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Topologies Defined by Binary Relations

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1. Introduction.

A consumer's preferences can be conceptualized as a binary relation on a set of alternatives. In different contexts, economists can assume different properties for binary relations expressing consumers' preferences. For example, if a consumer is required to rank alternatives from best to worst, preferences constitute a linear order. In another situation, for example one where bounded rationality comes into play, it might be difficult for a consumer to formulate complete or consistent preferences. Then in an extreme case it might be useful to place no restrictions on binary relations expressing preferences. Furthermore, allowing incomplete or inconsistent binary relations to express preferences is one way around the common complaint that economists assume, unreasonably, that all economic agents are utility maximizers.

The binary relation provides a starting point in the process of understanding the consumer's preferences. This understanding can be enhanced by studying not only the given binary relation, but also the topology it induces. The induced topology on a set of alternatives on which a binary relation has been defined is defined as the topology with a subbasis containing two sets for each alternative: the set of alternatives preferred to it and the set of alternatives to which it is preferred. The added structure of an induced topology on the set of alternatives provides a notion of nearness or proximity between alternatives, and does so without the benefit of a distance function defined on the set of alternatives. The induced topology also makes it possible to ask if the given preferences possess some form of continuous representation. Continuous representation are valuable conceptually and as optimization tools.

The role of induced topologies in illuminating preferences is the motivation for this paper, in which we investigate the nature of induced topologies. We will characterize induced topologies and produce topological versions of two classical preference representation theorems. We will apply these results to construct topologies that are not induced topologies, and to construct induced topologies associated with various forms of representation. We focus on utility functions and two-function representations.

We have used the following to motivate the study of topologies induced by arbitrary binary relations: 1) the complaint that economists too often assume that all economic agents are utility maximizers, 2) the fact that a preference-induced topology provides a notion of proximity of alternatives and 3) the fact that induced topologies make possible the search for continuous representations of preferences. But what is the motivation for the particular approach we will use–characterizing induced topologies and then searching for preferences that induce a given topology? A general motivation is that characterizing induced topologies is a natural first step in their study. But could there be a situation where a social scientist has a topology in hand and wishes to find preferences that induce that topology? Consider the following example.

The Four Senators. Each of four senators C, D, E, and F from four different states is asked to place herself in a peer group, the subset of $\{C, D, E, F\}$ in which she fits best. The result: C: $\{C, D, E, F\}$, D: $\{D, E, F\}$, E: $\{E, F\}$, and F: $\{F\}$. The following question arises: "Are these groupings both consistent and complete?" At first glance the answer seems to be no; for example, if it is considered consistent that $\{E, F\}$ and $\{F\}$ are peer groups, it seems that completeness would require $\{E\}$ to be a peer group. However, on further reflection, the question seems too vague. One natural, more concrete version is as follows: "Is there a voter or group of voters whose preferences induce the topology $\{\emptyset, \{C, D, E, F\}, \{D, E, F\}, \{E, F\}, \{F\}\}$?" The answer to this second question is yes if and only if there are preferences such that each member of the topology (except possibly $\emptyset$ and $\{C, D, E, F\}$) and no other set is either the set of senators preferred to a senator or the set of senators to which a senator is preferred.[1] Therefore the reported groupings would be considered a consistent and complete collection of peer groups by a voter or group of voters with preferences that induce the topology in question. The second question is one concrete version of the first question.[2]

We could end our discussion of this example here, since our scenario has led to the

[1] For pedagogical reasons, we chose to start with a collection of nested sets, since then the topology formed has the property that every subbasis contains every member of the topology except possibly $\emptyset$ and $\{C, D, E, F\}$. This simplifies slightly the argument that the second question is a concrete version of the first question

[2] It is an artifact of the definition of a subbasis, which doesn't require the universal set

type of question we wanted to motivate. However, there is something else to be learned by answering the question. Again, at first glance the answer is no, since the preferences of a voter asked to rank the senators linearly could not induce the given topology. However, there are many questions a voter can be asked. Suppose for each ordered pair of senators (x, y) we ask voters in y's state whether x would be a good replacement for y (or, if $x = y$, whether a well-known out-of-state politician very like y would be a good replacement for y). Suppose also that the senators' abilities as rated on a scale from 1 to 10 by constituents is given by $f(C) = 7$, $f(D) = 5$, $f(E) = 3$, $f(F) = 1$ and their abilities as rated by out-of-state voters is given by $g(C) = 2$, $g(D) = 4$, $g(E) = 6$, $g(F) = 8$. Then the voters will answer yes if $g(x) > f(y)$ since to the voter's asked, y is in-state and x is out-of-state. Define B on $\{C, D, E, F\}$ by xBy if $g(x) > f(y)$. Then a simple calculation (which we will omit since several such calculations will be made in the body of the paper) shows that B induces the given topology, so that the groupings are consistent and complete. The second lesson learned from the scenario is that simple questions like the one asked the voters can lead to preferences described by unlikely binary relations (for example, B is neither reflexive nor irreflexive), binary relations that induce unlikely topologies.

