $r \in [1, \beta]$, we get:

$$\frac{1}{\delta h'(z - c(z))} \leq \int_1^\beta rG(r)dr = (Er) \tag{A3}$$

since u' is decreasing in its argument. Using the form of h, this yields:

$$\frac{B}{[1 + (z - c(z))]^2} = h'(z - c(z)) \geq \frac{1}{\delta E(r)}$$

so that:

$$(z - c(z)) \leq [\delta E(r)B]^{\frac{1}{2}} - 1 \equiv \eta \tag{A4}$$

Thus, we obtain the following upper bound on z :

$$z = h(z - c(z)) \leq h(\eta) = \frac{B\eta}{(1+\eta)} \tag{A5}$$

the third inequality in (A5) following from (A4), and the fact that h is increasing in its argument.

The upper bound on z in (A5) has the implication that for all $r \in [1, \beta]$,

$$rz \leq \frac{\beta B\eta}{(1+\eta)} = \frac{\beta B\eta}{(1+\eta)(B-1)}(B-1) < (B-1) \tag{A6}$$

because of the restriction (7).

Step 3: [Estimating the MRS in (A2)]

We now introduce the inverse function of h, call it J. For $y \in [0, B)$, the function:

$$J(y) = \frac{y}{B-y}$$

is well-defined and is the inverse function of h.

For $r \in [1, \beta]$, we have $rz \in (0, B-1)$ by (A6), and so:

$$J(rz) = \frac{rz}{B - rz} \tag{A7}$$

This yields:

$$\frac{J(rz)}{rz} = \frac{1}{B - rz}$$

and so:

$$1 - \frac{J(rz)}{rz} = 1 - \frac{1}{B - rz} = \frac{(B-1) - rz}{B - rz} \tag{A8}$$

Since $h(z - c(z)) = z$, we have $z - c(z) = h^{-1}(z) = J(z)$, and so:

$$\frac{c(z)}{z} = 1 - \frac{J(z)}{z} = \frac{(B-1) - z}{B - z} \tag{A9}$$

For $r \in (1, \beta]$, we have $rz \in (z, B-1)$, and so $h(rz - c(rz)) < rz$. Thus, $rz - c(rz) < h^{-1}(rz) = J(rz)$, and:

$$\frac{c(rz)}{rz} > 1 - \frac{J(rz)}{rz} = \frac{(B-1) - rz}{B - rz} \tag{A10}$$

by using (A8). Note that the right hand side expression in (A10) is positive for all $r \in (1, \beta]$.

Using (A9) and (A10), we can proceed to estimate the MRS in (A2) as follows. We have by the form of u :

$$\frac{u'(c(rz))}{u'(c(z))} = \frac{(c(z))^\rho}{(c(rz))^\rho}$$

and by (A9) and (A10),

$$\frac{rc(z)}{c(rz)} \leq \left[\frac{(B-1)-z}{B-z}\right] / \left[\frac{(B-1)-rz}{B-rz}\right] \equiv \phi(r,z) \tag{A11}$$

Now, we need to calculate the right-hand side of (A11), tedious though it might be. We have:

$$\begin{aligned}\phi(r,z) &= \frac{[(B-1)-z][B-rz]}{[B-z][(B-1)-rz]} \\ &= \frac{[B(B-1)-Bz-(B-1)rz+rz^2]}{[B(B-1)-Brz-(B-1)z+rz^2]} \\ &= 1+\frac{(r-1)z}{[B(B-1)-Brz-(B-1)z+rz^2]} \end{aligned} \tag{A12}$$

To estimate the last line of the right hand side of (A12), we write:

$$B(B-1)-Brz-(B-1)z+rz^2 = B(B-1)-B(1+r)z+z+rz^2] \tag{A13}$$

and note that, by using (A5),

$$\begin{aligned} B(1+r)z &\leq [B(1+\beta)]\left[\frac{B\eta}{(1+\eta)(B-1)}\right](B-1) \\ &= \left[\frac{B(1+\beta)\eta}{(1+\eta)(B-1)}\right]B(B-1) \\ &= \mu B(B-1) \end{aligned} \tag{A14}$$

Using (A14) in (A13), we obtain:

$$\begin{aligned} B(B-1)-Brz-(B-1)z+rz^2 &= B(B-1)-B(1+r)z+z+rz^2 \\ &\geq (1/\mu)B(1+r)z-B(1+r)z+z+rz^2 \\ &= [(1-\mu)/\mu]B(1+r)z+z+rz^2 \\ &\geq z\{1+[(1-\mu)/\mu]B(1+r)\} \\ &\geq z\{1+[(1-\mu)/\mu]2B\} \end{aligned} \tag{A15}$$

Now, using (A15) in (A12), we get:

$$\phi(r,z) \leq 1+\theta(r-1) \tag{A16}$$

where:

$$\theta = \frac{1}{\{1+[(1-\mu)/\mu]2B\}} \tag{A17}$$

Clearly, $\theta \in (0,1)$.

