Soc Choice Welfare manuscript No.
(will be inserted by the editor)

Generalizing the Gibbard-Satterthwaite theorem: Partial preferences, the degree of manipulation, and multi-valuedness.

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May 15, 2008

Abstract The Gibbard-Satterthwaite theorem is generalized in three ways: firstly, it is proved that the theorem is still valid when individual preferences belong to a convenient class of partial preferences; secondly, it is shown that every non-dictatorial SCF that respects unanimity is not only manipulable, but it can be manipulated in such a way that some individual obtains either his best or second best alternative; thirdly, we prove a variant of the theorem where the outcomes of the SCF are subsets of the set of alternatives of an a priori fixed size. In addition, all results are proved not only for finite, but also for countable infinite sets of alternatives.

1 Introduction

The well-known Gibbard-Satterthwaite (GS) theorem, proved independently by Gibbard (1973) and Satterthwaite (1975), states that a social choice function (SCF) that respects unanimity can only be strategy-proof if it is dictatorial, provided that there are at least three alternatives and that the domain of preferences is unrestricted. This result has rightly become the cornerstone in strategy-proof social choice theory because it characterizes in a simple, but general framework all SCFs that satisfy the desirable properties of strategy-proofness and respect of unanimity. But since its conclusion unfortunately is very negative and since its assumptions concerning preferences and the form of the SCF do not always appropriately model real votings, numerous papers have followed up the GS theorem and analyzed its robustness to variations of the basic framework. In some cases, non-dictatorial strategy-proof social choice indeed turned out to be possible, but on the whole, the conclusion of the GS theorem has proved to be more far-reaching than is obvious from its original formulation. This tradition is continued in this paper, and we consider and generalize the GS theorem

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in three respects, which so far do not have deserved much attention and which will be explained in the remainder of this introduction.

In the GS theorem, voters' preferences over the set of available alternatives are assumed to be complete, transitive, and unrestricted in the sense that every possible preference is allowed as an argument in the SCF. It has been argued previously that it may be inappropriate to assume the domain of preferences to be unrestricted, because as a consequence of the nature of alternatives or as a result of cultural or social forces, it is likely that the set of preferences exhibits some structure (see, for instance, Sen (1970, p. 165)). Moreover, every available preference that can be hold by a voter puts restrictions on an SCF in order to be strategy-proof and makes it more difficult to find non-dictatorial strategy-proof SCFs. This observation has given rise to a number of studies analyzing whether the conclusion of the GS theorem can be escaped by restricting the domain of preferences in some reasonable way, the most well-known example probably being domains of single-peaked preferences, on which non-dictatorial strategy-proof SCFs indeed turn out to exist. In Section 3, we will in a similar way question the assumption that preferences are complete, because on the one hand, this assumption may, as we will argue, be inappropriate for several reasons, and on the other hand, every ranking of two alternatives that holds by an admissible preference makes it more difficult for an SCF to be strategy-proof. We will therefore analyze whether non-dictatorial strategy-proof social choice becomes possible when the completeness assumption is replaced by the more realistic assumption that preferences belong to some convenient class of partial preferences. It turns out, however, that non-dictatorial strategy-proof SCFs can only be obtained when the completeness assumption is relaxed considerably, and Theorem 3.1 shows instead that the GS theorem also applies for a large class of domains of partial preferences, which do not require more than that preferences contain a best and a second best alternative.

In Section 4, we move then our attention to the conclusion of the GS theorem, which can be re-formulated in the practically more relevant way that every non-dictatorial SCF that respects unanimity must be manipulable. This conclusion is purly qualitative, and one can think that an SCF that allows some individual to obtain at most his seventh instead of his eighth alternative by misrepresentation is less vulnerable for strategic voting than an SCF where some individual by manipulation can get his third instead of his fourth alternative. It is therefore natural to ask whether the conclusion of the GS theorem can be made quantitative by specifying which ranks of alternatives always can be obtained by means of manipulation. This question is answered in Theorem 4.1, which states that every non-dictatorial SCF that respects unanimity can be manipulated in such a way that some individual obtains at least his second best alternative.

