THE BOURGUIGNON AND CHAKRAVARTY MULTIDIMENSIONAL POVERTY FAMILY:
A CHARACTERISATION

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ABSTRACT

The family of multidimensional poverty indices introduced by Bourguignon and Chakravarty (Journal of Economic Inequality, 2003) has attracted a great deal of interest in the field of measuring poverty.

In this paper we explore a number of properties fulfilled by the members of this family, related to the way in which the attributes are aggregated for each individual, and we show that the properties we highlight characterise the family.

Key Words: multidimensional poverty indices, Bourguignon and Chakravarty family, aggregativity.

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1. INTRODUCTION

This paper tackles the problem of multidimensional poverty. A number of multidimensional poverty indices has been proposed (see (UNDP (1997), Bourguignon and Chakravarty (1998, 2003), Chakravarty, Mukherjee and Ranade (1998), Tsui (2002), Chakravarty, Deutsch and Silber (2005), Maasoumi and Lugo (2006) among others), trying to measure this complex phenomenon. Although many of them have not been developed using an axiomatic approach, the setting of desirable axioms makes easier the normative analysis of this concept.

There exist in the literature two different procedures often used to construct indicators for measuring either deprivation or living standard in a multidimensional framework. The first combines the different attributes for each individual and then the summary indices are aggregated over individuals. The second summarizes for each attribute an index across individuals to construct, then, an indicator of all the attributes\(^1\).

Pattanaik et al. (2007) refer to these two approaches as “row-first” and “column-first” two stage procedures, and show that the latter procedure presents difficulties and “must lead to possibly untenable conditions”\(^2\). Therefore, the alternative procedure should be adopted to construct multidimensional indicators, in other words, a multidimensional index should aggregate the “summarized” deprivations or living standards of the individuals in the society.

Consequently, to construct a multidimensional index we first face the problem to aggregate, for each individual, his/her deprivations in the different attributes. For doing so different ways have been introduced. In this paper we explore the properties fulfilled by a family of multidimensional poverty measures derived by Bourguignon and

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1 This procedure is followed by the United Nations Development Programme’s Human Development Reports to construct both the Human Development Index and the Human Poverty Index.

2 We are indebted to Professor Peter Lambert for having introduced us to the Pattanaik et al. paper.
Chakravarty (2003) and show that these properties characterise the way of aggregation they propose. In a concluding section we discuss possible extensions.

2. NOTATION AND BASIC AXIOMS OF MULTIDIMENSIONAL POVERTY INDICES.

We consider a population consisting of \( n \geq 2 \) individuals endowed with a bundle of \( k \geq 1 \) attributes, such as income, health, education and so on. A multidimensional distribution among the population is represented by an \( n \times k \) real matrix \( X \), where the \( ij \)th entry denoted \( x_{ij} \) represents the \( i \)th individual’s amount of the \( j \)th attribute. The \( i \)th row of \( X \) is denoted \( \mathbf{x}_i \) and the \( j \)th column is denoted \( \mathbf{x}'_j \). We denote by \( M(n,k) \) the class of \( n \times k \) real matrices and let \( D = \bigcup_{n \in \mathbb{N}_+} \bigcup_{k \in \mathbb{N}_+} M(n,k) \).

As regards the identification of the poor through the specification of a poverty line, let’s consider \( z_j > 0 \) to be the minimum level of subsistence of the \( j \)th attribute and \( \mathbf{z} = (z_1, z_2, \ldots, z_k) \in \mathbb{R}^k_+ \) to be the \( k \)-vector of thresholds, such that \( \mathbf{z} \in Z \) where \( Z \) is a non empty subset of \( \mathbb{R}^k_+ \). In this paper we consider a person to be poor if he/she falls below the subsistence level of at least one attribute. Then for any \( X \in D \) and \( \mathbf{z} \in Z \), we denote the set of poor persons as \( Q(X,\mathbf{z}) = \{i \mid x_{ij} < z_j \text{ for some } j\} \) of cardinality \( q \). We define \( X(\mathbf{z}) \in M(q,k) \) as the matrix derived from \( X \) by selecting only rows \( \mathbf{x}_i \) such that \( i \in Q(X,\mathbf{z}) \).

Bourguignon and Chakravarty (2003) understand poverty in terms of relative shortfalls from the threshold level of attributes. We denote by \( u_{ij} = \max \left(1 - \frac{x_{ij}}{z_j}, 0\right) \) the
percentage of deprivation of the \(i\)th individual with respect to the \(j\)th attribute. Later we shall discuss possibilities for using absolute shortfalls in some circumstances.

