Measuring Inequality of Well-Being: A proposal based on a Multidimensional Gini Index

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Abstract

Individual well-being is inherently a multidimensional concept. An attempt to measure inequality in well-being should take this multidimensionality explicitly into account. In this paper we follow a normative procedure to derive measures of well-being inequality from their underlying multidimensional social evaluation function. The social evaluation function itself is characterized by an explicit two step approach. In a first step, an index of individual well-being is derived from a set of attractive properties. The second step aggregates the individual well-being indices into a measure of societal well-being. Hereby we allow the evaluation of well-being to depend both on its level and on the position of the individual in the distribution. A multidimensional single-parameter Gini inequality measure is derived from the multidimensional social evaluation function. In this context, we investigate the role of a multidimensional generalization of the Pigou-Dalton transfer principle and investigate the sensitivity of our measures to changes in the correlation between the dimensions. An application of the measure is presented using individual well-being data from Indonesian households in 1993 and 2000 and results are compared with those obtained applying a multidimensional Gini index that is not sensitive to correlations (2005).

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1 Introduction

Many argue that well-being is inherently a multidimensional concept (Rawls 1971, Sen 1985, Streeten 1994, among others). Dimensions of individual well-being can be typical outcome variables such as per capita income, health status, educational attainment, or housing characteristics. These dimensions are neither freely tradable nor perfectly correlated to income. As a result, an attempt to measure inequality of well-being should take this multidimensionality explicitly into account. In this paper we present a multidimensional generalization of the Gini coefficient to measure inequality in well-being.

The Gini coefficient is probably the best known inequality measure in economics. From its introduction in the economics literature in 1912 by the Italian statistician Corrado Gini (Gini 1912, Gini 1921), the coefficient has become increasingly popular as a tool to measure inequality.

The present paper uses a normative procedure to derive a measure of well-being inequality. Starting from a set of attractive properties, we characterize a multidimensional social evaluation function, from which we derive a measure of inequality. This procedure has the advantage of bringing the chosen properties and value judgements explicitly to the fore, which in turn, allows researchers to form a clear opinion on the attractiveness of the inequality measure for the problem at hand. The characterization of the social evaluation function is carried out in two distinct steps in order to make the multidimensional and complex characterization problem as tractable and intuitive as possible.

In a first step, we derive an objective well-being index that summarizes the outcomes of every individual across the dimensions of well-being. Contrary to the approaches based on subjective well-being or happiness, these well-being indices are not dependent of subjective individual preferences over the dimensions. Rather, the ‘objective’ well-being function reflects the preferences of the society (the social planner or external observer) over the different dimensions. The well-being function is thus common to all members of society and permits well-being judgements that are not purely subjective but interpersonally justifiable and comparable (Gaspart 1998).

The second step aggregates the well-being indices across individuals to obtain a

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1 For a recent survey of multidimensional normative inequality measures, the reader is referred to Weymark (2006).
2 For an overview of the literature on subjective well-being, we refer to (Kahneman & Krueger 2006) and the references therein.
3 A potential concern with the ‘objective’ well-being approach is that it is overly paternalistic or perfectionist, since it represents the preferences of the social planner about what constitutes a good life for the individuals, and not the preferences of the individuals themselves (Fleurbaey 2005). Fleurbaey and Trannoy (2003) prove the impossibility of combining differences in individual preferences with multidimensional egalitarianism. However, this critique is less prevalent when the preferences of the social planner are derived by a democratic process or when personal satisfaction or ‘happiness’ is included as a relevant dimension of well-being.
measure of the well-being of a society as a whole. An essential feature of this step is that we allow for the contribution of an individual’s well-being to the total societal well-being to depend on the position of that individual in the total distribution. From the obtained rank-weighted social evaluation function, a family of inequality measures flows rather naturally. The obtained family of inequality measures is a multidimensional generalization of the one-dimensional Gini coefficient. We show that multidimensionality is confronted twice throughout the process from individual attainments to inequality measures: in the aggregation of dimensions in the first step, and in the derivation of the inequality measure.

Our paper is related to the recent work of Gajdos and Weymark (2005) on normative multidimensional generalized Gini indices. Driven by different sets of properties, the authors obtain a multidimensional generalized Gini social evaluation function which turns out to be a weighted-average of the one-dimensional Gini indices of the different dimensions. The two-step approach of Gajdos and Weymark can be considered to be the mirror-image of the procedure used here. We will argue that first aggregating across dimensions and then across individuals is more attractive, since it is more in line with the conceptual framework of welfare economics (Dutta, Pattanaik & Xu 2003) and the literature on multidimensional inequality (Maasoumi 1999). Moreover, the procedure used in this paper does not exclude a-priori sensitivity to the correlation between the dimensions. The importance of correlation between the dimensions in the analysis of multidimensional inequality is brought under attention by Atkinson and Bourguignon (1982), Rietveld (1990) and Tsui (1999).

Outside the normative approach, two other broad strategies have been followed to generalize the Gini coefficient into multiple dimensions, each extending an alternative one-dimensional definition of the Gini coefficient. Koshevoy and Mosler (1996) introduced the Lorenz zonoid as q-dimensional generalization of the standard Lorenz curve. From the volume of the Lorenz zonoid a multidimensional Gini coefficient suggests itself automatically (Koshevoy & Mosler 1997). A more pragmatic strategy is followed by Arnold (1987), Koshevoy and Mosler (1997) and Anderson (2004), who extend the definition based on the absolute differences between individuals. In particular, they propose a multidimensional distance measure to measure the pairwise distances between the vectors of outcomes. It has the virtue of being easy to implement but the essential underlying value judgements are made implicit, which makes the measure opaque from a normative viewpoint.

