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Knightian Uncertainty and Endogenous Growth Key Ingredients to Climate Modeling*

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Abstract

This paper models Knightian uncertainty in an Integrated Assessment Model (IAM) with a carbon-intensive and a carbon-neutral sector and analyzes implications for the optimal allocation between the sectors. The main contributions of the paper are firstly that it introduces a robust control approach in an IAM which differs from existing 'risk analysis' in IAMs which forces the model builder to make a guess at the probability distributions of the uncertain parameters. The paper also introduces endogenous growth in two rather than one sector.

The household uses an optimal rule to update the climate model it is using to observed changes in climate data over time. A precautionary preference makes the household concerned about sudden increases in mean temperature that might imply that its assumed climate sensitivity was underestimated. This results in a more stringent policy as an insurance against possible realizations of high-impact low-probability outcomes.

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Keywords: climate change, integrated assessment models, differential game theory, Knightian uncertainty, robust control

JEL classification: C73, C61, Q54

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1 Introduction

Much of the current debate in climate economics has centered on the uncertainties surrounding future costs (Stern (2007), Weitzman (2007), Weitzman (2009*a*), Nordhaus (2009) and Yohe and Tol (2007). Climate change is subject to fundamental uncertainties concerning the underlying scientific information. The sources underlying this uncertainty span over a broad series of issues from converting emissions to atmospheric concentrations, converting concentrations to radiative forcing, modeling climate response to a given forcing, converting climate response into inputs for impact studies as well as uncertainty about the costs and efficacy of various measures for mitigation and adaptation . We focus here on uncertainty in one essential part of this chain: from concentrations to temperature. The summary measure of this is called the the climate sensitivity. The IPCC Executive Summary IPCC (2007*a*) stated *'The equilibrium climate sensitivity is a measure of the climate system response to sustained radiative forcing. It is... defined as the global average surface warming following a doubling of carbon dioxide concentrations. It is 'likely' to be in the range* $2.0^o C$ *to* $4.5^o C$ *with a best estimate of about* $3.0^o C$*, and is 'very unlikely' to be less than* $1.5^o C$*. Values substantially higher than* $4.5^o C$ *cannot be excluded, but agreement of models with observations is not as good for those values.'*

Moreover, we do not know what damages really depend on. It may well be that the rate of temperature change is itself quite important (we know that humans like many other species can live in both Patagonia and the Kalahari but that we are adversely affected by having too rapid change). Still, there is strong evidence that the frequency of storms, droughts, and the extent of sealevel rise, glacier melting and other factors that will directly cause distress are related to mean temperature rise and thus in climate-economics it is typically assumed damages are a function (e.g. quadratic) of the level or change in mean temperature. The physical relationship between gas concentrations, radiative forcing and temperature are also fairly well understood. However there is considerable uncertainty due to a large number of feedbacks. Thus the critical uncertainty is in many ways the climate sensitivity.

When faced with uncertainties, we typically make recourse to one of a number of methods. The first of these is sensitivity testing and the second is to assume the uncertain variable is random and has a probability distribution so that expected outcomes can be discussed, possibly with some correction for risk aversion. Having got this far, most researchers appear to have a natural preference for ease of calculation and choose to follow standard practice which is of course - as in so many other cases - to assume a normal distribution. Naturally a number of objections can be raised. The most fundamental of these is that this is not a true random number - just an unknown. There may be a relationship in the latest millions of years between gas content of the atmosphere and temperature (and assuming certain models this can be estimated statistically). However in the past climate composition and temperature co-evolved very slowly in reaction to other external forces. We do not know to what extent the current surge in climate gases is comparable.

As analysts we like to think we can master uncertainties. However, in all honesty, if we assume normal errors and little or no risk aversion then the expected utility results will not be very different from their certainty equivalents. It is presumably as a reaction to this fact that Weitzman (2009*a*) developed the 'dismal' theory saying that if events in the tail of the distribution are a) sufficiently serious and b) if these tails themselves are 'fat' then they cannot be disregarded - and in fact depending on definitions, their value may be arbitrarily or infinitely large. This is why the theorem is dismal. The dismal theorem has of course received its share of harsh criticism, essentially because we cannot deal with infinite numbers. We believe we need to incorporate the fundamental behavioral element of how people actually think about true uncertainty as distinct from risk when probabilities are known.

There is no generally accepted definition of a fat-tailed distribution. In general, however, it makes extreme events more likely. It may or may not mean that the sample mean fails to converge. Nordhaus has a pedagogical example of Weitzman's use of 'fat-tails' that compares the optimal building regulations that are tailored to the worst earthquake that is expected once in a hundred years and then compare this with the regulation to instead accommodate the worst earthquake expected once in two hundred years. Only if the difference is big (and does not converge) do we have a problem of strong tail dominance ('fat tails'). Building on work by Geweke (2001), Nordhaus goes on to show that the conditions for 'fat-tailed' dominance in Weitzman's case depend on not only the power of PDF for the error distribution but also the concavity of the utility function. His conclusion is that a combination of very 'fat' tails and very high risk aversion (concavity) are required and that this is unlikely.

As pointed out in Weitzman (2009*b*), the last quarter century, the carbon content has risen by roughly 40 ppm. In the geological record of the last 800 000 years there is no period of 10 000 years with such a big increase. Weitzman goes on to criticize the fact that many cost-benefit analyzes (CBAs) or 'Integrated assessment models' (IAMs) essentially use some form of expected utility framework that has relatively little effect on their final numbers meaning that the uncertainty involved did not per se make much difference. Ultimately the discussion is hard to resolve since it may be a matter of judgment whether the climatic change we are causing is very exceptional and very serious.

An analysis by Roe and Baker (2007) shows that the climate sensitivity probability distribution is highly sensitive to uncertainties in underlying physical feedback factors. Besides that the climate sensitivity probability distribution is uncertain in mean and variance, it also tends to be skewed with thicker high-temperature tails that are not likely to be reduced despite scientific progress in understanding (reducing variance in) underlying feedback factors. Roe and Baker (2007) do not therefore expect the climate sensitivity range presented in the next IPCC report to be different from that in the 2007 report despite scientific progress.

We present in this paper, an alternative approach which recognizes explicitly that we know neither the exact climate sensitivity nor the exact probability distribution of this climate sensitivity. (In fact the notion of knowing exactly the probability distribution for an unknown constant requires a stretch of the imagination). Instead we build on behavioral economics and posit a mental process for the decision maker facing true uncertainty. We refer to uncertainty as ´Knightian', when not even probabilities are known. The weight placed on the worst outcome could be influenced by the decision maker's concern about the magnitude of associated costs and maybe which probability is more or less likely. This mental process takes the form of a game between the decision maker who tries to optimize given his uncertainty and Fate or Nature which is an (imaginary) player that chooses the worst-case scenario for the decision maker. This is one intuitive image of the 'precautionary' behavior that explains why people pack umbrellas even if they do not expect rain on an outing. Instead of guessing probabilities (a futile exercise under Knightian uncertainty) they choose a robust strategy and circumvent the uncertainty by bringing the umbrella. We believe that this model has some resonance with behavioral economics and with introspection. We find that we are able to reproduce (by varying parameters of the game, such as the extent of attention paid to the minimizer) a number of special cases that include both expected utility maximization and infinity in the same variable as in the Dismal theory as special cases.

We believe Knightian uncertainty is more relevant in climate modeling than traditional concepts of expected utility theory and risk aversion, that rely on known probabilities. We also believe it captures better the reasoning behind the precautionary principle as in the Rio Declaration, which adopts a 'precautionary principle' in article 15: *'Where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation.'*

So far, Knightian uncertainty has been neglected in IAMs - such as the DICE model (Nordhaus, 1992), the PAGE model (Hope, 2003), also used by the Stern Review, and the FUND model (Tol, 1999) - mainly due to complexities that require demanding computer resources. Instead 'risk' has been analyzed by 'sensitivity and Monte-Carlo analyses'. But this still begs the question from which probability distributions one draws the input numbers. The result from a Monte Carlo analysis may be more obscure but obviously depends fully on the exact distribution of parameters chosen.