In the preference literature, a particular type of induced topology appears, the order topology. An order topology is a topology induced by a total preorder. Order topologies sometimes appear only implicitly. The treatment in Debreu's (1964) classic paper on continuous utility functions is typical. He assumes that the set of alternatives is already endowed with a topology and that for each alternative the set of alternatives preferred to it and the set of alternatives to which it is preferred both belong to the given topology. In other words, all results apply for topologies at least as fine as the order topology. Similarly, Chateauneuf (1987) explores continuity of two-function representations when the set of alternatives comes endowed with a topology at least as fine as the induced topology, again without explicit mention of the induced topology. Order topologies sometimes receive explicit mention. A prime example is Section 1.6, Order and Topology, in Bridges and

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(in this case $\{C, D, E, F\}$) to be a union of subbasis members, that in order for questions one and two to mesh perfectly, question 2 should really include a clause dealing with the universal set separately.

Mehta (1995).

One reason for the interest in topologies finer than the induced topology is that the set of available alternatives may be a subset of a well-known space such as a Euclidean space, as in Sprumont (2001). Then it may be more convenient to work with this well-known space than with the induced topology.

Also, for a binary relation that actually possesses representations of a certain type, passing to a finer topology increases the chances of finding a *continuous* representation of that type; however, it does so at a cost, a loss of information. In the extreme, passing to the finest topology, the discrete topology, to which every subset of alternatives belongs, guarantees that any representation is continuous, but yields no information about the given preferences. Therefore in this paper we focus on the nature of induced topologies without any consideration of pre-existing finer topologies.

Very like one of our results in spirit is Eilenberg's (1940) statement (6.1)[3] , which states that a *connected* topology on X can be mapped *one to one* and continuously into the reals if and only if it is Hausdorff and separable, and $X \times X - \{(x, x)\colon\ x \in X\}$ with the relative product topology is disconnected. Our Proposition 5 is more general in that it leaves out the words "connected" and "one-to-one" (and therefore has very different characterizing properties).

Most recently, Campión et al. (2009) have characterized topologies induced by total preorders, as a by-product providing a characterization of topologies induced by binary relations represented by utility functions (they also characterize topologies that coincide with the lower topology of some total preorder). Their Theorem 3.1 and Lemma 4.1 combine to give a characterization like our Proposition 5. Nevertheless, we include Proposition 5 because it fits perfectly between our Propositions 2 and 6, thereby providing our paper with

[3] Eilenberg's result (6.1) is incorrect as stated in his paper; after commenting early on that his results require the given topology to be Hausdorff, he forgets to include that condition in the statement of his results. Although its main role is to illustrate our Proposition 4, Example 3 in Section 4.1 also demonstrates that the Hausdorff condition cannot be left out of Eilenberg's statement (6.1); that is, it is not only necessary, but also independent of the other two conditions.

a unified approach. As a bonus, Propositions 5 and 6 demonstrate a surprising closeness between utility representability and two-function representability, a closeness not apparent in representability theorems in the literature.

The following comment by Campión et al. (2009) in the section titled "open problems" describes the hole in the literature into which our paper fits: "*But nothing similar is known for topologies induced by other kinds of binary relations ...*" The italics are theirs.

The paper is organized as follows. After a section of topological and preference-related definitions, Section 3 contains characterizations of topologies that are, and examples of topologies that are not, induced by binary relations. Section 4 contains topological versions of preference representation theorems and illustrative examples. As described above, there exist in the literature characterizations of topologies induced by binary relations represented by utility functions. We nevertheless include our Proposition 5 for the reasons stated in the last paragraph but one. Propositions 2 and 6 and the examples, especially Examples 1 and 2, are our main results. Section 5 concludes.

2. Preliminaries.

A *topology* τ on a set X is a collection of subsets of X such that $\emptyset \in \tau, X \in \tau$ and τ is closed under arbitrary unions and finite intersections. A *subbasis* S of a topology τ is a subset of τ such that $O \in \tau - \{X, \emptyset\}$ implies O is a union of finite intersections of elements of S. A *basis* β of a topology τ is a subset of τ such that $O \in \tau - \{\emptyset\}$ implies O is union of elements of β. A topology is *second countable* if it has a countable basis, and *first countable* if for $x \in X$ there is a countable subset $\{O^i\}$ of τ such that $x \in O \in \tau$ implies $x \in O^i \subseteq O$ for some i.

If τ is a topology on a finite set X, define $\{O_x\}_{x \in X}$ by setting $O_x = \cap_{x \in O \in \tau} O$. Then O_x is the smallest element of τ containing x. In general, when we index a collection of subsets of X by the elements of X, that is, when we write $\{A_x\}_{x \in X}$, it will not be the case that we require $x \in A_x$.