The rest of the paper is structured as follows. Section 2 introduces the notation. Section 3 derives an individual well-being index that aggregates individual’s attainments

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across all dimensions of well-being. Section 4 characterizes the single-series Gini social evaluation function to aggregate the well-being indices across individuals. Distributional concerns are introduced in section 5, paying special attention to a multidimensional generalization of the one-dimensional Pigou-Dalton transfer principle and the effect of changes in correlation between dimensions. From the obtained social evaluation function two multidimensional single-parameter Gini inequality measures are derived in section 6. Section 7 illustrates the use of these measures using household data from Indonesia in 1993 and 2000[5]. The exercise shows that the ranking of Indonesian regions by inequality level is sensitive to the multidimensional index used. Specifically, we compare rankings based on the measure proposed in this paper – which accounts for the interdependence between dimensions – with the measure proposed by Gajdos and Weymark (2005), which does not. Section 8 concludes the paper.

2 Notation

Let \( N = \{1, \ldots, n\} \) be the set of individuals \( i \) and \( Q = \{1, \ldots, q\} \) the set of dimensions \( j \). The population size \( n \geq 2 \) is assumed to be fixed in the first sections and then allowed to change. A distribution matrix \( X \) is an \( n \times q \) strictly positive real valued matrix whose element \( x_{ij} \) represents the attainments of individual \( i \) on dimension \( j \). When \( q = 1 \) matrix \( X \) is a one-dimensional vector. The domain of the distribution matrices is denoted \( \mathcal{D} \) and is restricted to the set of strictly positive real-valued distribution matrices.

Define \( x_i \) as the row vector of distribution matrix \( X \) that represents the attainments of individual \( i \) and \( x^j \) the column vector of the same matrix, representing the distribution of the \( j \)-th dimension of well-being. The set \( \mathcal{V} \) is the set of admissible attainment vectors, a \( 1 \times q \) subset of \( \mathcal{D} \).

The well-being relation \( \succeq \) is a binary relation on the set of vectors of attainments \( x_i \). It captures the preferences of the social planner with respect to bundles of attainments. By \( x_i \sim y_i \) we denote that the vector of attainments \( x_i \) is socially indifferent to vector \( y_i \). Strict social preference for \( x_i \) will be denoted \( x_i \succ y_i \). In the following section we identify the conditions under which the well-being relation \( \succeq \) can be represented by a real-valued function \( S(.) : \mathcal{V} \rightarrow \mathbb{R}^{++} \) which we call a well-being function. We refer to the image of the well-being function of vector \( x_i \) as the well-being index \( S(x_i) \).

The \( n \times 1 \) well-being vector \( S_X = [S(x_1), S(x_2), \ldots, S(x_n)] \) consists of the well-being indices for the \( n \) individuals, obtained from the \( n \) rows of distribution matrix \( X \). The set of the well-being vectors is denoted \( \mathcal{S} \) and is restricted to the set of strictly positive real vectors. The vector \( \tilde{S}_X \) is a permutation of vector \( S_X \) for which \( \tilde{S}(x_1) \geq \tilde{S}(x_2) \geq \)

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5To be completed.
... ≥ \( S(x_n) \). In other words, vector \( \vec{S_X} \) is ordered from the highest well-being index to the lowest. Let \( S^D \) denote the set of the ordered well being vectors \( \vec{S_X} \).

Let us define a social evaluation relation \( R \) as a binary relation on the set of well-being vectors \( S \). By \( S_X \mathcal{P} S_Y \) we denote that the vector of well-being indices \( S_X \) is strictly socially preferred to \( S_Y \). Social indifference between \( S_X \) and \( S_Y \) will be denoted \( S_X \not\mathcal{I} S_Y \). The social evaluation function \( W(.) : S \rightarrow \mathbb{R} \) is a real-valued function that represents the relation \( R \). The social evaluation index \( W(S_X) \) represents the societal evaluation of the well-being vector \( S_X \).

Finally, let \( R_\succ \) denote the compound social evaluation relation defined on the set of distribution matrices \( D \). We write \( X \mathcal{R}_\succ Y \), to denote that distribution matrix \( X \) is socially preferred to \( Y \). The compound social evaluation function \( W(.) : D \rightarrow \mathbb{R} \) is a real-valued function that represents the relation \( R_\succ \).

### 3 Aggregation across Dimensions of Well-Being

In this section we characterize an objective well-being function \( S(.) \) from a set of attractive properties.

The well-being function aggregates attainments across the multiple dimensions of well-being and captures different value judgements regarding trade-offs between dimensions and admissible transformations, among others. We crystalize those value judgements by presenting them as a set of explicit properties. We do not claim that the set of properties laid down here is the only one possible, \textit{au-contraire}, but we suggest that it represents an attractive set for the problem of measuring well-being. The problem of finding a well-being function that satisfies a set of properties is similar to the standard social choice problem of aggregating individuals’ preferences into a social welfare function, hence we can make use of existing social choice results.

The first three properties are standard ones.

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**Property 1. Ordering (ORD\(S\))** The binary relation \( \succeq \) is reflexive, complete and transitive on \( V \).

**Property 2. Continuity (CONT\(S\))** The sets \( \{ y_i \in V \mid y_i \succ x_i \} \) and \( \{ y_i \in V \mid x_i \succ y_i \} \) are open for all \( x_i \in V \).

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\(^6\)We do not discuss which dimensions of well-being should be included, rather we assume that these are either obtained by a democratic process or given by philosophical reasoning like the primary goods defined by Rawls (1971), the list of ‘functionings’ proposed by Nussbaum (2000), or the basic needs approach advocated by Streeten (1994).