Two early examples based on extensions of DICE (Nordhaus, 1992) resulted in 2 to 4 times higher carbon cost than the certainty case, reflecting the benefit of reducing risk of high future climate change costs, see Schauer (1995) and Nordhaus and Popp (1997). Another example modeled catastrophic events by altering the probability distributions of damages as temperature increases (Nordhaus and Boyer, 2000). The Stern Review Stern (2007) uses the PAGE2002 (Hope, 2003) where several parameters are represented as probability distributions, to explore consequences of e.g. increasing climate sensitivities of 2.4^{o}C - 5.4^{o}C for the $5 - 95\%$ interval.

Knightian uncertainty was introduced in IAMs by Hennlock (2008*b*) and Hennlock (2009). Weitzman also introduced Knightian (deeper) uncertainty, though not in an IAM setting, in a draft to his Review of the Stern Report and then in an early working paper of Weitzman (2009*a*). The main results in Hennlock (2008*b*) and Weitzman (2009*a*) told the same story - uncertain probability distributions can justify large measures taken. In Hennlock (2008*b*) the results emerged as a 'shadow precautionary premium' inducing more stringent policy measures. When a policymaker expresses a *perfect* precautionary preference his expected shadow carbon cost becomes infinite, and hence, he stops all carbon-generating production. In Weitzman's analysis, based on a static linear relationship between a utility function and a parameter with unknown probability distribution, Bayesian learning in a two-period analysis results in an infinite expected marginal utility at zero consumption.[1]

In this paper we incorporate Knightian uncertainty into a two-goods IAM with two sectors: a carbon-intensive and a carbon-neutral one.[2] Another important modification is

[1]Nordhaus (2009) also commented on Weitzman (2009*a*) and how the the result can depend on fat tails in the (posterior) probability distribution.

[2]Sterner and Persson (2008) presented a two-good IAM presented in their comment on the Stern review but with consumption as a choice variable rather than a stock variable. They noted that introducing a relative price of a second (environmental) stock in DICE justifies a significantly more stringent policy than in the Stern Review.

that both of these are characterized by endogenous growth.

Instead of looking at an 'optimal' planner-type solution, the model is 'solved' by considering the choice for the representative household under Knightian uncertainty. This choice takes the shape of a game in which the consumer tries to maximize outcome against "Nature" which tries to minimize it. The choice the consumer faces is whether to slow global climate change by decreasing carbon content in consumption, increasing abatement or investing in carbon-neutral technology or (more efficient) carbon-intensive technology. The purpose is not to perform a simulation, but to present the essential analytical results in connection to the discussion following the Stern Review. The advantage of analytical solutions over computer-based simulations is that the former have better reliability and allow for a deeper understanding of the underlying mechanisms.

We find that deep uncertainty and precautionary preferences generate behavior that goes outside that predicted by the 'sensitivity-analysis approach' in IAM simulations. Among the main results we find that precaution to Knightian uncertainty changes the nature of climate model updating with respect to new climate information. New observations, for instance of high temperature or drought make the household concerned that the climate sensitivity is higher and thus it updates its beliefs that even greater global warming and climate impact might occur in the future from a given increase in CO2 concentration.

With a robust or precautionary approach to potential disaster we easily get an infinite shadow cost of carbon much like Weitzman's Dismal Theory but a striking result of our model is that this does not necessarily imply corner solutions in the switch from the carbon-intensive to carbon-neutral production sector. This makes our most drastic case somewhat easier to relate to than the equivalent in Weitzman's analysis which is a negatively infinite utility.

1.1 Ambiguity and Ambiguity Aversion

In the literature on decision theory, Knightian uncertainty is not new. Knight (1971) claimed that for many choices the assumption of known probability distributions is too strong and therefore distinguished between 'measurable uncertainty' (risk) and 'unmeasurable uncertainty', reserving the latter denotation to include also unknown probabilities. Unmeasurable uncertainty has later been named Knightian uncertainty, deeper uncertainty or simply uncertainty to distinguish it from risk.

Also Keynes (1921), in his treatise on probability, put forward the question whether we should be indifferent between two scenarios that have equal probabilities, but the first scenario has subjective probabilities while the second has objective probabilities. Savage's Sure-Thing principle (Savage, 1954) argued that we could, while Ellsberg's experiment (Ellsberg, 1961) showed that individuals facing two lotteries - the first one with known probabilities and the second one with unknown probabilities - tended to prefer to bet on outcomes in the first lottery than to bet on outcomes in the second lottery where they had to rely on subjective probabilities, thus contradicting the Sure-Thing principle. This behavior was referred to as ambiguity (or uncertainty) aversion as a broader aversion than risk aversion.

Since then ambiguity has been much studied in experimental research on decision making; Ellsberg's experimental setup has been repeated several times, supporting ambiguity aversion. In e.g. Fox and Tversky (1995) subjects were asked for their willingness to pay, resulting in much higher willingness to pay for the urn with known probabilities than for the ambiguous urn. Other experiments by e.g. Curley et al. (1986) showed that fear of negative evaluation when others observe the choice and may judge the decision-maker for it, increases his ambiguity aversion, which reminds us how social norms may affect policymaking, see also Trautmann et al. (2008).

One of the most influential ways to model aversion to ambiguity in the presence of Knightian uncertainty is by Gilboa and Schmeidler (1989) who formulated a maximin expected decision criterion, by weakening Savage's Sure-Thing Principle.[3] Instead, the decision-maker faces a set of probability distributions and maximizes expected utility under the belief that the worst-case probability distribution is true, which in effect implies assigning more weight to bad outcomes. That individuals tend to assign more weight to low-probability extreme outcomes than explained by expected utility was also supported empirically by Kahneman and Tversky (1979).

Henry (2006) provides two examples, the first is the link between bovine spongiform encephalopathy (BSE) in cows and Creutzfeld-Jacob Disease (CJD) in humans and the second is the link between asbestos and lung disease, to illustrate the relevance of a precautionary principle or aversion to ambiguity in the presence of uncertainty. Henry refers to work by Maccheroni et al. (2006) that provide a formal description of the precautionary principle and a generalization of the maximin decision criterion with multiplier preferences in Hansen and Sargent (2001). This maximin decision criterion has before been applied in dynamic models with applications to e.g. water management in Roseta-Palma and Xepapadeas (2004), climate change in Hennlock (2008*b*) and Hennlock (2009) and biodiversity management in Vardas and Xepapadeas (2008).[4]

This paper is structured as follows: Section 2 presents the main features of the IAM in a deterministic setting. Section 3 introduces Knightian uncertainty and multiple priors in radiative forcing and climate sensitivity in the climate modeling. Section 4 presents optimal polices under precautionary preferences and discusses the major outcomes which is followed by a concluding summary in 5. The appendix contains the analytical solution to the IAM.

2 The Integrated Climate-Economy Model

Consider a representative household problem as in Nordhaus' DICE model but modified and extended with a carbon-intensive and a carbon-neutral (environmental and ecosystem)

[3] The Choquet expected utility (CEU) model by Schmeidler (1989) is another example.

[4] The maximin criterion has also been applied in static models, e.g. Chichilnisky (2000) and Bretteville Froyn (2005)

production sector, both having endogenous technology growth in a similar setting as in the endogenous growth theory by Romer (1990). The inclusion of endogenous technical change in growth theory has been an important addition and is now becoming fairly standard in the literature.[5]

The household maximizes objective (1)

$$\max_{C,G,q,s,r} \int_0^\infty \frac{1}{1-\eta} \left[(1-\omega) C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} e^{-\rho t} dt \tag{1}$$

which describes how the carbon-intensive good C_t and the carbon-neutral good G_t compose the final good by a CES function with constant elasticity of substitution σ, elasticity of marginal utility of consumption η and share parameter $\omega \in [0, 1]$.[6] This objective function is also used in Sterner and Persson (2008) though in their model the second argument is a stock variable rather than intertemporal consumption in terms of a choice variable as in (1).