In this paper a multidimensional poverty index is defined as a non-constant function \(P : D \times Z \rightarrow \mathbb{R}\) satisfying the following basic properties:

i) **Restricted Continuity**: \(P(X, z)\) is a continuous function in any poor individual’s attributes and in the poverty line.

ii) **Symmetry**: \(P(X, z) = P(\Pi X, z)\), for all \(n \times n\) permutation matrices \(\Pi\).

iii) **Monotonicity**: \(P(Y, z) \leq P(X, z)\) whenever \(Y\) is derived from \(X\) by increasing any one attribute with respect to a person who is poor.

iv) **Replication Invariance**: \(P(Y, z) = P(X, z)\) if \(Y\) is obtained from \(X\) by a replication.

v) **Normalization**: \(P(X, z) = 0\) if and only if \(x_{ij} \geq z_j\) for all \(i\) and for all \(j\).

vi) **Focus**: The poverty index remains unchanged if any attribute such that \(x_{ij} \geq z_j\) is increased for person \(i\).

vii) **Scale Invariance**: \(P(X, z)\) is scale invariant if for any \(X \in D\) and \(z \in Z\)

\[P(X, z) = P(X \Lambda, z \Lambda)\] for all diagonal matrices \(\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_k)\) with \(\lambda_j > 0\).

Later we shall discuss the possibilities to modify restricted continuity and scale invariance.

The following definition ensures that a poverty index is sensitive to inequality within the poor:

**Definition.** A multidimensional poverty \(P\) satisfies the Uniform Majorization Principle (UM) if \(P(Y, z) \leq P(X, z)\) whenever \(Y\) is derived from \(X\) by multiplying \(X(z)\) by a
bistochastic matrix $B$, such that $Y(z) = BX(z)$ is not a permutation of the rows of $X(z)$ and the attributes of the non-poor remain unchanged.

In empirical applications, if the population in which we want to measure poverty is split into groups according to certain socioeconomic or demographic characteristics, an appropriate requirement is to demand that if poverty in a subgroup decreases, the overall poverty does not increase. This property proposed by Foster and Shorrocks (1991) in the unidimensional framework may easily be extended to the multidimensional setting:

**Definition:** A multidimensional poverty index $P(X,z)$ is **Subgroup Consistent** if for any $X_1,Y_1 \in M(n_1,k)$ and $X_2,Y_2 \in M(n_2,k)$, with $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

$$P(Y,z) < P(X,z)$$ whenever $P(X_1,z) > P(Y_1,z)$ and $P(X_2,z) = P(Y_2,z)$.

By adapting the analysis in one dimension of Foster and Shorrocks (1991) it’s straightforward to prove that any subgroup consistent index can be expressed as:

$$P(X,z) = F \left[ \frac{1}{n} \sum_{i=1}^{n} \phi(x_i,z) \right]$$ (1)

where $\phi$ is an individual poverty function that indicates how the many aspects of poverty must be aggregated at the level of each person, and $F$ is a strictly increasing and continuous function.

3. **The Bourguignon and Chakravarty Multidimensional Poverty Family.**
Bourguignon and Chakravarty (2003) suggest a family of multidimensional poverty measures. They use an aggregator function of the relative shortfalls for the different attributes using the CES form. The specification for two attributes is the following

\[
P (X; z) = \frac{1}{n} \sum_{i=1}^{n} f \left[ \left( \frac{w_i u_{1i}^\theta + w_i u_{2i}^\theta}{\theta} \right)^{1/\theta} \right]
\]  

(2)

where \( f \) is an increasing and convex function such that \( f(0) = 0 \), \( w_j \) is the positive weight attached to the attribute \( j \) and the parameter \( \theta \) represents the elasticity of substitution between the relative shortfalls of the attributes for any person. They discuss the implications of various degrees of substitutability between the attributes according to different values of \( \theta \).

Taking as \( f \) the same functional form used by Foster et al. (1984) to derive the FGT family in the case of income poverty, Bourguignon and Chakravarty obtain the following family of multidimensional poverty measures.

\[
P^\alpha (X; z) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \left( \frac{w_i u_{1i}^\theta + w_i u_{2i}^\theta}{\theta} \right)^{\alpha/\theta} \right\}
\]  

(3)

where the positive parameter \( \alpha \) can be interpreted as the aversion of society against poverty. The larger \( \alpha \) is, the more sensitive is \( P^\alpha \) to the poorest\(^3\).

As Atkinson (2003) points out the crucial issues in the definition of this family are both the shape of the isopoverty contours, which depend only on \( \theta \), and the degree of concavity already present in the one dimensional case through the \( \alpha \)-parameter.

We shall analyze these two issues separately in order to better understand the behaviour of this family.