In Section 5, we turn then to the functional form of the SCF, and we will, instead of the single-valued SCFs in the original GS theorem, consider SCFs that are multi-valued in the sense that their outcomes are subsets of the set of alternatives of an a priori fixed size. Even though this kind of multi-valued votings is common in any democracy, SCFs of this form have not been considered in strategy-proof social choice theory until recently in Özyurt and Sanver (2008). If individuals are allowed to hold any possible preference over fixed-sized subsets of alternatives, then one can of course consider these subsets as new alternatives and apply the original GS theorem to this form of multi-valued SCFs to obtain an impossibility result. We will however assume that preferences over fixed-sized subsets are separable in the sense that if some alternative in a given subset is replaced by a preferred alternative, then the individual will prefer the new subset, or, put differently, if an individual prefers $\{a,c\}$ to $\{b,c\}$, then he will also prefer $\{a,d\}$ to $\{b,d\}$. Preferences over fixed-sized subsets will therefore belong to some restricted preference domain, and the question whether non-dictatorial strategy-proof social choice of fixed-sized subsets is possible becomes non-trivial. In Theorem 5.1, we will however show that also in the context of the social choice of fixed-sized subsets, strategy-proofness and non-dictatorship do in general exclude each other. We will then also consider the case when preferences over alternatives are single-peaked, and in this case the corresponding domain of preferences over fixed-sized subsets will turn out to be such that it admits non-dictatorial strategy-proof SCFs.

In addition, we extend the standard framework of the GS theorem in one more respect: The original GS theorem and most of its descendants are proved for a finite number of alternatives, but all of our results apply also for countable infinite sets of alternatives.

Note finally that the original GS theorem turns out as a special case of each of Theorem 3.1, Theorem 4.1, and Theorem 5.1, which motivates the title of this paper. It is also worth to point out already now that even though the three just mentioned theorems seems to deal with quite different problems, there is in fact a close relationship between them.

2 Some basic definitions and the theoretical starting point

In this paper, the following standard definitions and concepts from strategy-proof social choice theory will be used: We consider a finite society consisting of N individuals, indexed by the set $I = \{1,\ldots,N\}$, that is facing a countable set $\mathscr{A}$ of M alternatives, where M possibly equals infinity. The individuals are assumed to have transitive and asymmetric, but not necessarily complete preferences over the alternatives in $\mathscr{A}$, and individual i's preference is denoted by P_i.[1] The set of all admissible preferences is denoted by Γ and will normally be a proper subset of the set of all transitive and asymmetric preferences over $\mathscr{A}$. The preferences of all N individuals in the society will be collected in a (preference) profile $(P_1,\ldots,P_N) \in \Gamma^N$, and the profile $(P_1,\ldots,P_N)$ is sometimes, in order to focus on individual i's preference, re-written as (P_i,P_{-i}), where P_{-i} thus denotes the profile of all individuals except individual i.

Apart from Section 5, we assume that society chooses a single alternative from $\mathscr{A}$ using some SCF $f : \Gamma^N \to \mathscr{A}$ that assigns to every profile $(P_1,\ldots,P_N) \in \Gamma^N$ a unique $a \in \mathscr{A}$. We will say that $f : \Gamma^N \to \mathscr{A}$ is *manipulable* at the profile $(P_i,P_{-i}) \in \Gamma^N$ if there is some $P_i' \in \Gamma$ such that $f(P_i',P_{-i})\, P_i\, f(P_i,P_{-i})$, and if there is no profile $(P_1,\ldots,P_N) \in \Gamma^N$ at which f is manipulable, f is *strategy-proof*. We say further