The household maximizes objective (1) subject to the dynamics of the economic-climatic system (2) - (9), choosing carbon-intensive consumption C_t, carbon-neutral consumption G_t, abatement effort q_t, and research efforts r_t and s_t in the carbon-intensive and the carbon-neutral research sector, respectively. A list of all 31 model parameters of the model equations (1) - (9) is found in appendix A.2.

2.1 Carbon-Intensive Production Sector

Carbon-intensive capital accumulation is described by the carbon-intensive capital growth equation (2). Carbon-intensive capital K_t, accumulates by the production function $Y_{Kt} \equiv A_{Kt}^{\tau} K_t^{\alpha} L_t^{1-\alpha}$ minus research expenditure $r_t Y_{Kt}$ with research effort $r_t \in [0, 1]$, consumption of carbon-intensive good C_t, abatement cost and depreciation. Applying the polluter-pays-principle, the carbon-intensive sector pays for abatement effort q_t with a quadratic cost

[5] For example, Acemoglu et al. (2009) apply endogenous growth theory to climate change.

[6] See Hoel and Sterner (2007) for a discussion on how the CES function affects the so-called Ramsey-rule.

function due to capacity constraints as more effort is employed.

$$dK = \Big[(1 - r_t)A_{Kt}^{\tau}K_t^{\alpha}L_t^{1-\alpha} - cq_t^2 - C_t - \delta K_t\Big]\, dt \tag{2}$$

$$dA_K = \Big[\nu(r_tY_{Kt})^{\tau}A_{Kt}^{1-\tau} - \delta_K A_{Kt}\Big]\, dt \tag{3}$$

Carbon-intensive technology A_K in (3) increases carbon efficiency (greater output for a given amount of carbon used) and develops endogenously with research effort $r_t \in [0, 1]$ and the current technology A_{Kt} as inputs in the research process. Thus a research sector that has generated many ideas in its history has an advantage in generating new ideas relative to research sectors in less developed regions. The restriction $0 < \tau < 1$ in (3) suggests that it requires more than a doubling of researchers in order to double the number of ideas as researchers may come up with the same ideas see (Romer, 1990). The implementation of new discoveries in the production process, implies that some of the old knowledge cannot be used in the current production process. This imperfect substitution of knowledge over time is reflected by $\delta_K \geq 0$.

2.2 Carbon-Neutral Production Sector and Climate Models

The carbon-neutral consumption good G_t is produced by using carbon-neutral capital which can be considered as environmental or ecosystem capital E_t. The accumulation of environmental capital follows (4) which describes a technology-enhanced growth function $Y_{Et} \equiv A_{Et}^{\psi}E_t^{\phi}$ minus research expenditure s_tY_{Et} with research effort $s_t \in [0, 1]$, decay rate, climate impact and consumption of carbon-neutral good G_t.

$$dE = \Big[(1 - s_t)A_{Et}^{\psi}E_t^{\phi} - \frac{1}{\kappa}E_t - \Phi(T_t - T_0)E_t^{\phi} - \pi G_t\Big]\, dt \tag{4}$$

$$dA_E = \Big[\beta(r_t Y_{Et})^{\psi} A_{Et}^{1-\psi} - \delta_E A_{Et}\Big]\, dt \tag{5}$$

Carbon-neutral technology A_E, which develops endogenously in (5), improves carbon-neutral capital growth (and raises carrying-capacity), thus counteracting a negative impact from temperature increases in (4). Carbon-neutral technology A_{Et} can then also be seen as adaption technology where the parameter $0 < \psi < 1$ in (5) is a restriction on this technology's progress.

Climate impacts are often on natural capitals, such as agriculture, forestry, water resources, dry- and wetland (IPCC, 2007*b*), so we let natural capital E_t be damaged by an 'increasing-damage-to-scale' Cobb-Douglas function in (4) adopted from Hennlock (2005) and Hennlock (2008*a*) with Φ as a climate impact parameter.[7] A given mean temperature increase leads to a greater total damage (or gain for $\Phi < 0$) the greater is the capital stock.

Equations (6) - (9) adopt a modified continuous-time version of the climate model once used in DICE. The CO_2 evolution M_t in (6) is determined by input factors in K_t and L_t less abatement effort q_t and natural assimilation Ω. A modified equation for the radiative forcing R_t is described in (7).[8] The atmospheric mean temperature T_t and deep ocean mean temperature $\tilde{T}_t$ are described in (8) and (9), respectively.

$$dM = \Big[\epsilon\varphi K_t^{\alpha} L_t^{1-\alpha} - \mu q_t - \Omega M_t\Big]\, dt \tag{6}$$

$$R_t = \lambda_0 \frac{\sqrt{M_t/M_0}}{\sqrt{2}\gamma} + \hat{\lambda}_0 \frac{M_t/M_0}{2\gamma} \tag{7}$$

[7]Solutions are possible also when letting physical capital carry impact of climate change. However, separating stocks to damaging (physical) capital and damaged (environmental) capital, makes a unique solution possible corresponding to the verified value function.

[8]For analytical tractability of the Isaacs-Bellman-Flemming equation, we approximate the radiative forcing equation $R_t = \frac{\lambda_0 ln(M_t/M_0)}{ln(2)}$ used in e.g. DICE by the square-root approximation in (7) where γ can be calibrated to fit R_t.

$$dT = \frac{1}{\tau_1}\left(\lambda_2 R_t + O_t dt - \lambda_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)dt\right) \quad (8)$$

$$d\tilde{T} = \frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right)dt \quad (9)$$

The parameter λ_2 is essential for equilibrium climate sensitivity, τ_1 is the thermal capacity of atmosphere and upper ocean and τ_3 is the thermal capacity of deep ocean. $1/\tau_2$ is the transfer rate from the atmosphere and upper ocean layer to the deep ocean layer.[9]

3 Introducing Multiple Priors in Climate Modeling

In temperature equation (8) there are mainly three sources to uncertainty in probabilities over temperature outcomes that have been considered in IPCC (2007*a*) and the IAM literature - the radiative forcing parameter λ_0 in (7), the climate sensitivity parameter λ_2 in (8), and the climate feedback parameter λ_1 reflecting uncertainty in the underlying physical processes also in (8). All these are conclusive for equilibrium climate sensitivity and equilibrium mean temperature. For an illustrative straightforward tractable solution we simplify and look at a representative household that only forms multiple priors about equilibrium radiative forcing and equilibrium climate sensitivity $\Lambda_0 \equiv \lambda_0\lambda_2$ in (7) and (8). We follow Hennlock (2008*b*) and define the following unknown process for Λ_0:

$$B_{0t} = \hat{B}_{0t} + \int_0^t \Lambda_{0s} ds \quad \Lambda_{0s} \in [\Lambda_{0,min}, \Lambda_{0,max}] \quad (10)$$

where $d\hat{B}_0$ is the increment of the Wiener process $\hat{B}_{0t}$ on the probability space (Ξ_G, Φ_G, G) with variance $\sigma_v^2 \geq 0$ where $\{\hat{B}_{0t} : t \geq 0\}$. Moreover, $\{\Lambda_{0t} : t \geq 0\}$ is a progressively measurable drift distortion, implying that the probability distribution of B_{0t} itself is distorted

[9] The geophysical parameter values used in the discrete DICE climate model are $\Lambda_0 = 4.1$, $\Lambda_1 = 1.41$, $1/\tau_1 = 0.226$, $\tau_3/\tau_2 = 0.44$ and $1/\tau_2 = 0.02$ and $\Omega = 0.0083$. For a calibration of these parameters to continuous form see e.g. Smirnov (2005).

and the probability measure G_0 is replaced by another unknown probability measure Q_0 on the space (Ξ_G, Φ_G, Q). Substituting (7) in (8), the forcing-sensitivity process Λ_{0t} then modifies temperature equation (8) in the climate model to

$$dT = \frac{1}{\tau_1}\Bigg((\Lambda_{0t}dt + d\hat{B}_0)\frac{\sigma_v\sqrt{M_t/M_0}}{\gamma\sqrt{2}} + \frac{1}{2}\frac{\hat{\Lambda}_0}{\gamma}\frac{M_t}{M_0}dt + O_t dt \\ -\hat{\lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)dt\Bigg) \tag{11}$$

and hence, temperature equation (11) follows an analytically tractable Ito process.