Returning now to the estimation of MRS in (A2), we get by using (A11) and (A16):

$$\frac{u'(c(rz))}{u'(c(z))} = \frac{(c(z))^\rho}{(c(rz))^\rho} = \frac{[\phi(r,z)]^\rho}{r^\rho} \leq \left[\frac{1+\theta(r-1)}{r}\right]^\rho \tag{A18}$$

Step 4: [The Ramsey-Euler Inequality]

Using the Ramsey-Euler equation (A2) and (A18), we obtain the Ramsey-Euler inequality:

$$\begin{aligned}\frac{1}{\delta h'(z-c(z))} &\leq \left[\frac{\beta}{\beta-1}\right]\int_1^\beta \frac{u'(c(rz))}{u'(c(z))}dr \\ &\leq \left[\frac{\beta}{\beta-1}\right]\int_1^\beta \left[\frac{1+\theta(r-1)}{r}\right]^\rho dr \qquad \text{(A19)}\end{aligned}$$

Step 5: [Evaluating an Integral]

The idea now, given Step 4, is to show that the right hand side of (A16) is actually less than the left-hand side, when ρ exceeds the specified ρ'.This would be a contradiction to (A19) and would establish the Proposition.

To this end, we have to evaluate the integral appearing in the right-hand side of (A19). Let us denote $(r-1)$ by s for $r \in [1,\beta]$. Then, $s \in [0,\beta-1]$, and we have:

$$\begin{aligned}I &= \int_1^\beta \left[\frac{1+\theta(r-1)}{r}\right]^\rho dr \\ &= \int_0^{\beta-1} \left[\frac{1+\theta s}{1+s}\right]^\rho ds \\ &= \int_0^{\beta-1} \left[1-\frac{(1-\theta)s}{1+s}\right]^\rho ds \\ &\leq \int_0^{\beta-1} \left[1-\frac{(1-\theta)s}{\beta}\right]^\rho ds \qquad \text{(A20)}\end{aligned}$$

the last line following from the fact that $\theta \in (0,1)$, and $s \in [0,\beta-1]$. Denote $[(1-\theta)/\beta]$ by γ, and $t = (1-\gamma s)$ for $s \in [0,\beta-1]$. Since t is decreasing in s, and $\gamma(\beta-1) = [(\beta-1)(1-\theta)/\beta] \in (0,1)$, we have $t \in [1-\gamma(\beta-1),1]$, and:

$$\begin{aligned}\int_0^{\beta-1} \left[1-\frac{(1-\theta)s}{\beta}\right]^\rho ds &= \int_0^{\beta-1} [1-\gamma s]^\rho ds \\ &= (1/\gamma)\int_{1-\gamma(\beta-1)}^1 t^\rho dt \\ &= \left[\frac{1}{\gamma}\right]\left[\frac{1}{\rho+1}-\frac{(1-\gamma(\beta-1))^{\rho+1}}{\rho+1}\right] \\ &\leq \frac{1}{\gamma(\rho+1)} \qquad \text{(A21)}\end{aligned}$$

Using (A20) and (A21) in (A19), we obtain:

$$\frac{1}{\delta B} \leq \frac{1}{\delta h'(z-c(z))} \leq \left[\frac{\beta^2}{(\beta-1)(1-\theta)(\rho+1)}\right] \qquad \text{(A22)}$$

Transposing terms in (A22),

$$\rho \leq \left[\frac{\delta B\beta^2}{(\beta-1)(1-\theta)}\right] - 1 \equiv \rho' \tag{A23}$$

which contradicts the fact that the RRA (ρ) of the optimal growth model exceeds ρ'. ■