\(^7\)Also Tsui (1996) and Ebert and Welsch (2004) exploit the similarity with the social choice problem to derive improvement indices and meaningful environmental indices, respectively.

\(^8\)For clarity we will always subindex the properties of the well-being relation \( \succeq \) with an \( S \). In the next section we will subindex the properties of the social evaluation relation \( R \) by \( W \), the properties of the compound social evaluation function \( R_\succ \) are not subindexed.
Property 3. Monotonicity (MON$^S_S$) For all $x_i, y_i$ in $V$, if $x_i \neq y_i$ and $x_i^j \geq y_i^j$ for all dimensions $j \in Q$, then $x_i \succ y_i$.

The first property imposes some structure on the well-being relation $\succeq$, requiring it to be a complete preorder without cycles. We refer to a well-being ordering whenever the well-being relation satisfies property ORD$^S_S$. The second property ensures that the well-being ordering is continuous and, hence, not oversensitive to minor changes in the attainments, for example caused by measurement errors. Monotonicity captures the intuition that all dimensions are desirable. Regardless of the exact form of the preferences of the social planner, this third property requires that increasing the attainment in any dimension, without decreasing an attainment in any other dimension, leads to a more preferable attainment vector.

The next property introduces separability across dimensions. More specifically, when two attainment vectors display the same attainment for a certain dimension, then the exact level of the common attainment is not decisive to the well-being ordering. An example helps to clarify: suppose two individuals have the same attainment in the income dimension and different attainment levels in health and education, then we assert that the exact level of income is not important to order the individuals with respect to their well-being. This property imposes an additive separable structure to the well-being function $S$ and excludes, for instance, rank-weighted well-being functions where the contribution of a dimension depends on its ranking in the attainment vector of the individual.

Property 4. Dimension Separability (DSEP$^S_S$) For all $x_i, y_i$ in $V$, the relation $\succeq$ is dimension separable if $x_i \succeq y_i$ and $x_i^j = y_i^j = \alpha$ for a dimension $j \in Q$, then $x_i^{-j}(\alpha) \succeq y_i^{-j}(\alpha)$ for all $\alpha > 0$ where $x_i^{-j}(\alpha) = (x_i^1, \ldots, x_i^{j-1}, \alpha, x_i^{j+1}, \ldots, x_i^q)$.

The fifth property, homotheticity, asserts that the dimensions of an attainment vector can be rescaled without changing the well-being ordering\footnote{Homotheticity is a specific case of the invariance axioms introduced in the social choice literature by authors like Sen (1970). In social choice theory, homotheticity is commonly referred to as ratio scale measurability. We refer the reader to d’Aspremont and Gevers (2002) or Roberts (2005) for recent surveys.}. We consider two versions of this property. First, strong homotheticity allows the rescaling of every dimension with a dimension-specific rescaling factor without affecting the well-being order (Tsui 1995). The ordering of well-being can be obtained irrespective of the measurement units employed. For instance, one can order two attainment vectors, remaining completely agnostic about whether the income dimension is measured in dollars, dollarcents or euros. Weymark (2006) advocates the use of this property as the appropriate invariance property when the dimensions of well-being are different in nature, such as income expressed in dollars and life expectancy in years. However, strong homotheticity may seem to be

\footnote{Note that an ordering that satisfies SHOM$^S_S$ also satisfies DSEP$^S_S$.}
too strong a requirement, since the measurement units used for each dimension are often known and can be used as information when making comparisons between attainment vectors.

Therefore, a weaker version of the property is also proposed. Weak homotheticity allows a rescaling of all dimensions with the same proportional change. Bourguignon (1999) argues that weak homotheticity is more appropriate when the dimensions are naturally measured in comparable units.

Formally, we state:

**Property 5. Strong Homotheticity (SHOMS)** For all \(x_i, y_i \in V\), and for all positive \(q \times q\) diagonal matrices \(\Lambda\), \(x_i \succeq y_i \iff x_i \Lambda \succeq y_i \Lambda\).

**Property 6. Weak Homotheticity (WHOMS)** For all \(x_i, y_i \in V\), and for all \(\lambda > 0\), \(x_i \succeq y_i \iff x_i \lambda \succeq y_i \lambda\).

The following proposition brings together the above properties, and derives the sole well-being functional that satisfies them all.

**Proposition 1.** A well-being ordering \(\succeq\) satisfies

(a) \(\text{ORD}_S, \text{CONT}_S, \text{MON}_S, \text{SHOM}_S\), if and only if \(\succeq\) can be represented by a Cobb-Douglas well-being function \(S(x_i)\) with positive exponents. That is,

\[
S(x_i) = F\left[\prod_{j=1}^{q} (x_i^j)^{w_j}\right] \text{ where } w_j > 0, \text{ for all } j \in Q, \tag{1}
\]

(b) \(\text{ORD}_S, \text{CONT}_S, \text{MON}_S, \text{DSEP}_S, \text{WHOM}_S\), if and only if \(\succeq\) can be represented by a Constant Elasticity of Substitution function \(S(x_i)\). That is,

\[
S(x_i) = F\left[\left(\sum_{j=1}^{q} w_j (x_i^j)^{\beta}\right)^{(1/\beta)}\right] \text{ where } w_j > 0, \text{ for all } j \in Q, \tag{2}
\]

where \(F\) is any monotonically increasing function.