Since both mean and variance of the drift term Λ_{0t} are uncertain, (10) yields different statistics (priors) of equilibrium forcing-sensitivity in (11) where the interval $[\Lambda_{0,min}, \Lambda_{0,max}]$ indicates an arbitrarily large maximum model specification error, corresponding to the range that the household is willing to imagine. Setting $\sigma_v = 0$ yields the the 'benchmark model', based on $\hat{\Lambda}_0 \equiv \hat{\lambda}_0\hat{\lambda}_2$, that the household regards as an approximation to an unknown and unspecified global climate system that generates the true data. The 'benchmark model' is set to cover the scientific uncertainty ranges and with a model specification error that allows the unknown process to move outside these ranges.

The deeper uncertainty enters (11) when the unknown process in (10) unexpectedly changes both mean and probability distribution of B_{0t}, having probability measure Q_0, relative to the distribution of $\hat{B}_{0t}$ having measure G_0. The Kullback-Leibler distance between probability measure Q_0 and G_0 is then:

$$R(Q_0) = \int_0^\infty \varepsilon_{Q_0}\left(\frac{|\Lambda_{0s}|^2}{2}\right)e^{-\rho t}ds \tag{12}$$

As long as $R(Q_0) < \Theta_0$ in (14) is finite

$$Q_0\left\{\int_0^t |\Lambda_{0s}|^2 ds < \infty\right\} = 1 \tag{13}$$

which has the property that Q_0 is locally continuous with respect to G_0, implying that G_0

and Q_0 cannot be distinguished with finite data. Hence, probability distributions cannot be inferred by using current finite climate data. Statistically this mimics the scientific findings about uncertainty by Roe and Baker (2007) that current climate data from underlying physical processes is statistically insufficient to predict equilibrium climate sensitivity and equilibrium mean temperature probability distributions.

4 Optimal Policy under Precaution to Knightian Uncertainty

Robust control is a condition of analysis when specifications of the dynamics, in our case the climate model in (11), and therefore climate impacts in (4), are open to doubt by the decision-maker due to Knightian uncertainty or model uncertainty that may trigger precautionary preferences by the policymaker. However, precautionary preferences and ambiguity aversion may violate the Sure-Thing Principle by Savage (1954), which is essential for ensuring that conditional preferences are well-defined and consistent over time and also being a basis for Bayesian updating and traditional expected utility theory.

Instead, we assume that a rational decision-maker updates her beliefs to new information by a time consistent rule derived from backward induction using a dynamic maximin decision criterion adopted from robust control. We apply the concept by Gilboa and Schmeidler (1989) and Hansen et al. (2001) and add a hypothetical minimizer to objective function (1) that chooses the worst-case priors Λ_0^*. We introduce a measure of precautionary preference $1/\theta_0 \in [0, +\infty]$ assigning how much our household listens to this hypothetical minimizer.[10] The maximin criterion, with expectation operator ε, then takes the following form

$$\sup_{C,G,q,r,s} \inf_{\Lambda_0} \varepsilon \int_0^\infty \frac{1}{1-\eta} \left[C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} e^{-\rho t} dt + \theta_0 R(Q_0) \tag{14}$$

which can be formulated as a zero-sum game between the household (the maximizer) and the

[10]While Gilboa and Schmeidler (1989) view ambiguity aversion as a minimization of the set of probability measures, Hansen et al. (2001) set a robust control problem and let its perturbations be interpreted as multiple priors in max-min expected utility theory. Epstein and Schneider (2001) provides another updating process.

hypothetical minimizer choosing the worst-case prior path for the household. The last term contains a Lagrangian multiplier θ_0 and the finite entropy (Kullback-Leibler distance) $R(Q_0)$ as a statistical measure of the distance between the benchmark prior and the worst-case priors, generated by the process $\{\Lambda_{0s}\}$. Following Hansen and Sargent (2001), a maximin constraint problem like (14) can be rewritten as a zero-sum differential game

$$\max_{C,G,q,r,s} \min_{\Lambda_{0i}} \varepsilon \int_0^\infty \left\{ \frac{1}{1-\eta} \left[(1-\omega) C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} + \frac{\theta_0 \Lambda_{0t}^2}{2} \right\} e^{-\rho t} dt \tag{15}$$

subject to the dynamic system (2) - (6), (9) and (11). The quadratic term in (15) contains the mean distortions Λ_{0t}. The minimization with respect to Λ_{0t} creates a lower (worst-case) boundary of the value function. The corresponding optimal policy vector under precaution $(C_t^*, G_t^*, q_t^*, r_t^*, s_t^*)$ from maximization would then be robust to priors that the household could imagine within the endogenous range of priors $[0, \Lambda_{0t}^*]$.

Instead of a computer simulation as usual in IAMs, we idenfity the analytical solution and present the essential results in the next-coming sections. The maximin zero-sum differential game, as defined by objective (15) and the dynamic system (2) - (6), (9) and (11), is solved by forming its Isaacs-Bellman-Flemming (IBF) equation in (19) in appendix A.1.[11] Finding an analytically tractable solution to (19) by 'guessing-and-verifying' is tedious and left for appendix. In short, the procedure implies taking the first-order conditions of (19) and rearranging yielding robust policy feedback rules. In order to identify shadow prices and costs, a value function that solves the IBF-equation (19) needs to be identified by a guessing-and-verifying procedure. Once a value function is verified that solves (19) it can be differentiated with respect to state variables and so identify the shadow price partial derivatives.

Since, the objective function in (15) is time autonomous, any robust policy feedback rule and the worst-case beliefs will be time consistent (Dockner et al., 2000). Moreover, certainty

[11] For simplicity, the labor stock L_t is omitted hereinafter defining K_t as the amount of capital per unit labor.

equivalence makes the variance distortions in (10) irrelevant, thus only mean distortions are relevant (Hansen and Sargent, 2008). The optimal policy feedback rules of consumptions, abatement and research efforts under precaution are time consistent and found in (26) - (30) in appendix A.1 and we will only refer to them in the next-coming subsections when presenting the major results in the context of the discussions on discounting, the Dismal Theorem and relative prices that followed the Stern Review.

4.1 Precaution, Knightian Uncertainty and Climate Data Observations

In a deterministic model ($\sigma_v = 0$) the household uses the benchmark climate model assumed by the model builder in the optimization. Introducing Knightian uncertainty by the unknown process (10) and a precautionary preference $1/\theta_0 > 0$, make the household endogenously update the climate model over time to its observations in climate data.

The time consistent feedback rule $\Lambda_{0t}^*(M_t)$ in (31) indicates how a household with a precautionary preference $1/\theta_0$ in a time consistent way updates its upper boundary of the radiative forcing and climate sensitivity model parameter Λ_{0t} to observed changes in the CO_2 stock. The endogenous range $[0, \Lambda_{0t}^*]$ 'stakes out the corners' of the priors considered for policymaking in terms of the precautionary preference $1/\theta_0 \in [0, +\infty)$ and expected shadow cost of climate change.

Proposition 1 *A precautionary preference $1/\theta_0 > 0$ makes the household concerned about misreading sudden observed changes in CO_2 concentration and atmospheric mean temperature as sources to underestimated relationships in radiative forcing and climate sensitivity, and hence, it updates its beliefs that even greater global warming and climate impact might occur in the future from an observed increase in CO_2 concentration. On the contrary, a reduction in observed variables adjusts the household's worst-case beliefs downwards updating to a less pessimistic climate model that induces a less stringent policy. Both these behaviors are time consistent.*

Proof: Minimizing the IBF equation (19) with respect to Λ_0 gives the optimal feedback rule (31) and substituting the undetermined coefficient (41) gives the worst-case prior distortion path in terms of the household's damage from climate impacts.