\(^3\) In common with the FGT, if \( \alpha \) is raised ceteris paribus, measured poverty in any distribution falls. But in comparisons the situation of the poorest becomes more important.
4. **First Stage: Aggregating Attributes for Each Individual.**

Let’s consider the aggregation function proposed in the Bourguignon and Chakravarty framework. For each individual they propose to aggregate across two attributes in the following way:

\[
A(u_1, u_2; w_1, w_2) = \left( w_1 u_1^\theta + w_2 u_2^\theta \right)^{\frac{1}{\theta}}
\]

where \( u_j \), and \( w_j \) represent, respectively, the relative shortfall and the weight of attributes \( j \).

We are going to explore nine properties fulfilled by this aggregation function. In addition, these properties allow us to characterise what type of function \( A \) should be. The first five are ordinal properties, in the sense of telling us whether the level of individual deprivation should increase in response to a certain change, but without imposing a numerical effect.

First of all, in measuring the individual deprivation, the order or name of the attributes is irrelevant; that is, attributes are treated symmetrically.

1. **Symmetry:** \( A(u_1, u_2; w_1, w_2) = A(u_2, u_1; w_2, w_1) \) for all \( u_1, u_2, w_1, w_2 \).

Second, if the weight of the attribute whose shortfall is the greatest increases, the aggregation function also increases:

2. **Increasing in the (2\(^{nd}\)) weight:** If \( a < b \) then \( A(a, b; w_1, w_2) < A(a, b; w_1, w') \) whenever \( w_2 < w' \).

Third, if the shortfall of one attribute increases, the aggregate also increases:

3. **Increasing in the (2\(^{nd}\)) variable:** \( A(u_1, u_2; w_1, w_2) < A(u_1, u'_2; w_1, w_2) \) for \( u_2 < u'_2 \).
Moreover, the deprivation function is homogeneous of 0\textsuperscript{th} degree in the weights, in other words, if the weight on every attribute is modified in the same proportion, the aggregate does not change:

**4. Homogeneity (0\textsuperscript{th} degree) in the weights:** \( A(a,b;w_1,t,w_2) = A(a,b;w_1',w_2') \) for all values of \( w_1, w_2 \geq 0 \) and for \( w_1 + w_2, t > 0 \).

If possibly there is no weight attached to one attribute, then the following inequality holds:

**5. Internality:** If \( a < b \) then \( A(a,b;1,0) < A(a,b;w_1,w_2) < A(a,b;0,1) \) whenever \( w_1, w_2 > 0 \).

Finally a property which allows us to carry out multilevel decomposition by attributes. The individual deprivation function is consistent in multilevel decompositions by subgroups of attributes:

**6. Associativity:**

\[
A[A(u_1,u_2;w_1,w_2),u_3;w_1 + w_2,w_3] = A[u_1,A(u_2,u_3;w_2,w_3);w_1,w_2 + w_3]
\]

for all \( u_1,u_2,u_3;w_1,w_2,w_3 \).

Now, we are going to explore three cardinal properties. If for any given poor individual, the percentages of deprivation in all the attributes are the same, for instance, he/she suffers a percentage of deprivation of 60% in each attribute, then the aggregate deprivation coincides with this value:

**7. Reflexivity:** \( A(u,u;w_1,w_2) = u \) for all values of \( u, w_1, w_2 \).

In the case that the weight attached to one attribute is 1 whereas the weight attached to the other is 0, then the aggregate deprivation takes exactly the value of the first attribute, that is:

**8. Normalization:** \( A(a,b;1,0) = a \)
Finally, if for each individual, the shortfall of each attribute is modified in the same proportion, then the deprivation felt by that individual changes in the same proportion, that is:

**9. Homogeneity (1st degree) in the shortfalls:** \( A(\lambda a, \lambda b; w_1, w_2) = \lambda A(a, b; w_1, w_2) \) for all \( 0 \leq a \leq 1, 0 \leq b \leq 1, 0 \leq \lambda \leq 1 \) and \( \lambda^{-1} \leq \max(a, b) \)

As already mentioned these nine properties characterise, for any individual, the aggregative function in the Bourguignon and Chakravarty framework.

**Proposition 1:** Let \( A: \mathbb{R}_+^k \times \mathbb{R}_+^k \to \mathbb{R}_+ \). Conditions 1 through 9 extended to \( k \geq 2 \) attributes are necessary and sufficient for the function \( A \) to be of the form

\[
A(u_1, u_2, \ldots, u_k; w_1, w_2, \ldots, w_k) = \left( w_1 u_1^\theta + w_2 u_2^\theta + \ldots + w_k u_k^\theta \right)^{\frac{1}{\theta}}
\]  

(1)

where \( \theta \neq 0 \) is a real constant and \( w_1, w_2, \ldots, w_k \) are positive constants such that

\[
\sum_{j=1}^k w_j = 1.
\]

or as

\[
A(u_1, u_2, \ldots, u_k; w_1, w_2, \ldots, w_k) = Cu_1^{w_1}u_2^{w_2} \ldots u_k^{w_k}
\]  

(2)

where \( C > 0 \) and \( w_1, w_2, \ldots, w_k \) are real constants such that \( w_1w_2\ldots w_k \neq 0 \) and \( \sum_{j=1}^k w_j = 1 \).