A set X with topology τ is *disconnected* if there exist $U, V \in \tau$ such that $U \neq \emptyset, V \neq \emptyset$, $U \cap V = \emptyset$ and $X = U \cup V$. Otherwise, X is *connected*.

A binary relation B on a set X is a subset of $X \times X$. For convenience, if B is a binary relation we write xBy instead of $(x, y) \in B$. If B is a binary relation on a set X, $L_B(x) = \{y \in X\colon xBy\}$ and $G_B(x) = \{y \in X\colon yBx\}$; the complement of the converse of B is $\overline{B} = \{(x, y) \in X \times X\colon \textit{not}(yBx)\}$; B is asymmetric if xBy implies not(yBx), *transitive* if $xByBz$ implies xBz, *pseudotransitive* if $xBa\overline{B}bBy$ implies xBy, *separable* if there exists countable $Q \subseteq X$ such that xBy implies $x\overline{B}z\overline{B}y$ for some $z \in Q$ and $*$-*separable* if there exists countable $Q \subseteq X$ such that xBy implies $y \in L_B(z) \subseteq L_B(x)$ for some $z \in Q$.

A binary relation B on a set X induces a topology τ on X, the collection of subsets of X consisting of $X, \emptyset$ and all unions of finite intersections of $L_B(x)$'s and $G_B(x)$'s. In other words, the topology generated by B has $\{L_B(x), G_B(x)\}_{x \in X}$ as a subbasis. A topology induced by a binary relation is an *induced topology*.

We will be concerned with two forms of preference representation. A *utility function* representing a binary relation B on a set X is a function $u\colon X \to \Re$ such that xBy if and only if $u(x) > u(y)$. Notice that the existence of a utility function representation places severe restrictions on a binary relation; it must be the asymmetric part of a total preorder. A *two-function representation* for a binary relation on a set X is a pair of functions $f, g\colon X \to \Re$ such that xBy if and only if $f(x) > g(y)$.

3. Characterizing Order Topologies.

A simple observation leads to our first example, a topology that is not an induced topology.

Proposition 1. *An induced topology on a countable set is second countable.*

Proof. Suppose B is a binary relation on a countable set X. The topology induced by B has countable subbasis $S = \{L_B(x), G_B(x)\}_{x \in X}$ and therefore a countable basis consisting of X and all finite intersections of elements of S

Example 1. A topology that is not an induced topology. We describe Arens-Fort space (Arens, 1950) and show that it is not an induced topology. Let $X = Z^+ \times Z^+$ and define τ by $O \in \tau$ if $O = \emptyset$, or O is a singleton $\{(m, n)\}$ such that $(m, n) \neq (0, 0)$, or $(0, 0) \in O$ and $S_m = \{n \in Z^+\colon (m, n) \in X - O\}$ is finite for all but finitely many $m \in Z^+$, or O is any union of the above.

Arens-Fort space was originally constructed to provide an example of a topology on a countable set that is not first countable, and that is the property we need. For completeness, we provide a proof of that well-established fact (for example, see Steen and Seebach (1970), Example 26). Given a countable collection $\{O^k\}_{k=1}^{+\infty}$ such that $(0,0) \in O^k \in \tau$ for all k, we will construct $O \in \tau$ such that $(0,0) \in O$ and for all k, $O^k \nsubseteq O$. Choose (m_1, n_1) such that $(m_1, n_1) \in O^1$ and $m_1 > 0$; choose (m_2, n_2) such that $(m_2, n_2) \in O^2$ and $m_2 > m_1$; etc. Let $O = X - \{(m_1, n_1), (m_2, n_2), \ldots\}$. Then for all k, $O^k \nsubseteq O$.

Since Arens-Fort space is not first countable, it is not second countable and by Proposition 1 it is not induced by a binary relation.

An argument like that above cannot establish the existence of a topology on a *finite* set that is not an induced topology, since such an argument would proceed as follows: "Construct τ on a finite set X such that every subbasis has cardinality greater than $2|X|$. Then τ is not an induced topology, since an induced topology would have a subbasis $\{L_B(x), G_B(x)\}_{x \in X}$ of cardinality at most $2|X|$". However, it is not possible to construct such a topology τ, since a topology τ on a finite set X has subbasis $\{O_x\}_{x \in X}$ with cardinality at most $|X|$.

We therefore proceed by characterizing induced topologies. Propositions 2 and 3 are easily proven, but useful nonetheless; they will be used to construct a topology on a finite set that is not an induced topology.

Proposition 2. *A topology τ on a set X is an induced topology if and only if there exists a collection $\{U_x, V_x\}_{x \in X}$ of subsets of X such that*

$$\{U_x, V_x\}_{x \in X} \text{ is a subbasis for } \tau \tag{1}$$

and

$$\text{for } x, y \in X,\ x \in U_y \text{ if and only if } y \in V_x \tag{2}$$

Proof. Suppose τ is a topology on a set X.