As a precautionary measure, the household increases policy stringency as an increased insurance against ambiguity to avoid (if possible) a realization of future irreversible uncertainty over high-temperature outcomes. This behavior is in accordance with the 'precautionary principle' as formulated by the Rio Declaration in article 15, stating that uncertainty shall not be used as a reason for postponing measures. In presence of deeper uncertainty, a 'wait-n-see' strategy would violate a precautionary preference for avoiding exposing society to irreversible uncertainty. On the other hand, a reduction in observed CO_2 concentration would by the rule (31) adjust the household's worst-case beliefs downwards making it less pessimistic and it would therefore adopt a less stringent policy. Both these behaviors are time consistent.

When damage is linear in (4), a precautionary preference under Knightian uncertainty has a similar effect on the carbon cost path in (46) as has a low pure rate of time preference. This makes precautionary concerns another issue in the discussion on discounting that took place in the reviews following the Stern Review as one of the important reasons for the policy stringency in the Stern Review is the low utility discounting of 0.1 percent noted by e.g. Nordhaus (2006), Dasgupta (2006) and Weitzman (2007).[12]

Proposition 2 *When the cost of climate change T is linear, a precautionary preference $1/\theta_0 > 0$ increases expected shadow carbon cost in (46) in a similar manner as a low pure rate of time preference ρ.*

Proof: Solving (19) by guessing-and-verifying and identifying $\partial W/\partial M_t = fe^{-\rho t}$ by determining the undetermined coefficients in (43) and (44) and substituting in $\partial W/\partial M_t$ gives (46).

[12] A discussion on discounting and uncertainty is also found in Guo et al. (2006).

Stern himself, explicitly cites Knightian uncertainty and articles by Maccheroni et al. (2006) on robust control as one of the motivations for choosing a low discount rate in his model.

Although the Stern Review does not take the full step of using a model with robust control, the reasoning is that they approximate the results in an optimizing model by using a low discount rate. For the purposes of a big inquiry such as the Stern Review, this may be a reasonable kind of approximation. In fact we show that ambiguity aversion in our model, like low discount rates in the Stern Review both lead to recommendations of a more stringent climate policy. However it is worth pointing out two important differences. First of all a low discount rate gives higher prominence to both costs and benefits in the future while ambiguity aversion mainly leads us to take more seriously the possibility that we will be hit by worst-scenario costs in the future. Secondly the proof that these two have similar effect in proposition 2 is only shown for the case of linear damage. If damages are non-linear - which they are in the Stern Review, the case does not appear to be analytically tractable.

It is clear though that a quadratic formulation of temperature (4) would call for a value function with non-linear terms in T which, in turn, would call for a shadow carbon cost (46) being a function of M, T and $\tilde{T}$, making the policy feedback rules in (20) - (23) *directly* responsive to observed changes in CO_2 concentration and mean temperature besides the indirect effect from the updating of beliefs in (31) and the direct effect from $1/\theta_0 > 0$ on carbon cost in (46) that already exist in the linear damage case. However, imposing a quadratic temperature term jointly with the nonlinear differential equation system for K_t, A_{Kt}, E_t and A_{Et} results in demanding calculations. To keep the illustration analytical tractable straightforward, we leave this for future research.

Finally, a precautionary preference $1/\theta_0 > 0$ also has one additional effect that cannot be explained by a low utility discounting; it makes the household's worst-case beliefs about equilibrium radiative forcing and climate sensitivity responsive to changes in climate data observations over time as seen in proposition 1.

4.2 Precaution, Knightian Uncertainty and the Dismal Theorem

The Dismal Theorem in Weitzman (2009*a*), more than anything else, seems to highlight the importance of taking precaution seriously in climate policy. Though Weitzman's analysis is not an IAM analysis but rather based on a static relationship between a utility function and a 'climate-sensitivity parameter' with unknown probability distribution, it exhibits an infinite expected marginal utility at zero consumption level.

Hennlock (2008*b*) showed in an IAM that a policymaker who expresses a *perfect* precautionary preference also exhibits infinite shadow carbon cost and resorts to zero carbon-intensive production. In the two-sectoral IAM, a perfect precautionary preference ($\theta \to 0$) results in a complete shift from carbon-intensive consumption to carbon-neutral consumption.

Proposition 3 *Let the household express perfect precautionary preference, $\theta_0 = 0$, then its expected shadow cost of carbon-intensive capital approaches infinity, $\partial W/\partial K_t \to +\infty$, resulting in a zero carbon-intensive consumption as the consumption feedback rule approaches $\epsilon\,[\lim_{\theta_0 \to 0} C^*(K^*(t)) \equiv 0]$ regardless of the capital stock levels $K^*(t)$.*

Proof: Setting $\theta_0 = 0$ in (43) gives $\lim_{\theta_0 \to 0} f \to -\infty$ and $\lim_{\theta_0 \to 0} a \to +\infty$ in (39). Differentiating (25) with respect to K_t gives $\lim_{\theta_0 \to 0} \partial W/\partial K_t \to +\infty$ which in (20) yields proposition 3.

Proposition 3 is reminiscent of the dismal theorem by Weitzman (2009*a*). A perfect precautionary preference ($\theta \to 0$) reproduces Weitzman's zero consumption. The expected shadow carbon cost in (46) is infinite with an unbounded value function.[13] Weitzman (2009*a*) has received critique for the extreme result from e.g. Nordhaus (2009) and Yohe and Tol (2007); the latter admitting the importance of the Dismal Theorem, but labeled it *'Warning: Not to be taken to its logical extreme in application to real world problems.'*

[13]One important difference is that there is (Bayesian) learning in Weitzman (2009*a*) while there is no learning in our model.[14] However, the learning in Weitzman's two-period model is not *realized* until we are far away (200 years?) into the future by the arrival of the second period.

Our model is slightly distinct from Weitzman's logical approach, since it can raise the question what initial level of precaution the household should have in the presence of current scientific uncertainty. With infinite precaution we reproduce Weitzman's result on consumption and marginal utility. With no precaution we get expected utility. This begs the question of how to determine the appropriate level of precaution. One possible way to go about this is as follows: An upper boundary of precaution that makes it easy to statistically distinguish the household's worst-case climate sensitivity priors from the benchmark model sensitivity priors should be evidence of a degree of precaution that is too cautious. Thus, an upper boundary for precaution could be set to make it statistically difficult to distinguish the worst-case climate sensitivity priors from the benchmark sensitivity priors, using the current scientific range for climate sensitivity. In the case scientific discoveries narrow predictions of climate sensitivity in the future, $\hat{\theta}_{0t}$ should be adjusted, and so update the range of climate-sensitivity outcomes used as basis for optimal climate policy design under precaution.[15]

4.3 Precaution, Knightian Uncertainty and Endogenous Growth

Thanks to having a two sector model with substitution and endogenous growth, we find that we are also capable of analyzing the results of perfect precaution: Even an infinite shadow carbon cost, due to a perfect precaution to Knightian uncertainty does not necessarily imply corner solutions in the switch from carbon-intensive to carbon-neutral sectors as one might think at a first glance. Though there is an immediate switch from carbon-intensive to carbon-neutral consumption in proposition 3, abatement levels are still interior and finite and investment in carbon-intensive technology and (more efficient) carbon-intensive capital are still positive.

[15] Hansen and Sargent (2001) suggest that a robustness parameter θ should be set sufficiently high for it to take long time series to distinguish the benchmark model from worst-case models. By calculating likelihood ratio under benchmark and worst-case models Hansen and Sargent (2008) suggest calculating overall detection error probability using detection error probabilities conditional on each model, respectively. For $1/\theta_0 = 0$ models are identical and $p = 0.5$. In general the greater is $1/\theta_0$, the lower is then p.