[1] For the sake of clarity, we recall briefly the definitions of the following three well-known properties of preferences: A preference P on $\mathscr{A}$ is *complete* if for all $a,a' \in \mathscr{A}$, $a \neq a'$, either $a\,P\,a'$ or $a'\,P\,a$; P is *transitive* if $a\,P\,a'$ and $a'\,P\,a''$ implies $a\,P\,a''$ for all $a,a',a'' \in \mathscr{A}$; finally, P is *asymmetric* if, for all $a,a' \in \mathscr{A}$, $a\,P\,a'$ implies that $a'\,P\,a$ does not hold.

that $f : \Gamma^N \to \mathscr{A}$ *respects unanimity* if every $P \in \Gamma$ has a (unique) top alternative, denoted by $r_1(P)$, and the implication $[\exists\, a \in \mathscr{A}$ such that $r_1(P_i) = a \,\forall\, i \in \{1,\dots,N\}]$ $\Rightarrow [f(P_1,\dots,P_N) = a]$ holds for all $(P_1,\dots,P_N) \in \Gamma^N$. We say also that $f : \Gamma^N \to \mathscr{A}$ is *dictatorial* if there is some $i \in I$ such that $f(P_i, P_{-i}) = r_1(P_i)$ for all $(P_i, P_{-i}) \in \Gamma^N$. Finally, if a preference domain Γ is such that every strategy-proof SCF $f : \Gamma^N \to \mathscr{A}$ that respects unanimity is dictatorial, then Γ itself is called *dictatorial*.

Given this formal framework, the theoretical starting point for the considerations in this paper can be formulated as follows:

The Gibbard-Satterthwaite Theorem: *Let $\mathscr{A}$ be a finite set of at least three alternatives, and denote by Σ the set of all complete, transitive, and asymmetric preferences over $\mathscr{A}$. Then an SCF $f : \Sigma^N \to \mathscr{A}$ that respects unanimity is strategy-proof if and only if it is dictatorial.*

3 Strategy-proof social choice with partial preferences

In most economic models, completeness of preferences is a convenient assumption, which simplifies arguments, but which does not appear as the driving force for results. The unquestioned role of the completeness assumption is for example evident in Campbell and Kelly (2002a), who thoroughly investigate to what extent the impossibility result of Arrow's theorem can be escaped by relaxing its assumptions in different ways, but the only assumption not discussed is that of completeness.

But in strategy-proof social choice theory, the assumption of complete preferences plays in fact a fundamental role for results like the GS theorem because every ranking $a\,P\,b$ of two alternatives a and b that holds by an individual imposes restrictions on an SCF in order to be strategy-proof, which then in their entirety lead to an impossibility result. To be concrete, if for example two individuals choose from a set $\mathscr{A}$ using some SCF f, and $f(P_1,P_2) = a$ for some preferences P_1 and P_2, then f can only be strategy-proof if the restrictions

$$f(P_1',P_2) \notin \{a' \in \mathscr{A}; a'\,P_1\,a\} \quad \text{and} \quad f(P_1,P_2') \notin \{a' \in \mathscr{A}; a'\,P_2\,a\} \tag{3.1}$$

hold for all P_1' and P_2', and it is clear that the more rankings P_1 and P_2 consist of, the more extensive are the restrictions imposed on f by (3.1), and the more difficult it becomes to find a strategy-proof SCF.[2]

On the other hand, completeness of preferences can be an unjustified assumption for at least three reasons: Firstly, an individual's ranking of two alternatives does not reveal itself instantaneously and without effort, but requires a mental process, which sometimes can be demanding. It seems therefore reasonable that an individual ranks available alternatives only to such an extent this appears motivated, which in

[2] Stating conditions in social choice theory in the form of restrictions as in (3.1) can fruitfully be developed further, which has been demonstrated in Sethuraman et al. (2003). In this paper, the conditions for an Arrovian social welfare function (ASWF) on some given preference domain have been expressed as a system of linear inequalities, and every integer solution to this system has been shown to correspond to some ASWF on this domain. The strength of this approach is that, after some initial costs of formalizations, many of the well-known results in connection with Arrow's theorem can be derived easily within this integer programming framework.