($\Rightarrow$) If B is a binary relation on X inducing τ, for all $x \in X$ let $U_x = L_B(x)$ and $V_x = G_b(x)$. Then (1) follows from the definition of an induced topology and (2) follows from the definition of $L_B(x)$ and $G_B(x)$.

($\Leftarrow$) If $\{U_x, V_x\}_{x\in X}$ satisfies (1) and (2), define B on X by setting $L_B(x) = U_x$ for $x \in X$. By (2), $V_x = G_B(x)$. By (1) B induces τ. ∎

Proposition 3. *Suppose τ is a topology on a set X, $\{U_x, V_x\}_{x\in X}$ satisfies (1) and (2), and $x, y \in X$. If $x \in O \in \tau$ implies $y \in O$, then $U_x \subseteq U_y$ and $V_x \subseteq V_y$. If X is finite, then $y \in O_x$ implies $U_x \subseteq U_y$ and $V_x \subseteq V_y$.*

Proof. Given the hypotheses of Proposition 3, if $z \in U_x$ then by (2) $x \in V_z$. By (1), $V_z \in \tau$. Since $x \in V_z \in \tau$, $y \in V_z$. Therefore by (2), $z \in U_y$. This proves $U_x \subseteq U_y$. By an identical proof, $V_x \subseteq V_y$.

The statement when X is finite follows since $y \in O_x$ implies $y \in \cap_{x\in O\in\tau} O$ implies ($x \in O \in \tau$ implies $y \in O$). ∎

Example 2. A topology on a finite set that is not an induced topology. Let $X = \{a^1, a^2, a^3,$
$a^4, a^5, a^6, a^7, b, c, d, r, s, t\}$. Define τ by $O_{a^k} = X - \{a^1, a^2, \ldots, a^{k-1}\}$ for each a^k, $O_b = \{b, r, s\}$, $O_c = \{c, s, t\}$, $O_d = \{d, r, t\}$, $O_r = \{r\}$, $O_s = \{s\}$, $O_t = \{t\}$.

To see that τ is not an induced topology, suppose S is a subbasis of τ. If $2 \le k \le 7$, then $O_{a^k} \in S$, since $a^k \in O \in \tau - \{O_{a^k}\}$ implies $a^{k-1} \in O$. Also, $O_b \in S$, since $b \in O \in \tau - \{O_b\}$ implies $t \in O$. Similarly, $O_c, O_d \in S$.

Now suppose τ is an induced topology. By Proposition 2 there exists a subbasis $\{U_x, V_x\}_{x\in X}$ such that $y \in U_x$ if and only if $x \in V_y$. By Proposition 3

$$\begin{aligned} U_{a^1} \subseteq U_{a^2} \subseteq \ldots \subseteq U_{a^7} \subseteq U_b \cap U_c \cap U_d;\ U_b \subseteq U_r \cap U_s; \\ U_c \subseteq U_s \cap U_t \text{ and } U_d \subseteq U_r \cap U_t \end{aligned} \tag{3}$$

Since $\{O_b, O_c, O_d\} \subseteq \{U_x, V_x\}_{x\in X}$, by the symmetry of b, c and d with respect to τ and of $\{U_x\}_{x\in X}$ and $\{V_x\}_{x\in X}$ with respect to Propositions 1 and 2, we can assume $\{O_b, O_c\} \subseteq \{U_x\}_{x\in X}$. Since $O_b \not\subseteq O_c$ and $O_c \not\subseteq O_b$, by (3) $\{O_b, O_c\} \subseteq \{U_b, U_c, U_d, U_r, U_s, U_t\}$ and $U_b \cap U_c \cap U_d \subseteq O_b \cap O_c = \{s\}$. Therefore $U_{a^1} \subseteq U_{a^2} \subseteq \ldots \subseteq U_{a^7} \subseteq \{s\}$. Then $x \neq s$ implies $x \notin U_{a^k}$ implies $a^k \notin V_x$. Therefore $V_{a^1} \cup V_{a^2} \cup \ldots \cup V_{a^7} \cup V_b \cup V_c \cup V_d \cup V_r \cup V_t \subseteq \{b, c, d, r, s, t\}$. It must be that $\{O_{a^2}, O_{a^3}, \ldots, O_{a^7}\} \subseteq \{U_b, U_c, U_d, U_r, U_s, U_t, V_s\} - \{O_b, O_c\}$. Since the set to the left of the inclusion has 6 distinct elements and the set to the right has at most 5, our assumption that τ is an induced topology has led to a contradiction.

4. Induced topologies and preference representation.

We now turn from characterizing induced topologies to characterizing induced topologies generated by binary relations with a specified form of representation.

4.1 Utility functions.

Proposition 4. *A topology on a finite set X is induced by a binary relation represented by a utility function if and only if*

$$x, y \in X \text{ implies } O_x = O_y \text{ or } O_x \cap O_y = \emptyset \tag{4}$$

Proof.