Proof of Proposition 2. Denote $(y - c(y))$ by $x(y)$ for $y \geq 0$.Then, there exists a sequence $\{y_n\}$, with $y_n > 0$ for all $n \in \mathbb{N}$, and $y_n \to 0$ as $n \to \infty$, such that

$$h(x(y_n)) < y_n \text{ for all } n \in \mathbb{N} \tag{A24}$$

Denote $x(y_n)$ by x_n for all $n \in \mathbb{N}$. Then, $x_n \in (0, y_n)$, $h(x_n) < y_n$ for all $n \in \mathbb{N}$, so that:

$$u'(y_n - x_n) < u'(h(x_n) - x_n) \text{ for all } n \in \mathbb{N} \tag{A25}$$

The Ramsey-Euler equation, with initial stock y_n, is given by:

$$u'(y_n - x_n) = \delta \int_1^\beta [u'(rh(x_n) - x(rh(x_n)))rh'(x_n)]G(r)dr \tag{A26}$$

where G is the density of F on $[1, \beta]$.Using (A25) in (A26), we obtain for all $n \in \mathbb{N}$,

$$\begin{aligned} u'(h(x_n) - x_n) &> \delta \int_1^\beta [u'(rh(x_n) - x(rh(x_n)))rh'(x_n)]G(r)dr \\ &\geq \delta \int_1^\beta [u'(rh(x_n))rh'(x_n)]G(r)dr \end{aligned} \tag{A27}$$

so that:

$$\frac{1}{\delta} \geq \int_1^\beta r\frac{u'(rh(x_n))h'(x_n)}{u'(h(x_n) - x_n)}G(r)dr \text{ for all } n \in \mathbb{N} \tag{A28}$$

Given the form of G, we have:

$$\frac{(\beta-1)}{\delta} \geq \int_1^\beta r\frac{u'(rh(x_n))h'(x_n)}{u'(h(x_n) - x_n)}dr \text{ for all } n \in \mathbb{N} \tag{A29}$$

Pick any $\lambda \in (1/B, 1)$. Choose $\xi > 0$, such that for all $x \in (0, \xi) \equiv J$, we have:

$$[h(x)/x] \geq h'(x) \geq \lambda B \tag{A30}$$

Since $h'(x) \to B$ as $x \to 0$, it is always possible to choose such a ξ. Choose $n' \in \mathbb{N}$, such that $x_n \in J$ for all $n \geq n'$. Then, for $n \geq n'$,

$$\begin{aligned} h(x_n) - x_n &= h(x_n)[1 - \frac{x_n}{h(x_n)}] \\ &\geq h(x_n)[1 - \frac{1}{\lambda B}] = a(\lambda)h(x_n) \end{aligned} \tag{A31}$$

where $a(\lambda) \equiv \{[\lambda B - 1]/\lambda B\}$.Using (A31) in (A29),

$$\frac{(\beta-1)}{\delta} \geq \int_1^\beta r \frac{u'(rh(x_n))h'(x_n)}{u'(a(\lambda)h(x_n))} dr \text{ for all } n \geq n' \tag{A32}$$

Using (A30) and the form of the utility function, (A32) yields:

$$\frac{(\beta-1)}{\delta} \geq \int_1^\beta r \frac{[a(\lambda)]^\rho[\lambda B]}{r^\rho} dr \tag{A33}$$

This can be rewritten as:

$$\frac{(\beta-1)}{\delta[\lambda B][a(\lambda)]^\rho} \geq \int_1^\beta r^{1-\rho} dr \tag{A34}$$

Note that (A33) and (A34) are valid when $\rho \neq 1$, and also when $\rho = 1$.

Letting $\lambda \to 1$ in (A34), and denoting $a(1)$ by a, we get:

$$\frac{(\beta-1)}{\delta B a^\rho} \geq \int_1^\beta r^{1-\rho} dr \tag{A35}$$

When $\rho \neq 2$, we obtain from (A35),

$$\frac{(\beta-1)}{\delta B a^\rho} \geq \left[\frac{\beta^{2-\rho}-1}{(2-\rho)}\right]$$

and when $\rho = 2$, we obtain:

$$\frac{(\beta-1)}{\delta B a^\rho} \geq \ln \beta$$

This establishes the result. ■

Proof of Proposition 3. It is sufficient to prove the result for an economy, which belongs to a *subclass* of the class of economies $E \equiv (B, \beta, \delta)$ satisfying the restriction (1). We define a class of economies as follows:

$$\left.\begin{array}{ll} B(n) = n+2 & \text{for } n \in \mathbb{N} \\ \beta(n) = 2 & \text{for } n \in \mathbb{N} \\ \delta(n) = 1/(n+1) & \text{for } n \in \mathbb{N} \end{array}\right\}$$

This allows us to describe each economy in the class in terms of a single parameter $n \in \mathbb{N}$. Since β is fixed at 2 for all n, we have $E(r(n))$,the expected value of $r(n)$, also fixed for all $n \in \mathbb{N}$, and it is legitimate to write simply $E(r) = (3/2)$.