connection with a voting can mean that one is satisfied with finding out one's most preferred two or three alternatives and ranking these internally. Secondly, an individual might be unwilling to express any clear ranking of two alternatives because he ranks these alternatives differently along different dimensions (this occurs for example in the voting model considered in Section 5), and this indecisiveness is conveniently modelled with incomplete preferences. At first glance, it might be tempting to model an individual's ambiguity as indifference, but Footnote 9 provides a simple example showing that this latter approach might be logically inconsistent. Thirdly, in particular in strategy-proof social choice theory, it can be questioned whether the usual notion of preference is appropriate to decide when some SCF is manipulable, because even if an individual can obtain a preferred alternative by misrepresentation, he will not always exploit this opportunity since a successful manipulation is costly in the sense that the individual must form well-founded guesses about other individuals' votes, he must think about how to misrepresent his vote in order to obtain the preferred alternative, and finally, he must overcome his potential moral objections against misrepresentation. Hence, an individual's preference "x is preferred to y" does not necessarily imply "x is sufficiently much preferred to y to motivate for misrepresentation", and the latter, in the context of strategy-proofness more relevant relation is again conveniently modelled with partial preferences.

As a consequence of the preceding discussion, it may thus be possible that there exist SCFs that are classified as manipulable by the GS theorem based on the inappropriate assumption of complete preferences, but that turn out to be strategy-proof in practice because voters' preferences are actually incomplete. It is therefore of interest to study partial preferences in strategy-proof social choice theory, but the so far only paper where preferences are allowed to be incomplete is, to our knowledge, Ching and Zhou (2002), in which a special domain of partial preferences arises when (complete) preferences over alternatives in connection with an analysis of strategy-proof social choice rules[3] are extended to preferences over subsets of alternatives. In the following, we aim to fill this gap and present a general investigation of to what extent the conclusion of the GS theorem holds on domains of partial preferences.

Thereby, we will of course not simply omit the assumption of complete preferences, but we will replace it by a weaker, more realistic condition. From our objections against the completeness assumption above, it seems reasonable that people are often truly engaged in a few (k) alternatives, which they prefer to all other alternatives and which they rank internally, but they do not necessarily rank the remaining alternatives internally. This preference structure is formalized in the following definition.

Definition 3.1 (Top-k preferences and top-k domains) A transitive and asymmetric relation P on a set $\mathscr{A}$ is said to be a *top-k preference* if there exist k alternatives $r_1(P),\dots,r_k(P) \in \mathscr{A}$ such that (1) $r_i(P)\,P\,r_{i+1}(P)$ for all $i \in \{1,\dots,k-1\}$, and (2) $r_i(P)\,P\,a$ for all $i \in \{1,\dots,k\}$ and $a \in \mathscr{A} \setminus \{r_1(P),\dots,r_k(P)\}$. A set of top-$k$ preferences will be referred to as a *top-k domain*.

[3] In Ching and Zhou (2002), a *social choice rule* is defined to be a voting procedure that associates to every preference profile a subset of the set of alternatives from which the ultimate outcome will be selected later.

In the following, we will always assume that the domain of admissible preferences Γ is a top-k domain for some $k \geq 1$. Note that a top-k preference P must have more structure the larger k is, and if $k = M$, then P is actually complete. Conversely, a top-k preference is obviously also a top-k' preference for all $k' < k$, and the class of top-1 domains is thus the most general class which requires the weakest structure of preferences. It is therefore natural to start an investigation of whether non-dictatorial strategy-proof social choice is possible on domains of partial preferences by first considering top-1 domains, and if these domains do not exclude the existence of non-dictatorial strategy-proof SCFs, to increase then k successively until some kind of general impossibility result is obtained. Thereby, it is of course possible to work with the standard notions of manipulability and strategy-proofness as they are defined in Section 2, but it will turn out below that more informative results can be obtained with the following notions of manipulability and strategy-proofness, which are only sensitive to the first k alternatives of voters' preferences.