($\Rightarrow$) Suppose τ on X is induced by B represented by utility function u, $x \in X$, $u(x) \neq \max u(x)$ and $u(x) \neq \min u(X)$. Choose $z^1, z^2 \in X$ such that $u(z^1) = \max\{u(z)\colon\ u(z) < u(x)\}$ and $u(z^2) = \min\{u(z)\colon\ u(z) > u(x)\}$. Then $O_x = \{y \in X\colon\ u(y) > u(z^1)\} \cap \{y \in X\colon\ u(y) < u(z^2)\} = \{y \in X\colon\ u(y) = u(x)\}$. Therefore $O_x = O_y$ if $u(x) = u(y)$ and $O_x \cap O_y = \emptyset$ if $u(x) \neq u(y)$, so that (4) holds. Similarly, (4) holds if $x = \max u(X)$ or $x = \min u(X)$.

($\Leftarrow$) Let $\{O^1, O^2, \ldots, O^m\}$ be an enumeration of $\{O_x\}_{x \in X}$ with no duplicates, that is, such that $O^i \neq O^j$ if $i \neq j$. Define $u\colon\ X \to \Re$ by $u(x) = j$ if $x \in O^j$. By (4) $x \in O^j$ and $i \neq j$ together imply $x \notin O^i$ so that u is well-defined. Define B on X by xBy if $u(x) > u(y)$, so that u is a utility function representing B. Also B induces τ since if $O_x = O^j$, $L_B(x) = \cup_{i=1}^{j-1} O^i \in \tau$; $G_B(x) = \cup_{i=j+1}^{m} O^i \in \tau$; and $O^j = L_B(y) \cap G_B(z)$ where $y \in O^{j+1}$ and $z \in O^{j-1}$ if $1 < j < m$, $O^j = L_B(y)$ where $y \in O^2$ if $j = 1 < m$, $O^j = G_B(z)$ where $z \in O^{m-1}$ if $j = m > 1$, $O^j = X$ if $j = 1 = m$. ■

Example 3. Let $X = \{a, b\}$ and $\tau = \{\{a, b\}, \{a\}, \emptyset\}$. Then τ is an induced topology induced by $B = \{(a, a)\}$, but by Proposition 4, τ is not induced by a binary relation represented by a utility function, since $O_a \neq O_b$ and $O_a \cap O_b \neq \emptyset$.

Of course since $|X| = 2$, Example 3 could have been dealt with without appealing to Proposition 4.

Next we present a characterization that is valid whether X is finite or infinite. As was mentioned earlier, topologies induced by a binary relation represented by a utility function have already been characterized, for example by Campión et al. (2009), Theorem 3.1 and Lemma 4.1. Nevertheless, we include the following as a bridge between Propositions 2 and 6 and because together with Proposition 6 it demonstrates a strong similarity between utility representability and two-function representability.

Proposition 5. *A topology τ on a set X is induced by a binary relation represented by a utility function if and only if there exists a collection $\{U_x, V_x\}_{x\in X}$ of subsets of X satisfying (1),(2), (5), (6) and (7), where (5), (6) and (7) are as follows:*

$$\text{for } x, y \in X,\ U_x \subseteq U_y \text{ or } U_y \subseteq U_x,\ \text{ and } V_x \subseteq V_y \text{ or } V_y \subseteq V_x \tag{5}$$

$$\text{for } x \in X,\ U_x \cap V_x = \emptyset \text{ and } U_x \cup V_x = X - [x], \text{where}$$
$$[x] = \{y \in X\colon \text{ for all } O \in \tau,\ y \in O \text{ if and only if } x \in O\} \tag{6}$$

$$\text{there exists countable } \ Q \subseteq X \text{ such that}$$
$$\text{if } \ x, y \in X \text{and } y \in U_x,\ \text{ then } y \in U_z \subseteq U_x \text{ for some } z \in Q \tag{7}$$

Proof. ($\Rightarrow$) Suppose u is a utility function that represents binary relation B on X and B induces topology τ.

As in the proof of Proposition 2, $\{L_B(x), G_B(x)\}_{x\in X}$ satisfies (1) and (2). If $x, y \in X$ then $L_B(x) \subseteq L_B(y)$ and $G_B(y) \subseteq G_B(x)$ if $u(y) \geq u(x)$, and $L_B(y) \subseteq L_B(x)$ and $G_B(x) \subseteq G_B(y)$ if $u(y) < u(x)$, so that (5) holds.

If $x \in X$, then $L_B(x) \cap G_B(x) = \{y \in X\colon\ u(y) < u(x) < u(y)\} = \emptyset$ and $L_B(x) \cup G_B(x) = \{y \in X\colon\ u(y) < u(x) \text{ or } u(y) > u(x)\} = X - [x]$, so that (6) holds.