**Proof.** For (a) see Tsui and Weymark (1997) Theorem 4, and for (b) Blackorby and Donaldson (1982) Theorem 2.

The result of proposition (a) is, in some respects, disappointing and reveals how restrictive the requirement of \(\text{SHOM}_S\) is. In the presence of the other properties, choosing for \(\text{SHOM}_S\) leads inevitably to a Cobb-Douglas well-being function which has unit elasticity of substitution, i.e. \(\sigma = 1\). Other elasticities of substitution between the dimensions cannot be obtained without relaxing one of the properties (Tsui & Weymark 1997). A

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11 Examples include aggregating incomes at different periods of time (inter-temporal analysis and chronic poverty) or from different sources (earnings, transfers, profits, etc).
possible relaxation is to impose WHOM$_S$ instead of SHOM$_S$ which leads, in combination with DSEP$_S$ and the other properties, to proposition \(Ib\). Parameter \(\beta\) in expression \(2\) reflects the degree of substitutability between the dimensions of well-being. In particular, \(\beta\) is related to the elasticity of substitution between the dimensions, and equals \(1 - \frac{1}{\sigma}\).

When \(\beta = 0\), the limit case leads to expression \(1\). When \(\beta = 1\) the dimensions of well-being are seen as perfect substitutes. As \(\beta\) tends to \(-\infty\), dimensions tend to perfect complementarity; at the extreme, individuals are judged upon their worst attainment.

Proposition \(I\) characterizes a broad class of functions that are ordinally equivalent to expression \(1\) or \(2\). In other words, the above proposition specifies the \(q\)-dimensional iso-well-being curves\(^{13}\) but not the cardinal labels of these curves. For the purpose of measuring inequality in well-being, however, we are interested precisely in the inequality between those cardinal labels. The cardinalization is hence a last indispensable and delicate step.

A convenient cardinalization of the ordinally specified functions, is the equally distributed equivalent attainment, denoted \(\xi\). The equally distributed equivalent attainment is defined analogously to the equally distributed income in one-dimensional inequality

\(^{12}\)This extreme case is excluded by the monotonicity property, and should be considered as a limiting case.

\(^{13}\)A iso-well-being curve, is a \(q\)-dimensional hyperplane that connects the \(q\)-dimensional attainment vectors between which the the social planner is indifferent.
measurement (Kolm 1969, Atkinson 1970). It is the attainment level \( \xi_i \) of individual \( i \) that obtained equally in all dimensions leads to the same level of well-being as the observed attainment vector. Formally, \( \xi_i \) is the value that satisfies \( S(\xi_i 1_k) = S(x_i) \), where \( 1_k \) is a \( k \)-dimensional vector of ones. Figure 1 represents the attainments of two individuals, A and B, in two dimensions of well-being: health and income. The points \( a \) and \( b \) depict their respective attainment vectors. The iso-well-being curves connect the attainment vectors between which the social planner is indifferent. The equally distributed equivalent labels the iso-well-being curves by the value of their intersection with the 45° line. The \( \xi_a \) and \( \xi_b \) are the corresponding equally distributed equivalent attainment levels for individuals A and B, respectively. Opting for the equally distributed equivalent attainment as the method for cardinalization may be a convenient but, admittedly, an arbitrary choice to label the iso-well-being curves. Two alternatives present themselves. First, we can select rays passing through the origin that are different from the 45° line to label the iso-well-being curves. One option is to choose, for instance, the ray trough the origin closest to the most current situations in the population. By the homotheticity of the well-being function \( S \), whichever angle is chosen, the labels of the iso-well-being curves are a rescaling of the labels attached to the 45° line. A second alternative is to shift the zero-point along the ray. One may argue that indeed the minimum level of well-being is not achieved at the point where all dimensions are zero but at some other point. However, people may disagree about this zero point and would, by consequence, label differently the iso-well-being curves. Along the same ray, disagreement about the minimal level leads to cardinalizations that are translations of each other. An example of such a rescaling and translation can be found in figure 2. Instead of using the 45° line to cardinalize the iso-well-being curves, an alternative ray through the origin is chosen, denoted reference line 2. Moreover, the iso-well-being curve, referred to as ‘minimal level’ reflects the minimal well-being level equal to zero. A possible cardinalization of the well-being levels belonging to \( a \) and \( b \) is \( \tau_a \) and \( \tau_b \) (or \( \tau_a' \) and \( \tau_b' \)). In general \( \xi_a \) and \( \xi_b \), \( \tau_a \) and \( \tau_b \) or \( \tau_a' \) and \( \tau_b' \), lead to different cardinal values that are a linear transformation of each other.

In the next section we impose a linear invariance property at the level of the well-being levels. In particular, we assert that a rescaling or translation of the well-being levels will not affect the social evaluation ordering. In consequence, the well-being ordering of two well-being vectors will be invariant to the exact choice of the ray through the origin or the minimal level.

The well-being function characterized by proposition 1 and cardinalized by its equally distributed equivalent attainment is a popular measure of well-being in the literature. The Human Development Index advocated by the UNDP, for instance, is a special case of expression 2, with \( \beta = 1 \), and weights \( w_j \) equal to 1/3. Other examples can be found in the literature on multidimensional inequality measurement. Maasoumi
(1986, 1999) derives a CES well-being function based on different considerations rooted in information theory.

4 Aggregation across Individuals

In this section we characterize a class of social evaluation functions $W$ to aggregate the well-being indices across individuals. We follow a procedure that is standard in the literature on one-dimensional inequality measurement and is similar to the one of the previous section (Ebert 1988). However, because the nature of the problem of aggregation across individuals differs from that of aggregating across dimensions, the set of attractive properties and the resulting aggregation function is different.

The first three properties are similar to their namesakes in the previous section. Also the interpretation is analogous, so we restrict ourselves to a summary treatment.