Proposition 4 *Let the household express perfect precautionary preference, $\theta_0 = 0$, then its expected shadow cost of carbon approaches infinity,* $\lim_{\theta_0=0} \partial W/\partial M_t \to -\infty$, *with the result that abatement* $\epsilon\left[\lim_{\theta_0 \to 0} q^*(t)\right] \geq 0$ *and finite and research effort* $\epsilon\left[\lim_{\theta_0 \to 0} r^*(t)\right] \geq 0$ *and finite in carbon-intensive technology.*

Proof: Setting $\theta_0 = 0$ in (43) gives $\lim_{\theta_0 \to 0} f \to -\infty$ and $\lim_{\theta_0 \to 0} a \to +\infty$ in (39) and $\lim_{\theta_0 \to 0} b \to +\infty$ in (40). Differentiating (25) with respect to M_t, K_t and A_{Kt} yields $\partial W/\partial K_t \to +\infty$, $\partial W/\partial M_t \to -\infty$ and $\partial W/\partial A_{Kt} \to +\infty$ which in (21) and (22) reproduce proposition 4.

In other words, despite that both shadow prices of atmospheric carbon and carbon-intensive capital explode in proposition 4 - the ratio - the relative shadow price of the atmospheric carbon stock to the carbon-intensive capital stock

$$\epsilon\left[\lim_{\theta_0 \to 0} -\frac{\frac{\partial W}{\partial M_t}}{\frac{\partial W}{\partial K_t}}\right] = -\frac{2f}{a}(K^*(t))^{1/2} \geq 0 \tag{16}$$

converges to a finite non-negative value for all finite levels of carbon-intensive capital $K^*(t)$. As results, abatement and research effort in carbon-intensive technology converge to positive and finite levels despite the 'uncut' worst-case mean distortions and the unbounded value function due to a perfect precautionary preference. These results are mainly due to the assumptions that the accumulated capital in the carbon-intensive production sector is used in the endogenous carbon-intensive research process as well as the abatement in accordance with the polluter-pays-principle.

5 Concluding Comments

The presence of Knightian uncertainty in climate modeling and climate impacts suggests that the traditional concepts of expected utility theory and risk aversion are insufficient to capture the essential behavioral response à la Ellsberg that is codified in the precautionary principles in climate policy decision-making as formulated in e.g. the Rio Declaration.

Still, integrated assessment models, such as DICE, PAGE and FUND, have so far neglected Knightian uncertainty, nevertheless the presence of Knightian uncertainty in climate modeling and climate impacts usually requires model builders to make sometimes subjective ad hoc guesses about probability distributions for model parameters in the simulations of IAMs.

Still, integrated assessment models have so far neglected Knightian uncertainty. Instead the presence of uncertainty in climate modeling is usually dealt with by assuming probability distributions for uncertain model parameters in IAM simulations. These distributions are however also uncertain or unknown.

A contribution of this paper is that it introduces a robust control approach in an IAM which differs from existing 'risk analysis' in IAMs in that it does not fix one or another model builder's use of probability distributions to uncertain parameters. Instead it lets a household with a precautionary preference face the Knightian uncertainty in the climate model. Statistically this mimics the scientific findings about uncertainty that current climate data from underlying physical processes is statistically insufficient to predict future mean temperature probability distributions. We applied this approach to a two-good IAM with a carbon-intensive and a carbon-neutral production sector and Knightian uncertainty in radiative forcing and climate sensitivity.

We identified an analytical solution using a closed-loop information structure resulting in time consistent behavior. The insights from analytical solutions can serve more complex analytical approaches in future research than computer-based simulations seen so far in IAMs. Analytical solutions usually have better reliability, and also allow for deeper understanding as trajectories can be traced down to their explicit functional forms. In summary, we found the following results:

The household updates the climate model it is using over time to observed climate data changes. A precautionary preference makes the household concerned about sudden observed increases in mean temperature and interprets them as underestimated climate sensitivity in the climate model. This results in a more stringent policy as an insurance against realizations of high-impact low-probability outcomes. On the other hand, a reduction in observed temperature adjusts the household's worst-case beliefs downwards, inducing a less stringent policy than before.

We concur with the essential idea in Weitzman that deep uncertainty must be reflected in more caution and more stringent climate policy. An infinite expected carbon cost and zero (carbon-intensive) consumption as in Weitzman (2009*a*) is however problematic as it does not provide a realistic guidance how to act under deeper uncertainty. In our model, a robust IAM approach with two sectors and endogenous growth allows us to tie and calibrate the level of precaution to the current level of scientific uncertainty and by this obtain guidance in selecting upper boundaries to precautionary concerns. Precaution should not be greater than to make it statistically difficult to distinguish worst-case scenarios in climate models from benchmark models in the scientific range using detection error probabilities.

Even when precautionary concern is taken to its extreme, resulting in an infinite expected shadow carbon cost and an unbounded value function similar to the Weitzman result, we do not necessarily find corner solutions. Consequently, abatement levels and investment in (more efficient) carbon-intensive capital and technology may still be finite and positive.

Appendix

Analytically tractable solutions to non-linear differential games, using guessing-and-verifying methods in dynamic programming, are usually extremely difficult to identify. To obtain the solution in this model, some simplifications have been made; (i) specifications have been carefully chosen for 6 of the 31 parameters, that amongst other things make the objective function additively separable, while the remaining 25 parameters are free to be varied for sensitivity analysis, (ii) linear damage only in the non-carbon-generating sector (two-sector damage still results in analytically tractable solutions but they are not unique for the guessed-and-verified value function) and (iii) a slightly modified radiative forcing and temperature equation as used in the DICE model makes temperature to follow an Ito process.

A.1. The Dynamic Programming Problem

This section presents a solution to the model with a parameter-setting that allows for an analytically tractable solution to the zero-sum differential game defined by the objective (15) and the dynamic system (2) - (6), (9) and (11).

Definition 1 *If there exist a value function $W(K, A_K, E, A_E, M, T, \tilde{T}, t)$ that satisfy*

$$\begin{aligned} & W(K, A_K, E, A_E, M, T, \tilde{T}, t) = \\ & \varepsilon\Big\{ \int_0^\infty \Big\{ \frac{1}{1-\eta}\Big[(1-\omega)(C_t^*)^{\frac{\sigma-1}{\sigma}} + \omega (G_t^*)^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{(1-\eta)\sigma}{\sigma-1}} + \frac{\theta_0(\Lambda^*)^2_{0t}}{2}\Big\} e^{-\rho t} dt\Big\} \\ & \geq \varepsilon\Big\{ \int_0^\infty \Big\{ \frac{1}{1-\eta}\Big[(1-\omega)C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{(1-\eta)\sigma}{\sigma-1}} + \frac{\theta_0(\Lambda^*)^2_{0t}}{2}\Big\} e^{-\rho t} dt\Big\} \end{aligned} \tag{17}$$

for strategies $C^*(K,t) \subseteq R^1$, $G^*(E,t) \subseteq R^1$, $q^*(K,t) \subseteq R^1$ $r_K^*(A_K,t) \subseteq R^1$ and $s^*(A_E,t) \subseteq R^1$ and $\Lambda_0^*(M,t) \subseteq R^1$ given that $\Lambda_0^*(M,t) \equiv \arg\min W(K, A_K, E, A_E, M, T, \tilde{T}, t)$ and *which satisfy state equations (2) - (6), (9) and (11),then*

$$\Gamma_t^* = (C_t^*, G_t^*, q_t^*, r_t^*, s_t^*, \Lambda_{0t}^*) \tag{18}$$

provide a robust feedback Nash equilibrium solution of the game defined by (15), (2) - (6), (9) and (11) (Basar and Olsder, 1999).