Definition 3.2 (Top-*k* manipulability and top-*k* strategy-proofness) An SCF $f : \Gamma^N \to \mathscr{A}$ defined on a top-k domain is said to be *top-k manipulable* if there exist $(P_i, P_{-i}) \in \Gamma^N$ and $P_i' \in \Gamma$ such that $f(P_i', P_{-i})\, P_i\, f(P_i, P_{-i})$ and $f(P_i', P_{-i}) = r_j(P_i)$ for some $j \in \{1, \ldots, k\}$. If f is not top-k manipulable, f is *top-k strategy-proof.*

Note that (ordinary) strategy-proofness implies top-k strategy-proofness, and top-k manipulability implies (ordinary) manipulability, but the converse implications do not necessarily hold.

Consider now first the class of top-1 domains. This class contains the least demanding domains that put the fewest restrictions on an SCF in order to be strategy-proof, and the following example shows that it indeed is possible to find top-1 domains that admit fully democratic strategy-proof SCFs.

Example 3.1 Suppose that Γ is a top-1 domain containing only top-1 preferences that are strict in the sense that every $P \in \Gamma$ has a top alternative which is preferred to any other alternative, but P does not rank any two of the remaining alternatives. Suppose further that social choice is made using a majority type SCF where every individual votes for one alternative and society elects the alternative with most votes, solving a possible tie by choosing among the tieing alternatives the leftmost according to some a priori fixed linear order $\prec$ of $\mathscr{A}$.[4] The SCF defined in this way respects obviously unanimity and it turns also out to be (top-1) strategy-proof because an individual can only gain from misrepresentation if this changes the outcome of the voting to his top alternative, but if he can obtain his top alternative by misrepresentation, then he can certainly also obtain it by sincere voting.

The assumption of strict top-1 preferences in the previous example is of course very restrictive, and it seems in general more appropriate to assume that individuals rank also at least a second alternative, which means that preferences belong to a top-2 domain. For top-2 domains, our conclusions are however not equally positive. It is of course possible to find top-2 domains that admit non-dictatorial strategy-proof SCFs (simply because there exist restricted domains of complete preferences, such

[4] A *linear order* $\prec$ of $\mathscr{A}$ is a transitive, asymmetric, and complete binary relation on $\mathscr{A}$.

as single-peaked domains, on which non-dictatorial strategy-proof social choice is possible), but it turns out that if a top-2 domain Γ satisfies the not very strong criterion of the following definition, then Γ imposes sufficiently many restrictions on a strategy-proof SCF that it must be dictatorial.

Definition 3.3 (Connectedness and linked top-2 domains) Let Γ be a top-2 domain. Two alternatives $a, a' \in \mathscr{A}$, $a \neq a'$, are said to be *connected* in Γ if there exist $P, P' \in \Gamma$ such that $r_1(P) = r_2(P') = a$ and $r_1(P') = r_2(P) = a'$. If a and a' are connected, this will be denoted by $a \sim a'$. Further, Γ is said to be a *linked* top-2 domain if the alternatives in $\mathscr{A}$ can be ordered in a list $a_1, a_2, a_3, \ldots$ such that

$$(1)\ a_1 \sim a_2, \text{ and } (2)\ \forall \ell \in \{3, \ldots, M\}\, \exists i, j < \ell, i \neq j :\ a_\ell \sim a_i, a_\ell \sim a_j. \tag{3.2}$$

Definition 3.3 generalizes the notion of *linked domains* introduced in Aswal et al. (2003) to top-2 domains. Aswal et al. (2003) showed that if voters' preferences belong to a linked domain of complete preferences over a finite set of alternatives, then every strategy-proof SCF that respects unanimity must be dictatorial, provided that the number of alternatives is at least three. The main result of this section states that a similar conclusion holds also for linked top-2 domains over countable sets of alternatives, and we have the following generalization of the result in Aswal et al. (2003), which we will prove in the remainder of this section.