**Property 7. Ordering (ORD$_W$)** The binary relation $R$ is reflexive, complete and transitive on $S$.

**Property 8. Continuity (CONT$_W$)** The sets \( \{S_Y \in S \mid S_Y \preceq S_X\} \) and \( \{S_Y \in S \mid S_X \succeq S_Y\} \) are open for all $S_X \in S$. 

![Figure 2: Two dimensional case](image-url)
Property 9. Monotonicity (\text{MON}_W) For all \( S_X, S_Y \in S \), if \( S_X \neq S_Y \) and \( S(x_i) \geq S(y_i) \) for all individuals \( i \in N \), then \( S_X \succeq S_Y \).

The next property ensures that individuals with equal overall well-being indices are treated equally. Information outside the well-being vector \( S_X \) which relates to the identity of the individuals does not matter for the social evaluation. Imposing this property restricts the social evaluation relation to a symmetric one. Note that we have not imposed anonymity on the well-being relation in the previous section. In our framework, individuals cannot be labeled, whereas dimensions can be and some can be considered to be more important than others.

Property 10. Anonymity (\text{ANON}_W) For all \( n \times n \) permutation matrices \( \Pi \) and all \( S_X \in S \), \( S_X \not\sim \Pi S_X \).

One can impose a similar separability property here as in the aggregation across dimensions. This would imply that the contribution of individual well-being to the social welfare is judged without reference to the well-being of others. This leads to a class of additively-separable social evaluation functions. A prominent example is the Atkinson social welfare function (Atkinson 1970). However, one can argue in favour of a social evaluation of individuals’ well-being that depends crucially on the levels of the well-being of others (Sen 1973, p.41) as documented in recent surveys on subjective well-being (Ferrer-i Carbonell 2005, Luttmer 2005). If so, a weaker version of the separability property can be called upon, denoted \( S^D \)-restricted individual separability. The property was introduced in the literature on income inequality measurement by Ebert (1988) and allows taking both the well-being level and its ranking into account. Specifically, the property imposes strict separability of the social evaluation in the rank-ordered subspace of well-being vectors \( \tilde{S}_X \), where a rank-ordered well-being vector is one in which individuals are ordered from high well-being to low well-being. Therefore, in the comparison of two rank-ordered well-being vectors, their social evaluation is not affected by the magnitude of equal well-being indices in both rank-ordered well-being vectors as long as the initial ranking is maintained.

Property 11. \( S^D \)-Restricted Individual Separability (\text{RISEP}_W) For all \( \tilde{S}_X, \tilde{S}_Y \) in \( S^D \), the well-being relation \( R \) is \( S^D \)-restricted individual-separable if \( \tilde{S}_X \not\sim \tilde{S}_Y \) and \( \tilde{S}(x_i) = \tilde{S}(y_i) = \tilde{\alpha} \) for an individual \( i \) implies that \( \tilde{S}(x_{-i})(\tilde{\alpha}) \not\sim \tilde{S}(y_{-i})(\tilde{\alpha}) \) for all \( \tilde{\alpha} \): \( \tilde{S}(x_{i-1}) \leq \tilde{\alpha} \leq \tilde{S}(x_{i+1}) \) and \( \tilde{S}(y_{i-1}) \leq \tilde{\alpha} \leq \tilde{S}(y_{i+1}) \) where \( \tilde{S}(x_{-i})(\tilde{\alpha}) = (\tilde{S}(x_1), ..., \tilde{S}(x_{i-1}), \tilde{\alpha}, \tilde{S}(x_{i+1}), ..., \tilde{S}(x_n)) \).

At the end of the previous section, we introduced different alternative cardinalizations of the well-being function \( S \) that are linear transformations of each other. By imposing the next property we assert that the social evaluation relation is invariant to...
common linear transformations of the well-being vectors.

Property 12. Linear Invariance (LINV$_W$) For all $S_X, S_Y \in S$, and for all $\lambda, \kappa > 0$, $S_X \ R \ S_Y \iff S_X \lambda + \kappa 1_n \ R \ S_Y \lambda + \kappa 1_n$.

The following proposition is a standard result in social choice theory and derives the sole social welfare function that satisfies all the above properties.

Proposition 2. A Social Evaluation Relation $R$ satisfies ORD$_W$, CONT$_W$, MON$_W$, ANON$_W$, RISEP$_W$, LINV$_W$, if and only if $R$ can be represented by

$$W(S_X) = F \left[ \sum_{i=1}^{n} a_i S(x_i) \right], \quad (3)$$

with $a_i > 0$, $\sum_{i=1}^{n} a_i = 1$ and $F$ any monotonically increasing function.

Proof. See Ebert 1988, Corollary 5. \hfill \square

From expression (3) it is clear that in the case with only one dimension of well-being $q = 1$, collapses to the standard one-dimensional generalized Gini social evaluation function (Weymark 1981). The $n \times 1$ vector of non-negative weights $a_i$ contains the weights attached to individual well-being indices in the social evaluation. By choosing the weights $a_i$ we can reflect different ethical viewpoints the social planner might hold regarding the aggregation across individuals. Two extreme cases are of interest: when $a_i = 1/n$ everybody’s well-being contributes equally to the social evaluation function which becomes a simple average of the well-being indices of the individuals, consistent with the utilitarian perspective; on the other hand, when $a_n = 1$ and all other weights are zero the social planner considers solely the situation of the worst-off individual, which reflects a Rawlsian perspective. An intermediate approach is, for example, obtained by considering $a_i = (2i - 1)/n^2$, which makes $W$ equivalent to a Gini social evaluation function.