Notice that for all $n \in \mathbb{N}$, we have:

$$\delta(n)E(r)B(n) = \frac{3(n+2)}{2(n+1)}$$

and so:

$$\eta(n) = \sqrt{\delta(n)E(r)B(n)} - 1 = \sqrt{\frac{3(n+2)}{2(n+1)}} - 1$$

Thus, $\eta(n) \to \eta'$ as $n \to \infty$, where:

$$\eta' = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$$

Then, we have:

$$\frac{\eta'}{(1+\eta')} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}$$

and consequently:

$$\mu(n) \equiv \frac{(1+\beta(n))\eta(n)B(n)}{(\eta(n)+1)(B(n)-1)} \to \sqrt{3}[\sqrt{3}-\sqrt{2}] = 3-\sqrt{6} \equiv \mu' \text{ as } n \to \infty$$

Since $\mu' \in (0,1)$, there is $N \in \mathbb{N}$, such that for all $n \geq N$, $\mu(n) < \mu'' \equiv 3-\sqrt{5} < 1$. Denote $\{n \in \mathbb{N} : n \geq N\}$ by $\mathbb{M}$. Henceforth, we restrict our attention to the *subclass* of economies $\{B(n), \beta(n), \delta(n)\}_{n\in\mathbb{M}}$. Every economy in this subclass satisfies restriction (1) and $\mu(n) < \mu'' \in (0,1)$. Thus, we have $\theta(n) \in (0,1)$ for all $n \in \mathbb{M}$, and:

$$\theta(n) = \frac{1}{\{1+[(1-\mu(n))/\mu(n)]2B(n)\}} \to 0 \text{ as } n \to \infty$$

Suppose the proposition is false. Then, for every $n \in \mathbb{M}$, there is $y(n) > 0$ such that the optimal consumption policy function, c_n, corresponding to n, satisfies:

$$h_n(y - c_n(y)) \geq y \text{ for } y = y(n) \tag{A36}$$

Note that for $y > K(n) \equiv B(n) - 1$, we have:

$$h_n(y - c_n(y)) \leq h_n(y) < y$$

Thus, we have:

$$z(n) \equiv \sup\{y > 0 : h_n(y - c_n(y)) \geq y\} \tag{A37}$$

to be well-defined, and by continuity of h_n and c_n, we have:

$$h_n(z - c_n(z)) \geq z \text{ for } z = z(n)$$

On the other hand, since the definition of $z(n)$ in (A37) entails that:

$$h_n(y - c_n(y)) < y \text{ for all } y > z(n)$$

continuity of h_n and c_n also imply that:

$$h_n(z - c_n(z)) \leq z \text{ for } z = z(n)$$

Thus, we must have:

$$h_n(z - c_n(z)) = z \text{ for } z = z(n) \tag{A38}$$

That is, $z(n)$ is a fixed point of the map $H_n(y,1) \equiv h_n(y - c_n(y))$, and indeed is the largest fixed point of $H_n(y,1)$.

We now break up the next part of the proof into four steps. For this purpose, we will fix $n \in \mathbb{M}$, and denote $A(n)$ by A, $\delta(n)$ by δ, $\beta(n)$ by β, $\eta(n)$ by η, $\mu(n)$ by μ, $\theta(n)$ by θ, h_n by h, g_n by g, H_n by H and $z(n)$ by z. This is merely to ease the writing, and the reader should be able to check that the dependence of the parameters, functions and fixed point (respectively) on n do not destroy the validity of the four steps. The four steps are merely exact repetitions of the four steps of the proof in Proposition 1, and they lead to the inequality:

$$\frac{(\beta - 1)}{\delta B} \leq \int_1^{\beta} \frac{[1 + \theta(r-1)]^{\rho}}{r^{\rho - 1}} dr \tag{A39}$$