Finally, let Y be a countable subset of X such that $u(Y)$ is dense in $u(X)$ and let A be a countable subset of X such that $x, y \in X$, $u(x) > u(y)$ and $=]u(y), u(x)[\cap u(X) = \emptyset$ together imply $u(x) = u(z)$ for some $z \in A$. Let $Q = Y \cup A$. Now suppose $y \in L_B(x)$.

Then $u(x) > u(y)$. Therefore there exists $z \in Q$ such that $u(x) \geq u(z) > u(y)$. Therefore $y \in L_B(z) \subseteq L_B(x)$ and (7) holds.

($\Leftarrow$) Given a topology τ on a set X and $\{U_x, V_x\}_{x \in X}$ satisfying (1), (2), (5), (6) and (7), as in the proof of Proposition 2 define B on X by letting $L_B(x) = U_x$ for $x \in X$. Then as in the proof of Proposition 2, B induces τ. It remains to show that B can be represented by a utility function. We first establish some properties of B.

Asymmetry. Suppose $x, y \in X$ and xBy. Then $y \in L_B(x)$ and by (6) $y \notin V_x = G_B(x)$ by (2), so that not(yBx). Therefore B is asymmetric.

Transitivity of the complement of the converse of B. Let $\overline{B}$ on X be defined by $x\overline{B}y$ if not(yBx). Suppose $x\overline{B}y\overline{B}z$. Then $y \notin G_B(x)$. By (6) $y \in [x]$ or $y \in L_B(x)$. If $y \in [x]$, then using $y\overline{B}z$ and the definition of $[x]$, $x\overline{B}z$. If $y \in L_B(x)$, then $x \in G_B(y)$. Since $x \in G_B(y)$ and by asymmetry $x \notin G_B(x)$, by (5) $G_B(x) \subseteq G_B(y)$. Since $y\overline{B}z$ and $G_B(x) \subseteq G_B(y)$, $x\overline{B}z$. Therefore $\overline{B}$ is transitive.

Separability. We must show there exists countable $Q \subseteq X$ such that xBy implies $x\overline{B}z\overline{B}y$ for some $z \in Q$. Choose Q as guaranteed in (7). Suppose xBy. Then $y \in L_B(x)$. Therefore $y \in L_B(z) \subseteq L_B(x)$ for some $z \in Q$. Since $x \notin L_B(x)$ by asymmetry of B, $x \notin L_B(z)$ and $x\overline{B}z$. Since $y \in L_B(z)$, $z \notin L_B(y)$ by the asymmetry of B, so that $z\overline{B}y$.

The asymmetry and separability of B and the transitivity of $\overline{B}$ together imply the existence of a utility function representing B (see Debreu (1964) or Bridges and Mehta (1995) whose first five chapters contain a number of characterizations of binary relations representable by utility functions). ■

Example 4. The Euclidean topology on $\Re^2$ is not induced by a binary relation B represented by a utility function u. Suppose the contrary. By the proof of Proposition 5 ($\Rightarrow$), $\{L_B(x), G_B(x)\}_{x \in \Re^2}$ satisfies (1), (2), (5), (6) and (7). Clearly, $u(\Re^2)$ must contain more than two points. Therefore there exists $x \in \Re^2$ such that $L_B(x) \neq \emptyset$ and $G_B(x) \neq \emptyset$. By (6), $L_B(x) \cap G_B(x) = \emptyset$ and $L_B(x) \cup G_B(x) = \Re^2 - [x] = \Re^2 - \{x\}$. Therefore $\Re^2 - \{x\}$ with the Euclidean topology is disconnected. But $\Re^2 - \{x\}$ is connected. We have arrived at a contradiction. This example has been dealt with previously by Candeal et al. (1993).

Notice that the Euclidean topology on $\Re^2$ is an induced topology, since it is generated by the strong Pareto relation xPy if $x_1 > y_1$ and $x_2 > y_2$.

Example 5. Let $X = [0, 1]$ and let $\tau = \{[0, r[\ : 0 \leq r \leq 1\} \cup \{X\}$. Then τ is not induced by a binary relation represented by a utility function. Suppose the contrary. By Proposition 5 there exists $\{U_x, V_x\}_{x \in X}$ satisfying (1), (2), (5), (6) and (7). Since $\{U_x, V_x\}_{x \in X}$ is a subbasis of τ, there must be $x \in X$ such that $U_x = [0, r[$ with $0 < r < 1$ or $V_x = [0, r[$ with $0 < r < 1$. Without loss of generality, assume the former. By (6), $V_x = [r, 1] - [x] = [r, 1] - \{x\} \notin \tau$. We have arrived at a contradiction.

Notice that τ is an induced topology; for $x \in X$ let $U_x = V_x = [0, 1 - x[$. Then $\{U_x, V_x\}_{x \in X}$ satisfies (1) and (2) (since $y \in U_x$ if and only if $y < 1 - x$ if and only if $x < 1 - y$ if and only if $x \in V_y$). By Proposition 2, τ is an induced topology. More directly, let xBy if $y < 1 - x$. Then $\{L_B(x)\}_{x \in X} = \{G_B(x)\}_{x \in X} = \tau - \{X\}$, so that $\{L_B(x), G_B(x)\}_{x \in X}$ is a subbasis for τ.