For future notational convenience, we introduce the weighting function $f$, that denotes the cumulative weights in the social evaluation function $W$, $f : N \to [0, 1], f(i) = \sum_{k=1}^{i} a_k$. Hence expression (3) becomes

$$W(S_X) = F \left[ \sum_{i=1}^{n} \{f(i) - f(i - 1)\} \widehat{S}(x_i) \right]. \quad (4)$$

Finally, we relax the initial assumption of a fixed population of size $n$. The following two properties allow us to order the vectors of well-being indices of different length
by imposing further structure on the weighting function \( f \). Let \( R^n \) denote the social evaluation relation between two vectors of well-being of size \( n \). The strict binary social evaluation relation and the equivalence relation are defined analogously.

The principle of population states that if two vectors of well-being of population size \( n \) are equivalent, then the \( m \)-times replicated vectors are equivalent as well. By imposing this property, the weighting function \( f \) becomes a function of the relative position, \( i/n \).

**Property 13. Principle of Population (POP)\(_W\)** For all \( S_X, S_Y \in S \), such that \( S_Y I^n S_X \) and any scalar \( m \geq 2 \), we have that \( S_Y^{(m)} I^{(n-m)} S_X^{(m)} \), where \( S_Y^{(m)} \) and \( S_X^{(m)} \) are \( m \)-fold replications of \( S_Y \) and \( S_X \).

Donaldson and Weymark (1980) introduce a second property to allow comparisons between populations with different sizes. Restricted aggregation asserts that the well-being of the \( k \) better-off individuals can be replaced by their equally distributed equivalent well-being, \( \zeta_k \), that is, the average well-being among these \( k \) individuals. The procedure of substituting the actual well-being indices by the respective equally distributed equivalent is a procedure of aggregation. The aggregation is restricted to the subgroup of the \( k \) better-off individuals, and therefore its name.

**Property 14. Restricted Aggregation (RA)\(_W\)** For all \( \tilde{S}_X \in S^D \), and all \( k \leq n \) we have \( \tilde{S}_X I^n [\zeta_k, \ldots, \zeta_k, S(x_{k+1}), \ldots, S(x_n)] \) with \( \zeta_k \) the equally distributed equivalent well-being of \( [S(x_1), \ldots, S(x_k)] \).

Donaldson and Weymark (1980) prove the following result, leading to the single-parameter Gini (S-Gini) social evaluation function.

**Proposition 3.** A Social Evaluation Relation \( R \) satisfies ORD\(_W\), CONT\(_W\), MON\(_W\), ANON\(_W\), RISEP\(_W\), LINV\(_W\), POP\(_W\), RA\(_W\) if and only if \( R \) can be represented by

\[
W^\delta_R(S_X) = F \left[ \sum_{i=1}^{n} \left\{ \left( \frac{i}{n} \right)^\delta - \left( \frac{i-1}{n} \right)^\delta \right\} \tilde{S}(x_i) \right],
\]

with \( \delta > 0 \) and \( F \) any monotonically increasing function.

**Proof.** See Donaldson and Weymark (1980), Theorem 2. \( \square \)

The obtained class of social evaluation functionals is a convenient one, since it allows the sequence of weights and the associated value judgements to be captured by a single parameter \( \delta \). It is the underlying social welfare function of the popular class of S-Gini inequality indices (Donaldson & Weymark 1980, Kakwani 1980, Yitzhaki 1983). Parameter \( \delta \) captures the bottom-sensitivity of the social evaluation function. If parameter \( \delta \) equals 1, the social evaluation function is the unweighted average of the well-being
indices of the individuals in the utilitarian tradition. The higher $\delta$, the more weight is given to the bottom of the distribution, with as limiting case $\delta = +\infty$, which leads to a Rawlsian social evaluation function. If $\delta$ is smaller than 1, more weight is given to the best-off individuals. The standard Gini social evaluation function is obtained by setting $\delta = 2$.

5 Distributional concerns

Bringing together the results of the previous two sections, the compound multidimensional social evaluation function $W$ is ordinally equivalent to the following expression:

$$W_{n,\beta}(X) = \sum_{i=1}^{n} \left( \frac{i}{n} \right)^{\delta} - \left( \frac{i-1}{n} \right)^{\delta} \sum_{j=1}^{q} w_j(x_j)^{\beta} \left( \frac{1}{\beta} \right). (6)$$

In this expression, three parameters have to be specified. Parameter $\beta$ captures the degree of substitutability between the dimensions of well-being, the $q$-dimensional vector $w$ reflects the weights attached to attributes in the aggregation across dimensions, while $\delta$ specifies the weight given to individuals in the aggregation across the individuals.

The reader will note that so far we have not introduced any property that captures distributional concerns. In the standard one-dimensional analyses, distributional sensitivity is obtained by imposing some form of the Pigou-Dalton transfer principle. The principle states that a transfer of income from a poorer to a richer individual leads to a decrease in social welfare. Some proposals have been made to generalize the one-dimensional Pigou-Dalton principle to the multidimensional setting. In this section we focus on a popular generalization and investigate the effect of imposing it within the multidimensional framework laid down in the previous sections and captured by expression (6). We also introduce a distributional property which is specific to the multidimensional setting related to the correlation between the dimensions. These two distributional concerns are defined on the space of the distribution matrices and take full account of the multidimensionality of the problem.

The first property asserts that the same mean-preserving transfer is carried out across dimensions such that dispersion is reduced in all of them, then the resulting

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16 Examples of multidimensional generalizations of the Pigou-Dalton principle can be found in Kolm (1977) or more recently in Weymark (2006). These generalizations are rooted in the theory of multidimensional majorization (Marshall & Olkin 1979, chapter 15).