**Proof.** It is straightforward from Azcel (1966, p.242) and Eichhorn (1978, Theorem 2.2.1, p.32).

Q.E.D.
5. SECOND STAGE: COMBINING INDIVIDUAL DEPRIVATIONS.

Let’s go back to the Bourguignon and Chakravarty family of multidimensional poverty measures. Using the aggregation function, (3) can be rewritten

\[ P^\alpha(X; \tilde{z}) = \frac{1}{n} \sum_{i=1}^{n} A_i^{\alpha} \]  

(7)

Notice that if the deprivations of all individuals are multiplied by the same constant \(0 < \lambda \leq 1\) and \(\lambda^{-1} \leq \max_{1 \leq i \leq n} (A_i)\), then the poverty level is multiplied by to the \(\alpha\)-power, that is:

\[ \frac{1}{n} \sum_{i=1}^{n} (\lambda A_i)^{\alpha} = \frac{\lambda^{\alpha}}{n} \sum_{i=1}^{n} A_i^{\alpha} = \lambda^{\alpha} P^\alpha(X; \tilde{z}) \]

To be able to characterize formally the subgroup consistent families of multidimensional indices of poverty, let’s think now about, how we should combine the individual deprivations for a group of \(n\) persons? It seems to be natural think that for any \(X, Y \in D, \tilde{z} \in Z\), such that \(P(X; \tilde{z}) \leq P(Y; \tilde{z})\) if the aggregate deprivation of all people, in both distributions, are modified in the same proportion, this modification should not have influence on poverty rankings, this is, the multiplicative factor and \(P(X; \tilde{z})\) must work independently. We can articulate this idea in the following way through the use of the functions \(\phi\) and \(A_i\).

\[ \frac{1}{n} \sum_{i=1}^{n} \phi(\lambda A_i) = C(\lambda) \frac{1}{n} \sum_{i=1}^{n} \phi(A_i) \]  

(8)

where \(C(\lambda)\) is a positive function, \(0 < \lambda \leq 1\) and \(\lambda^{-1} \leq \max_{1 \leq i \leq n} (A_i)\).

Setting \(A_1 = A_2 = \ldots = A_n\), we obtain

\[ \phi(\lambda A_i) = C(\lambda)\phi(A_i) \]  

(9)

which is a Pexider equation whose solution (Azcél 1966, p.145) implies that there exist nonnegative constants \(a\) and \(\alpha\) such that \(\phi(A_i) = a A_i^{\alpha}\) and \(C(\lambda) = \lambda^{\alpha}\).
Therefore the follow proposition holds

**Proposition 2:** \( P(X; z) \) is a subgroup consistent, relative multidimensional poverty index constructed using an attribute-first two-stage procedure satisfies the condition (9) if and only if there exists a strictly increasing and continuous function \( F : \mathbb{R} \rightarrow \mathbb{R}_+ \) such that

\[
P_\alpha (X; z) = F \left( \frac{1}{n} \sum_{i=1}^{n} A_i^\alpha \right)
\]

(10)

where \( A_i \) denotes the individual deprivation function

If the individual deprivation functions \( A_i \) satisfy also the conditions 1 through 9, it follows immediately that:

**Corollary:** \( P(X; z) \) is a subgroup consistent, relative multidimensional poverty index constructed using an attribute-first two-stage procedure satisfies the condition (9) and the individual deprivation functions \( A_i \) satisfy the conditions 1 through 9 if and only if there exists a strictly increasing and continuous function \( F : \mathbb{R} \rightarrow \mathbb{R}_+ \) such that

\[
P(\mathbf{X}; \mathbf{z}) = F \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \sum_{j=1}^{k} w_j u_{ij}^\theta \right] \right\}
\]

(11)

where \( \theta \neq 0, \alpha > 0 \) and \( w_1 > 0, w_2 > 0, \ldots, w_k > 0 \), such that \( \sum_{j=1}^{k} w_j = 1 \).

or

\[
P(\mathbf{X}; \mathbf{z}) = F \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \prod_{j=1}^{k} w_j^\alpha \right] \right\}
\]

(12)

where \( \alpha > 0 \) and \( w_1 > 0, w_2 > 0, \ldots, w_k > 0 \), such that \( \sum_{j=1}^{k} w_j = 1 \)
These indices are the multidimensional generalization of the Foster, Greer and Thorbecke (1984) family of subgroup decomposable poverty indices. The functional form (11) is precisely family of multidimensional poverty measures suggested by Bourguignon and Chakravarty (2003).

REFERENCES