4.2 Two-function representations.

Proposition 6. *A topology τ on a set X is induced by a binary relation with a two-function representation if and only if there exists a collection $\{U_x, V_x\}_{x \in X}$ of subsets of X satisfying (1), (2), (5) and (7).*

Proof ($\Rightarrow$) Suppose τ on X is induced by a binary relation B with two function representation f, g. By the proof of Proposition 2 ($\Rightarrow$), $\{L_B(x), G_B(x)\}_{x \in X}$ satisfies (1) and (2).

Suppose $x, y \in X$ and $L_B(x) \not\subseteq L_B(y)$. Then there exits $z \in X$ such that $z \in L_B(x) - L_B(y)$, so that $f(x) > g(z) \geq f(y)$. If $w \in L_B(y)$, then $f(y) > g(w)$ which implies $f(x) > g(w)$. Since $w \in L_B(y)$ implies $w \in L_B(x)$, $L_B(y) \subseteq L_B(x)$. Similarly, for $x, y \in X$ $G_B(x) \subseteq G_B(y)$ or $G_B(y) \subseteq G_B(x)$. Therefore (5) holds.

Next let Y be a countable subset of X such that $f(Y)$ is dense in $f(X)$; let A be a countable subset of X such that $x \in X$, $\epsilon > 0$ and $]f(x) - \epsilon, f(x)[\cap f(X) = \emptyset$ together imply $f(z) = f(x)$ for some $z \in A$; and let $Q = Y \cup A$. Now suppose $y \in L_B(x)$. Then $f(x) > g(y)$ so that $f(x) \geq f(z) > g(y)$ for some $z \in Q$. Then $y \in L_B(z) \subseteq L_B(x)$ so that

(7) holds.

($\Leftarrow$) Given τ and $\{U_x, , V_x\}_{x \in X}$ satisfying (1), (2), (5) and (7), define B on X by setting $L_B(x) = U_x$ for each $x \in X$. Then by (1) and (2), B induces τ. It remains to show that B has a two-function representation. First we establish two properties of B.

pseudotransitivity. We need to show that for $x, a, b, y \in X$, $xBa\overline{B}bBy$ implies xBy. Suppose $xBa\overline{B}bBy$. Then $a \in L_B(x) - L_B(b)$ and $y \in L_B(b)$. By (5), $a \in L_B(x) - L_B(b)$ implies $L_B(b) \subseteq L_B(x)$. Since $y \in L_B(b) \subseteq L_B(x)$, xBy.

$*$-separability. Notice from the definition section that $*$-separability is identical to (7).

By Proposition 9 of Doignon et al. (1984)[5], a pseudotransitive, $*$-separable binary relation has a two-function representation. ■

Example 6. Let $X = \{b, c, d, r, s, t\}$ and define τ by $O_b = \{b, r, s\}$, $O_c = \{c, s, t\}$, $O_d = \{d, r, t\}$, $O_r = \{r\}$, $O_s = \{s\}$, $O_t = \{t\}$. Then τ is not induced by a binary relation with a two-function representation. Suppose to the contrary that B induces τ and B has a two-function representation. It is easy to see that O_b is an element of every subbasis of τ, since $b \in O \in \tau - \{O_b\}$ implies $t \in O$. Similarly O_c and O_d are elements of every subbasis of τ. By the proof of 5($\Rightarrow$), $\{U_x, V_x\}_{x \in X}$ satisfies (1), (2) and (5). Therefore $\{O_b, O_c, O_d\} \subseteq \{L_B(x), G_B(x)\}_{x \in X}$. Without loss of generality, $\{O_b, O_c\} \subseteq \{L_B(x)\}_{x \in X}$. By (5), for $x, y \in X$, $L_B(x) \subseteq L_B(y)$ or $L_B(y) \subseteq L_B(x)$. Therefore $O_b \subseteq O_c$ or $O_c \subseteq O_b$, a contradiction.

Example 5 revisited . Let $X = [0, 1]$ and let $\tau = \{[0, r[\ : 0 \le r \le 1\} \cup \{X\}$. For $x \in X$ let $U_x = [0, 1 - x[\ ,\ \ V_x = [0, 1 - x[$. Then $\{U_x, V_x\}_{x \in X}$ satisfies (1), (2) and (5). To see that (7) is satisfied, let Q be the set of rationals in $[0, 1]$. If $x, y \in X$ and $y \in U_x$, then

[5] Actually the pseudotransitivity and $*$-separability characterization is a slight simplification of Doignon et al. (1984), Proposition 9, and follows from their Proposition 9, the first paragraph of their proof of Proposition 8, (ii) implies (i) and their Proposition 2(iii), a if and only if b. There are other characterizations of binary relations with two-function representation. See for example Theorem 1 of Bosi et al. (2001) for nine such characterizations

$x < 1$ and $y < 1 - x$. There exists $z \in Q$ such that $y < 1 - z < 1 - x$. Then $y \in U_z \subseteq U_x$. Since (1), (2), (5) and (7) hold, by Proposition 6 τ is induced by a binary relation with a two-function representation.