Theorem 3.1 *Let Γ be a linked top-2 domain over a countable set $\mathscr{A}$ containing at least three alternatives, and assume that the SCF $f : \Gamma^N \to \mathscr{A}$ respects unanimity. Then f is top-2 strategy-proof if and only if f is dictatorial.*

The main message of Theorem 3.1 is that even if the assumption of complete preferences, which has been identified as crucial above, is relaxed considerably, the conclusion of the GS theorem cannot be escaped. Instead, the weak but realistic assumption that individual preferences contain a best and a second best alternative is sufficient for the conclusion that there exist no non-dictatorial strategy-proof SCF that respects unanimity, provided that sufficiently many alternatives are connected in the preference domain. Since the smallest possible linked top-2 domains contain as few as $4M - 6$ preferences, Theorem 3.1 shows further that far from all $M(M-1)$ possible combinations of first and second ranked alternatives are required in the preference domain for a dictatorial result. Note that Theorem 3.1 applies of course also if preferences have additional structure and belong to some top-k domain with $k > 2$ that satisfies (3.2).

The criterion of linked top-2 domains is of course of technical nature, and it is used here to demonstrate how little structure is needed to preclude the existence of non-dictatorial strategy-proof SCFs. But Theorem 3.1 can also be used as a tool to decide whether preference domains that come up in an economically motivated context admit non-dictatorial strategy-proof SCFs. A simple example of this is the original GS theorem, which follows almost immediately from Theorem 3.1, and a less trivial application can be found in Section 5 in connection with our study of strategy-proof social choice of fixed-sized subsets.

It is finally also worth to point out that the condition of linked top-2 domains in (3.2) is only sufficient, but not necessary for a top-2 domain to be dictatorial. This follows for example from the study on strategy-proof social choice rules in Ching and Zhou (2002), in which every complete, transitive, and asymmetric preference over alternatives is associated with a (partial) preference over subsets of alternatives, and the preferences domain obtained in this way turns out to be a top-2 domain. It can be shown that this domain is not linked, but from Theorem 1 in Ching and Zhou (2002) follows nevertheless that there exists no non-dictatorial strategy-proof SCF that respects unanimity in this domain.

The proof of Theorem 3.1

The rest of this section is devoted to prove Theorem 3.1. The proof will follow a line of proofs for the GS theorem that make systematically use of the fact that a strategy-proof SCF satisfies the following two properties, which are elementary to prove (see Svensson (1999) for example):

Monotonicity: *Let $f : \Sigma^N \to \mathscr{A}$ be a strategy-proof SCF. If $f(P_1,\ldots,P_N) = a$ for some profile $(P_1,\ldots,P_N) \in \Sigma^N$ and $(P'_1,\ldots,P'_N) \in \Sigma^N$ is another profile such that, for all $i \in I$ and $b \in \mathscr{A}$, $a P'_i b$ holds whenever $a P_i b$, then also $f(P'_1,\ldots,P'_N) = a$.*[5]

Pareto optimality: *Let $f : \Sigma^N \to \mathscr{A}$ be a strategy-proof SCF that satisfies unanimity. If $a,b \in \mathscr{A}$ and $a P_i b$ for all $i \in I$, then $f(P_1,\ldots,P_N) \neq b$.*

Using these two properties, it is possible to prove the GS theorem either directly (e.g., Svensson (1999)) or indirectly by establishing a link to Arrow's theorem (e.g., Mas-Colell et al. (1995, p. 874f)). The two properties can, however, not be used without modifications in the present context because a strategy-proof SCF satisfies the monotonicity property only if preferences are complete and Pareto optimality requires in addition to completeness also an unrestricted preference domain. Therefore, we need first to find reformulations of the two properties that hold for top-2 strategy-proof SCFs defined on domains of top-2 preferences. Concerning monotonicity, it will be sufficient for our purposes to cover the case when the chosen alternative moves to the top of some preference:

Monotonicity for top-moving alternatives: *Let Γ be a top-2 domain and suppose that $f : \Gamma^2 \to \mathscr{A}$ is a top-2 strategy-proof SCF. If $f(P_1,P_2) = a$ and $P'_1 \in \Gamma$ is such that $r_1(P'_1) = a$, then also $f(P'_1,P_2) = a$.*