17 An alternative approach would be to impose a Pigou-Dalton transfer in the space of the well-being vectors, as it is done implicitly by Masoumi (1986, 1999). Such an approach reduces the multidimensional problem to a one-dimensional one in the space of the well-being indices. It is a standard result from the one-dimensional inequality literature, that such a one-dimensional Pigou-Dalton principle will restrict the $\delta$ parameter to be not smaller than 1 (Ebert 1988).
distribution matrix is socially preferred to the original one.

**Property 15. Uniform Majorization (UM)** For all \( X, Y \in D \), and all bistochastic matrices \( B \), if \( Y = BX \), then \( Y \succeq X \).

UM imposes restrictions on the parameters of the compound social evaluation function \( W \).

**Proposition 4.** A compound social evaluation relation \( R \succeq \) satisfying the properties specified in Propositions \( 7 \) and \( 3 \) respectively, also satisfies UM if and only if the representation of \( R \succeq \) satisfies \( \beta \leq 1 \).

**Proof.** See Appendix.

Atkinson and Bourguignon (1982) point to another distributional concern. They argue that a social evaluation ordering should be sensitive to possible correlation between the dimensions. Tsui (1999) formalized this notion of correlation between the dimensions into the normative framework, by defining a correlation increasing transfer.

**Definition 1. Correlation Increasing Transfer (CIT)** For all \( X, Y \in D \), \( Y \) is obtained from \( X \) through a CIT if \( X \neq Y \), \( X \) is not a permutation of \( Y \), and there are at least two individuals \( k \) and \( l \) such that, (i) \( y_k = \max \{ x^j_k, x^j_l \} \) for all dimensions \( j \), (ii) \( y_l = \min \{ x^j_k, x^j_l \} \) for all dimensions \( j \) and (iii) \( y_i = x_i \) for all \( i \notin \{ k, l \} \).

The next property incorporates the idea that a distribution matrix \( Y \) that is obtained from \( X \) by a finite series of correlation increasing transfers, should be socially inferior (Tsui 1999). If two distribution matrices have identical marginal distributions, the one with lower correlation between the dimensions is preferred. The property captures the idea of compensating inequalities among different dimensions, hence implicitly assuming that dimensions are substitutes.

**Property 16. Correlation Increasing Majorization (CIM)** For all \( X, Y \in D \), if \( Y \) is obtained from \( X \) by a finite series of correlation increasing transfers, then \( X \succeq Y \).

We investigate the introduction of the correlation increasing majorization criterion within the multidimensional S-Gini framework developed in the previous sections.

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\(^{18}\) Although this generalization of the one-dimensional Pigou-Dalton transfer principle seems to be the one most often used in the literature on multidimensional inequality (Tsui 1995, Tsui 1999, Weymark 2006), it is not uncontroversial. Dardanoni even calls it “uninformative for evaluating the amount of inequality in society” (Dardanoni 1996, p. 202). Indeed, uniform majorization is a strong condition by requiring that the same mean preserving decrease in dispersion is applied in every dimension of well-being.

\(^{19}\) Bourguignon and Chakravarty (2003) suggest that depending on the nature of the dimensions, the opposite property could be considered.
Proposition 5. A compound social evaluation relation $R_{s}$ satisfying the properties specified in Propositions 1 and 3 satisfies CIM, if the representation of $R_{s}$ satisfies $\delta \geq \delta^\star$.

\[ \delta^\star = f[a^i, x^i, \beta] \]

Proof. See Appendix.

A formal proof of this proposition is left to the Appendix. Intuitively, Proposition 5 states that there is always a bottom-sensitivity level such that the social evaluation function would be lower after a transfer between individuals that increases the degree of correlation among dimensions. In general terms, as a result of any correlation-increasing transfer invariably at least one person has gained, in terms of well-being, and another has lost. Moreover, after the transfer the looser has, by definition, lower well-being level than the winner. Whether the gains of the winner are smaller than the losses of the looser in the eyes of the social planner depends on the amount of weight that is given to the bottom part of the distribution, relative to the rest. For the single-parameter Gini social evaluation the bottom-sensitivity is captured by parameter $\delta$\textsuperscript{20}. In the extreme, a social evaluation function that is arbitrary close to the Rawlsian-case would always ensure that CIM is satisfied. Without going that far, the value of $\delta^\star$ is determined by three factors. First, it depends on the ranks of the individuals involved in the correlation-increasing transfer ($l$ and $k$), that is, the bottom-sensitivity alluded before. The larger the difference between $a_l$ and $a_k$, the smaller the $\delta^\star$. Second, $\delta^\star$ depends on the attainment levels of these individuals ($x^l$ and $x^k$) before and after the transfer and on the degree of substitutability between the dimensions ($\beta$). Finally, given that some re-ranking can take place between individuals involved and those not involved in the transfer ($i \neq l, k$), the gains and losses in terms of well-being related to these re-ranking should also be considered\textsuperscript{21}.

6 Two Inequality Measures

In the one-dimensional normative approach, an inequality measure is derived from a social evaluation ordering. Using an Atkinson-Kolm-Sen approach, the inequality measure is defined as the fraction of total welfare wasted due to inequality (Atkinson 1970, Kolm 1969, Sen 1973). This approach leads to a relative inequality measure. Kolm (1969) proposes an alternative approach, leading to an absolute inequality measure, by defining the inequality measure as the total amount of welfare wasted due inequality.\textsuperscript{20}

\textsuperscript{20}It is possible to view a correlation-increasing transfer in terms of a ‘leaky-bucket’ transfer in the well-being space. Duclos (2000) investigates the effects of a leaky-bucket transfer on a one-dimensional single-parameter Gini social evaluation function similar to the $W$ we obtained in proposition 3.