More directly, let $f(x) = 1 - x$ and $g(x) = x$. Then (f, g) is a two-function representation for a binary relation that induces τ.

Example 4 revisited . The Euclidean topology on $\Re^2$ is not induced by a binary relation B with a two-function representation. Suppose the contrary. By the proof of Proposition 6 ($\Rightarrow$), $\{L_B(x), G_B(x)\}_{x\in\Re^2}$ satisfies (1), (2), (5) and (7). Choose $a^1, a^2, a^3 \in \Re^2$ with open disks $D(a^1), D(a^2), D(a^3)$ centered at a^1, a^2, a^3 respectively such that the closures $\overline{D(a^1)}, \overline{D(a^2)}, \overline{D(a^3)}$ of the disks are mutually disjoint. By (1) and (2), each a^i is contained in a finite intersection of $L_B(x)$'s and $G_B(x)$'s which in turn is contained in $D(a^i)$. By (5) each finite intersection is an $L_B(x)$, a $G_B(x)$ or an intersection $L_B(x) \cap G_B(y)$. Again by (5), for at most one a^i is the intersection an $L_B(x)$, and for at most one a^i is the intersection a $G_B(x)$. Therefore for one a^i, say a^1, $a^1 \in L_B(x) \cap G_B(y) \subseteq D(a^1)$, $L_B(x) \not\subseteq D(a^1)$ and $G_B(y) \not\subseteq D(a^1)$.

Also, $L_B(x) \cap (\Re^2 - \overline{D(a^1)}) \neq \emptyset$ since $L_B(x) \not\subseteq D(a^1)$ and by (1) $L_B(x) \in \tau$. Similarly $G_B(y) \cap (\Re^2 - \overline{D(a^1)})) \neq \emptyset$. Next, $(L_B(x) \cap (\Re^2 - \overline{D(a^1)})) \cap (G_B(y) \cap (\Re^2 - \overline{D(a^1)})) = \emptyset$, since $L_B(x) \cap G_B(y) \subseteq D(a^1)$. Finally $(L_B(x) \cap (\Re^2 - \overline{D(a^1)})) \cup (G_B(y) \cap (\Re^2 - \overline{D(a^1)})) = \Re^2 - \overline{D(a^1)}$ since if $b \in (\Re^2 - \overline{D(a^1)}$ and $b \notin L_B(x) \cup G_B(y)$, then by (1), (2) and (5) $b \in L_B(z) \cap G_B(w) \subseteq D(b)$ where $L_B(x) \subseteq L_B(z)$, $G_B(y) \subseteq G_B(w)$ and $\overline{D(b)} \cap \overline{D(a^1)} = \emptyset$. Then $a^1 \in L_B(x) \cap G_B(y) \subseteq L_B(z) \cap G_B(w)$ but $L_B(z) \cap G_B(w) \cap \overline{D(a^1)} = \emptyset$, a contradiction.

We have now written $\Re^2 - \overline{D(a^1)}$ as the union of two disjoint non-empty elements of the Euclidean topology on $\Re^2 - \overline{D(a^1)}$. Therefore $\Re^2 - \overline{D(a^1)}$ is disconnected. Our assumption that the Euclidean topology on $\Re^2$ is induced by a binary relation with a two-function representation has led to a contradiction.

5. Concluding Remarks.

Our goal was a new perspective on preferences and preference representation arrived at via a study of topologies induced by binary relations. The first results obtained consisted of characterizations of induced topologies, of topologies induced by binary relations represented by utility functions and of topologies induced by binary relations with two-function representations. One observation that can be derived from these characterizations concerns the relationship between utility representability and two-function representability. The standard characterizations of these two forms of representability are quite different–a binary relation represented by a utility function must be asymmetric, but one with a two-function representation need not be; and the latter must be pseudotransitive, while the complement of the converse of the former must be transitive. However, our topological characterizations of these two forms of representability are quite similar; Propositions 5 and 6 differ only in that one extra condition, the simple condition (6), is required to hold in Proposition 5.

The second line of results consists of examples of topologies that are not induced topologies, of topologies that are not induced by binary relations with two-function representations, and of topologies induced by binary relations with two-function representations, but not by binary relations represented by utility functions. Most of these examples would not have been discovered without the characterization theorems. Furthermore, direct verification of the properties claimed for some of the examples would have been extremely difficult or impossible; for example, direct verification in Example 2 would have required examining 2^{169} binary relations.

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