Using (15), (2) - (6), (9) and (11) and definition 1 yield the Isaacs-Bellman-Fleming (IBF) dynamic programming equation (see Fleming and Richel, 1975):

$$\begin{aligned}
-\frac{\partial W}{\partial t} &= \\
\max_{C,G,q,r,s} \min_{\Lambda_0} &\left\{ \frac{1}{1-\eta}\left[(1-\omega)C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{(1-\eta)\sigma}{\sigma-1}} + \frac{\theta_0 \Lambda_{0t}^2}{2} \right\} e^{-\rho t} \\
&+ \frac{\partial W}{\partial K}\left[(1-r_t)A_{Kt}^{\tau}K_t^{\alpha} - cq_t^2 - C_t - \delta K_t\right] \\
&+ \frac{\partial W}{\partial A_K}\left[\nu(s_t Y_{Kt})^{\tau} A_{Kt}^{1-\tau} - \delta_K A_{Kt}\right] \\
&+ \frac{\partial W}{\partial E_K}\left[(1-s_t)A_t^{\Psi}E_t^{\phi} - \frac{1}{\kappa}E_t - \Phi(T_t - T_0)E_t^{\phi} - pG_t\right] \\
&+ \frac{\partial W}{\partial A_E}\left[\ \beta(r_t Y_{Et})^{\psi} A_{Et}^{1-\psi} - \delta_E A_{Et}\right] + \frac{\partial W}{\partial M}\left[\epsilon\varphi K_t^{\alpha} L_t^{1-\alpha} - \mu q_t - \Omega M_t\right] \\
&+ \frac{\partial W}{\partial T}\frac{1}{\tau_1}\left[\Lambda_{0t}\frac{\sigma_v\sqrt{M_t/M_0}}{\sqrt{2}\gamma} + \hat{\Lambda}_0 \frac{M_t/M_0}{2\gamma} + O_t - \hat{\lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right] \\
&+ \frac{1}{2}\frac{\partial^2 W}{\partial T^2}\sigma_v^2 M_t + \frac{\partial W}{\partial \tilde{T}}\left[\frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right)\right]
\end{aligned} \tag{19}$$

The robust control vector $\Gamma_t^* = (C_t, G_t, q_t, r_t, s_t)$ is given by maximizing (19) with respect to policy variables and minimizing with respect to Λ_{0t} and solving for the robust feedback Nash rules.

$$C^*(K(t)) = \left(\frac{1-\omega}{\frac{\partial W}{\partial K_t}}\right)^2 e^{-2\rho t}, \quad G^*(E(t)) = \left(\frac{\omega}{\pi \frac{\partial W}{\partial E_t}}\right)^2 e^{-2\rho t} \tag{20}$$

$$q^*(K(t), M(t)) = -\frac{\epsilon\mu}{2c}\frac{\frac{\partial W}{\partial M_t}}{\frac{\partial W}{\partial K_t}} \geq 0 \tag{21}$$

$$r^*(A_K(t), K(t)) = \frac{A_{Kt}^{1-\tau}(\nu\tau)^{\frac{1}{1-\tau}}}{K_t^{\alpha}}\left(\frac{\frac{\partial W}{\partial A_{Kt}}}{\frac{\partial W}{\partial K_t}}\right)^{\frac{1}{1-\tau}} \in [0,1] \tag{22}$$

$$s^*(A_E(t), E(t)) = \frac{A_{Et}^{1-\psi}(\beta\psi)^{\frac{1}{1-\psi}}}{E_t^{\phi}}\left(\frac{\frac{\partial W}{\partial A_{Et}}}{\frac{\partial W}{\partial E_t}}\right)^{\frac{1}{1-\psi}} \in [0,1] \tag{23}$$

$$\Lambda_{0t}^*(M(t), T(t)) = -\frac{\partial W}{\partial T}\frac{\sigma_v\sqrt{M_t/M_0}e^{\rho t}}{\theta_0\tau_1\gamma\sqrt{2}} \geq 0 \quad \Lambda_{0t}^* \in [\Lambda_{0,min}, \Lambda_{0,max}] \tag{24}$$

Proposition 5 *The value function* $W(K, A_K, E, A_E, M, T, \tilde{T}, t)$

$$= \Big(aK^{1-\alpha} + bA_K^{\tau} + dE^{1-\phi} + eA_E^{\psi} + fM + gT + h\tilde{T} + k\Big)e^{-\rho t} \tag{25}$$

satisfy the differential equation system formed by (19).

Proof: Substituting (20) to (24) into (19) and collecting terms forms the indirect Isaacs-Bellman-Fleming equation. An analytically tractable solution is possible by setting $\sigma = 2$ and $\eta = \tau = \alpha = \phi = \psi = 1/2$ while the remaining 25 parameters, listed in appendix A.2., can be set free. Guessing the value function (25) and taking the first-order condition of the IBF-equation (19) with respect to the policy vector $(C_t^*, G_t^*, q_t^*, r_t^*, s_t^*)$ and rearranging, yield optimal policy feedback rules (26) - (30) under precaution. The carbon-intensive consumption feedback Nash rule is

$$C^*(K_t) = \frac{4(1-\omega)^2}{a^2}K_t \geq 0 \tag{26}$$

where a is defined in (39). The carbon-neutral consumption feedback Nash rule is

$$G^*(E_t) = \frac{4\omega^2}{(\pi d)^2} E_t \geq 0 \tag{27}$$

where d is defined in (41). The abatement feedback Nash rule is

$$q^*(K_t) = -\frac{\epsilon\mu}{c}\frac{f}{a} K_t^{1/2} \geq 0 \tag{28}$$

where a is defined in (39) and f in (43). The carbon-intensive research effort feedback Nash rule is

$$r^*(K_t) = \left(\frac{b\nu}{2a}\right)^2 \left(\frac{K_t}{A_{Kt}}\right)^{1/2} \in [0,1] \tag{29}$$

where b is defined in (40). The carbon-neutral research effort feedback Nash rule is

$$s^*(E_t) = \left(\frac{e\beta}{2d}\right)^2 \left(\frac{E_t}{A_{Et}}\right)^{1/2} \in [0,1] \tag{30}$$

where e is defined in (42). Minimizing the IBF equation (19) with respect to Λ_0 gives the optimal feedback rule identifying the household's worst-case mean distortion path, $\Lambda_0^*(M_t, T_t)$ in terms of its precautionary preference $1/\theta_0 \in [0, \infty)$ and expected shadow cost of climate change:

$$\Lambda_{0t}^*(M(t)) = \frac{\frac{d}{2}\Phi}{\rho + \frac{\hat{\lambda}_1}{\tau_1} + \frac{\tau_3}{\tau_1\tau_2}(1 - \frac{1}{1+\rho\tau_2})} \frac{\sigma_v\sqrt{M_t/M_0}}{\theta_0\tau_1\gamma\sqrt{2}} \geq 0 \quad \Lambda_{0t}^* \in [\Lambda_{0,min}, \Lambda_{0,max}] \tag{31}$$

The influence on policy enters via the undetermined coefficients a, b, e, f, g and h. Substituting (26) - (31) in (19) yield the equation system

$$\rho a = \frac{2(1-\omega)^2}{a} - a\frac{\delta}{2} + \frac{a\nu^2}{32(\rho + \delta_K/2)^2} + \frac{(f\epsilon\mu)^2}{2ac} + f\epsilon\varphi \tag{32}$$

$$\rho b = \frac{a}{2} - b\frac{\delta_K}{2} \tag{33}$$

$$\rho d = \frac{2\omega^2}{\pi} + \frac{(e\beta)^2}{8d_i} - \frac{1}{2\kappa} \tag{34}$$

$$\rho e = \frac{d}{2} - e\frac{\delta_E}{2} \tag{35}$$

$$\rho f = -\frac{1}{2\theta_0 M_0}\left(\frac{g\sigma_v}{\tau_1\gamma\sqrt{2}}\right)^2 - f\Omega + \frac{g\hat{\Lambda}_0}{2\tau_1\gamma M_0} \tag{36}$$

$$\rho g = -\frac{d\Phi}{2} - g_i\frac{\hat{\Lambda}_1}{\tau_1} - g\frac{\tau_3}{\tau_1\tau_2} + h\frac{1}{\tau_2} \tag{37}$$

$$\rho h = g\frac{\tau_3}{\tau_1\tau_2} - h\frac{1}{\tau_2} \tag{38}$$

Solving the equation system (32) - (38) for undetermined coefficients gives the coefficients in terms of parameter values

$$a = \frac{f\epsilon\varphi}{2\left(\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}\right)} \tag{39}$$
$$+\frac{1}{2}\sqrt{\left(\frac{f\epsilon\varphi}{\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}}\right)^2 + \frac{8(1-\omega)^2 + 2\frac{(f\epsilon\mu)^2}{c}}{\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}}}$$