\textsuperscript{21}Simulations based on the dataset at hand can shed some light on how large $\delta$ should be to make sure that a correlation-increasing transfer leads to a decrease in social welfare.
Both the relative and absolute approaches can be generalized into the multidimensional setting. In a seminal article, Kolm (1977) generalizes the relative definition where this is understood as a measure of the fraction of the aggregate amount of each dimension of a given distribution matrix that could be destroyed if every dimension of the matrix is equalized while keeping the resulting matrix socially indifferent to the original matrix according to $R_\succeq$. Formally, a relative inequality measure $I_R(X)$ is defined as the scalar that solves

$$[1 - I_R(X)]X_\mu \succeq X,$$

where $X_\mu$ is the equalized distribution matrix defined such that all the elements in the $j$-column of the matrix are the dimension-wise mean $\mu(x^j)$. Applying the compound social evaluation function obtained in (6),

$$I_R(X) = 1 - \sum_{i=1}^n a_i \left[ \sum_{j=1}^q w_j(x_{ij})^\beta \right]^{1/\beta} \left[ \sum_{j=1}^q w_j \left( \frac{1}{n} \sum_{i=1}^n x_{ij} \right)^\beta \right]^{1/\beta}.$$ \hspace{1cm} (8)

When $a_i = \left( \frac{i}{n} \right)^\delta - \left( \frac{i-1}{n} \right)^\delta$ the inequality index belongs to the family of single-parameter Gini measures. For $\beta \leq 1$ the measure decreases after a pre-multiplication by a bistochastic matrix, and for $\delta \geq \delta^\star$ inequality increases after a correlation-increasing transfer.

The generalization of the absolute measure of inequality is provided by Tsui (1995). An absolute multidimensional inequality index is defined as the amount of each attribute that could be destroyed if every dimension of the matrix is equalized, while keeping the resulting matrix socially indifferent to the original matrix according to $R_\succeq$. Formally,

$$[X_\mu - I_A(X)1_{n\times q}] \succeq X,$$

where $X_\mu$ is the equalized distribution matrix defined above. The compound social evaluation function obtained in (6) leads to the following expression for $I_A$\footnote{The same conditions apply for compliance with UM and CIM.}

$$I_A(X) = \sum_{i=1}^n a_i \left[ \sum_{j=1}^q w_j(x_{ij})^\beta \right]^{1/\beta} - \left[ \sum_{j=1}^q w_j \left( \frac{1}{n} \sum_{i=1}^n x_{ij} \right)^\beta \right]^{1/\beta}.$$ \hspace{1cm} (10)
7 Empirical Application

To be completed.

8 Conclusion

In this paper we used an intuitive two step procedure to characterize a multidimensional social well-being function. A weak rank-dependent separability axiom is introduced allowing individuals’ well-being to be evaluated in comparisons to others in the society. This weak separability property, in combination with standard properties, leads to a multidimensional S-Gini social evaluation function.

Conditions for the compliance to a multidimensional Pigou-Dalton transfer-principle as well as to a correlation increasing majorization are derived. The latter condition shows that researchers willing to work in a rank-dependent framework with a welfare measure decreasing after a correlation-increasing transfer, may be forced to use a highly inequality-averse aggregation procedure across individuals.

The social evaluation function derived in the paper allows us to define absolute and relative multidimensional normative measures of inequality. The main advantage of the measures developed here to their closest relative developed by Gajdos and Weymark (2005), is that they do not a priori exclude sensitivity to changes in correlation.
Appendix (Preliminary versions - to be completed)

**Proposition 4.** A compound social evaluation relation $R_{\succeq}$ satisfying the properties specified in Propositions 1 and 3 respectively, also satisfies UM if and only if the representation of $R_{\succeq}$ satisfies $\beta \leq 1$.

*Proof.* By definition of Schur-concavity, the ordering $R_{\succeq}$ satisfies UM, if its representation $W$ is Schur-concave (Marshall & Olkin 1979).

If $S$ is Schur-concave and $W$ is increasing, then $W$, defined as $W(X)=W[S(x_1), ..., S(x_n)]$, is Schur-concave. Due to MON$_W$, $W(.)$ is a strictly increasing function of its arguments. From Ebert (1988, Proposition 6) $S$ is Schur-concave if and only if $\beta \leq 1$.

Therefore, $W(.)$ is Schur-concave and $R_{\succeq}$ satisfies UM whenever $\beta \leq 1$.

\[\square\]

**Proposition 5.** A compound social evaluation relation $R_{\succeq}$ satisfying the properties specified in Propositions 1 and 3 satisfies CIM, if the representation of $R_{\succeq}$ satisfies $\delta \geq \delta^*$.  

*Proof.* Consider the distribution matrices $Y$ and $X$, where $Y$ is obtained from $X$ after a correlation increasing transfer between individual $l$ and $k$ as defined in Definition 1.

From the definition of a correlation increasing transfer and the monotonicity of the well-being relation, it follows that

$$S(y_l) \leq S(y_k)$$

From monotonicity of the well-being relation (MON$_S$)

$$S(y_l) \leq S(x_l),$$

and

$$S(x_k) \leq S(y_k),$$

We can assume without loss of generality that

$$S(x_l) \leq S(x_k).$$

Therefore,

$$S(y_l) \leq S(x_l) \leq S(x_k) \leq S(y_k),$$

with at least one of the inequalities being strict.
The compound social evaluation relation \( R_{\succ} \) satisfies CIM if and only if \( X R_{\succ} Y \).

By increasing the \( \delta \) parameter one can give the worse-off individual \( l \), more weight than the better-off individual \( k \). Therefore, there always exists a \( \delta^* \) above which the decrease in well-being for \( l \) by the correlation increasing transfer will outweigh the increase in well-being for individual \( k \), and hence \( W(X) \geq W(Y) \).
References