Note that the four steps are valid for an arbitrary $n \in \mathbb{M}$. Now, we explicitly recognize the dependence of the parameters in (A39) on n, and write:

$$\frac{(\beta(n) - 1)}{\delta(n)B(n)} \leq \int_1^{\beta(n)} \frac{[1 + \theta(n)(r-1)]^{\rho}}{r^{\rho - 1}} dr$$

for all $n \in \mathbb{M}$. Using the fact that $\beta(n) = 2$ for all $n \in \mathbb{M}$, we have:

$$\frac{1}{\delta(n)B(n)} \leq \int_1^{2} \frac{[1 + \theta(n)(r-1)]^{\rho}}{r^{\rho - 1}} dr \tag{A40}$$

Letting $n \to \infty$, we see that $\delta(n)B(n) \to 1$, and $\theta(n) \to 0$, so (by the bounded convergence theorem, Cunningham (1967, p.180)) we obtain the crucial inequality:

$$1 \leq \int_1^{2} \frac{1}{r^{\rho - 1}} dr \equiv I \tag{A41}$$

However, the integral on the right-hand side of (A41) is clearly < 1 (since $\rho > 1$) and this contradiction establishes the result. ■

Proof of Proposition 4. Suppose $\rho \in (0,1)$ and there is some $E \equiv (B, \beta, \delta)$ satisfying the restriction (1), such that the optimal consumption policy function, c_E, corresponding to E, satisfies:

$$h_E(y - c_E(y)) < y \ \text{ for all } y > 0 \tag{A42}$$

We fix this E in what follows, and so we drop the subscript E and write h and c for h_E and c_E respectively. Thus, (A42) is written as:

$$h(y - c(y)) < y \ \text{ for all } y > 0 \tag{A43}$$

Consider the deterministic model corresponding to E. This is specified by (B, δ) and ρ. Since restriction (1) is satisfied, there is a unique solution $k \in (0, \infty)$ satisfying $\delta h'(k) = 1$. Denote the optimal consumption for the deterministic model by $\gamma(y)$ for $y \geq 0$. Then, we have:

$$h(y - \gamma(y)) > y \ \text{ for all } y \in (0, k) \tag{A44}$$

and $h(k - \gamma(k)) = k$. Comparing (A43) and (A44), we have:

$$\gamma(y) < c(y) \text{ for all } y \in (0, k] \tag{A45}$$

Denote by V the value function for the stochastic problem, given (B, β, δ) and ρ; denote by W the value function for the deterministic problem, given (B, δ) and ρ. Both functions are continuous on $\mathbb{R}_+$ and continuously differentiable on $\mathbb{R}_{++}$. Further, $V(0) = W(0) = 0$.

By the envelope theorem (see Mirman and Zilcha(1975, Lemma 1, p.332)), we have:

$$\left.\begin{array}{r} (i)\ V'(y) = u'(c(y)) \text{ for all } y > 0 \\ (ii)\ W'(y) = u'(\gamma(y)) \text{ for all } y > 0 \end{array}\right\} \tag{A46}$$

Thus, using (A45) we must have:

$$V'(y) < W'(y) \text{ for all } y \in (0, k] \tag{A47}$$

Pick any $b \in (0, k)$, and define:

$$\xi \equiv \int_b^k [W'(y) - V'(y)]dy \tag{A48}$$

By (A47), we have $\xi > 0$, and further:

$$\int_a^k [W'(y) - V'(y)]dy \geq \xi \text{ for all } a \in (0, b] \tag{A49}$$

Applying the Second Fundamental Theorem of Calculus (see Goldberg (1964, p.186)) for each $a \in (0, b]$, we get:

$$[W(k) - W(a)] - [V(k) - V(a)] = \int_a^k [W'(y) - V'(y)]dy \geq \xi$$

and this can be written as:

$$W(k) - V(k) \geq \xi + W(a) - V(a) \text{ for all } a \in (0, b] \tag{A50}$$

Letting $a \to 0$ in (A50) and noting the continuity of V and W on $\mathbb{R}_+$ and $V(0) = W(0) = 0$, we obtain:

$$W(k) - V(k) \geq \xi > 0 \tag{A51}$$

But, clearly $V(k) \geq W(k)$, since the deterministic optimal consumption function γ can be used in the stochastic case, and will yield at least as much consumption in every period (regardless of the value of r) as in the deterministic model, since $r \in [1, \beta]$, and γ is an increasing function. This contradicts (A51) and establishes the result. ■

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