$$b = \frac{a}{2\rho + \delta_K} \tag{40}$$

$$d = \omega \sqrt{\frac{2/\pi}{\rho + \frac{1}{2\kappa} - \frac{\beta^2}{32(\rho+\delta_E/2)^2}}} \tag{41}$$

$$e = \frac{d}{2\rho + \delta_E} \tag{42}$$

$$f = \frac{g\left(\hat{\Lambda}_0 - \frac{g\sigma_v^2}{2\theta_{0i}\tau_1\gamma}\right)}{2\tau_1\gamma(\rho+\Omega)M_0} \tag{43}$$

$$g = \frac{-\frac{d}{2}\Phi}{\rho + \frac{\hat{\lambda}_1}{\tau_1} + \frac{\tau_3}{\tau_1\tau_2}(1 - \frac{1}{1+\rho\tau_2})} \tag{44}$$

$$h = g\frac{\tau_3}{\tau_1\tau_2(\rho + \frac{1}{\tau_2})} \tag{45}$$

The coefficients in (39) - (45) are uniquely defined, the coefficient k in proposition 5 is uniquely determined by (39) - (45), and hence, the feedback rules (26) to (31) corresponding to the guessed value function (25) are unique and the solution is verified. **Q.E.D.**

A.1.2. Explicit Function of Expected Cost of Carbon under Precaution

The transition trajectory of expected social cost of carbon is found by calculating $\partial W/\partial M = fe^{-\rho t}$ from (25) and substituting undetermined coefficients (43) and (44) yielding:

$$\frac{-\hat{\Lambda}_0 \frac{\omega\Phi\sqrt{\frac{2/\pi}{\rho+\frac{1}{2\kappa}+\frac{\beta^2}{8(2\rho+\delta_E)^2}}}}{\rho+\frac{\hat{\lambda}_1}{\tau_1}+\frac{\tau_3}{\tau_1\tau_2}\left(1-\frac{1}{1+\rho\tau_2}\right)} - \frac{\hat{\sigma}_v^2}{2\theta_0\tau_1\gamma}\left(\frac{\omega\Phi\sqrt{\frac{2/\pi}{\rho+\frac{1}{2\kappa}+\frac{\beta^2}{8(2\rho+\delta_E)^2}}}}{\rho+\frac{\hat{\lambda}_1}{\tau_1}+\frac{\tau_3}{\tau_1\tau_2}\left(1-\frac{1}{1+\rho\tau_2}\right)}\right)^2}{2\tau_1\gamma(\rho+\Omega)M_0} \cdot e^{-\rho t} \tag{46}$$

A.1.3. Transitional Dynamics under Precaution

To find the optimal trajectories in the dynamic system, the feedback rules (26) to (31) are substituted in dynamic system (2) - (6), (9) and (11) which gives

$$
\begin{aligned}
dK = \Bigg[A_{Kt}^{1/2} K_t^{1/2} - \left(\frac{b}{a}\right)^2 \frac{\nu^2}{4} K_t - c\left(-\frac{\epsilon\mu}{c}\frac{f}{a}K_t^{1/2}\right)^2 \\
- \frac{4(1-\omega)^2}{a^2} K_t - \delta K_t \Bigg] dt
\end{aligned} \tag{47}
$$

$$
dA_K = \left[\nu \left(\left(\frac{b}{a}\right)^2 \frac{\nu^2}{4} K_t \right)^{1/2} A_{Kt}^{1/2} - \delta_K A_{Kt} \right] dt \tag{48}
$$

$$
\begin{aligned}
dE_i = \Bigg[A_t^{1/2} E_t^{1/2} - \left(\frac{e}{d}\right)^2 \frac{\beta^2}{4} E_t - \frac{1}{\kappa} E_t \\
- \Phi(T_t - T_0) E_t^{1/2} - \pi \frac{4\omega^2}{(\pi d)^2} E_t \Bigg] dt
\end{aligned} \tag{49}
$$

$$
dA_E = \left[\beta \left(\left(\frac{e}{d}\right)^2 \frac{\beta^2}{4} E_t \right)^{1/2} A_{Et}^{1/2} - \delta_E A_{Et} \right] dt \tag{50}
$$

$$
dM = \left[\epsilon\varphi K_t^{1/2} L_t^{1/2} + \mu \frac{\epsilon\mu}{c}\frac{f}{a} K_t^{1/2} - \Omega M_t \right] dt \tag{51}
$$

$$
\begin{aligned}
dT = \frac{1}{\tau_1}\Bigg(\frac{\sigma_v\sqrt{M_t/M_0}}{\sqrt{2}\gamma} d\hat{B}_0 - \frac{g\sigma_v^2 M_t/M_0}{2\theta_0\tau_1\gamma^2} dt + \hat{\lambda}_0\hat{\lambda}_2 \frac{M_t/M_0}{2\gamma} dt + O_t dt \\
- \hat{\lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t) dt \Bigg)
\end{aligned} \tag{52}
$$

$$d\tilde{T} = \frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right) dt \tag{53}$$

Besides computer-based methods, an analytical solution to (47) - (53) can be found by using transformations $\hat{K} \equiv K^{1/2}$, $\hat{A}_K \equiv A_K^{1/2}$, $\hat{E} \equiv E^{1/2}$ and $\hat{A}_E \equiv A_E^{1/2}$ transforming the system to a linear system. Applying certainty equivalence in (19) the steady states $(\bar{K}, \bar{A}_K, \bar{E}, \bar{A}_E, \bar{M}, \bar{T}, \bar{\tilde{T}})$ as $t \to \infty$ in the state space can be derived in terms of parameter values from (47) to (53). The corresponding steady state policy variables as $t \to \infty$ are found by substituting the steady state stock values in (26) - (30).

A.2. List of Parameters

The found analytically tractable solution required 6 of 31 parameters to be specified as below. The remaining 25 parameters below are free to be varied in the analytical solution.

Free Parameters

$\rho > 0$	pure time preference
$\omega \in [0, 1]$	share parameter in objective function
$1/\theta_0 \in [0, +\infty]$	degree of ambiguity aversion
$c \geq 0$	abatement cost parameter
Θ	climate impact parameter
$\pi \geq 0$	relative price carbon-neutral input good
$\nu \geq 0$	carbon-intensive research sector efficiency parameter
$\beta \geq 0$	carbon-neutral research sector efficiency parameter
$\delta \geq 0$	depreciation rate carbon-intensive capital
$1/\kappa \geq 0$	depreciation rate carbon-neutral capital
$\delta_K \geq 0$	depreciation rate carbon-intensive technology
$\delta_A \geq 0$	depreciation rate carbon-neutral technology
$\varphi \geq 0$	carbon-intensity
$\mu \geq 0$	abatement effort efficiency

Free Parameters in the Climate Model

$\hat{\lambda}_0 \geq 0$	radiative forcing benchmark model parameter
$\hat{\lambda}_1 \geq 0$	climate feedback benchmark model parameter
$\hat{\lambda}_2 \geq 0$	climate sensitivity benchmark model parameter
$\tau_1 \geq 0$	thermal capacity of atmospheric layer
$\tau_3 \geq 0$	thermal capacity of deep ocean layer
$1/\tau_2 \geq 0$	transfer rate from the upper layer to the deeper ocean layer
$1/\Omega \geq 0$	transfer rate of CO_2 from atmosphere to other reservoirs
$\epsilon \geq 0$	marginal atmospheric retention ratio
$M_0 \geq 0$	initial CO_2 concentration rate
$T_0 \geq 0$	initial atmospheric mean temperature
$\tilde{T}_0 \geq 0$	initial deep ocean mean temperature

Specified Parameters in the Analytically Tractable Solution

$\sigma = 2$	elasticity of substitution
$\eta = 0.5$	elasticity of marginal utility of final good
$\alpha = 0.5$	capital intensity carbon-intensive production
$\phi = 0.5$	capital intensity carbon-neutral production
$\tau = 0.5$	constraint carbon-intensive sector technology progress
$\psi = 0.5$	constraint carbon-neutral sector technology progress

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