Monotonicity for top-moving alternatives is a direct consequence of top-2 strategy-proofness because if $f(P'_1,P_2) \neq a$, then individual 1 can top-2 manipulate f by representing P_1 and obtain his top-alternative a; thus, we must have $f(P'_1,P_2) = a$. Also concerning Pareto optimality, we can merely obtain a significantly restricted variant, which applies only to preference profiles where some alternative is unanimously preferred to almost all other alternatives:

Pareto optimality at the top: *If Γ is a top-2 domain and $f : \Gamma^2 \to \mathscr{A}$ is a top-2 strategy-proof SCF that respects unanimity, then*

$$r_2(P_1) = r_1(P_2) \quad \Rightarrow \quad f(P_1,P_2) \in \{r_1(P_1), r_2(P_1)\}. \tag{3.3}$$

[5] Monotonicity and strategy-proofness are actually equivalent properties if the domain of preferences is unrestricted, which has been shown by Muller and Satterthwaite (1977).

Also the implication in (3.3) follows easily from top-2 strategy-proofness because if $f(P_1,P_2) \notin \{r_1(P_1), r_2(P_1)\}$, then individual 1 can top-2 manipulate f by representing some $P_1' \in \Gamma$ with $r_1(P_1') = r_2(P_1)$ since $f(P_1',P_2) = r_1(P_1')$ by unanimity.

Our proof of Theorem 3.1 consists essentially of two steps: First, we show that it suffices to prove the theorem for the case $N = 2$, which then is done in the second step. The first step is provided by the following proposition, which is of independent interest since it is more general than actually needed for our purpose in that it holds not only for linked top-2 domains but for all minimally rich[6] top-2 domains (note that a linked top-2 domain must be minimally rich as an almost immediate consequence of its definition), and since it continues a tradition of similar reduction results in social choice theory[7]; it is a direct generalization of Proposition 3.1 in Aswal et al. (2003) to top-2 domains over countable sets of alternatives.

Proposition 3.1 *Suppose that Γ is a minimally rich top-2 domain over a countable set $\mathscr{A}$ of alternatives. If every top-2 strategy-proof SCF $f : \Gamma^n \to \mathscr{A}$ that respects unanimity is dictatorial for all n with $2 \leq n \leq N$, then also every top-2 strategy-proof SCF $f : \Gamma^{N+1} \to \mathscr{A}$ that respects unanimity is dictatorial.*

Proof Let $f : \Gamma^{N+1} \to \mathscr{A}$ be a top-2 strategy-proof SCF that respects unanimity. Define $g : \Gamma^2 \to \mathscr{A}$ by $g(P_1,P_2) = f(P_1,P_2,\dots,P_2)$, where P_2 thus is replicated N times in the argument of f. We claim that g is top-2 strategy-proof. It is clear that g is top-2 strategy-proof in its first argument since f is top-2 strategy-proof, and to prove that g is top-2 strategy-proof in its second argument, we argue by contradiction and suppose that

$$g(P_1,P_2) = a, \quad g(P_1,P_2') = b, \quad b\,P_2\,a, \quad \text{and} \quad b \in \{r_1(P_2), r_2(P_2)\}. \tag{3.4}$$

Then $f(P_1,P_2,\dots,P_2) = a$ and $f(P_1,P_2',\dots,P_2') = b$, and changing the argument of f successively from $(P_1,P_2,\dots,P_2)$ to $(P_1,P_2',\dots,P_2')$ for one individual at a time, i.e., first for individual 2, then for individual 3, etc., there must be an instance where

$$f(P_1,P_2',\dots,P_2',P_2,P_2,\dots,P_2) = c \tag{3.5}$$

for some $c \in \mathscr{A} \setminus \{b\}$, but $f(P_1,P_2',\dots,P_2',P_2',P_2,\dots,P_2) = b$. Note that $c \neq r_1(P_2)$, because if $c = r_1(P_2)$, then monotonicity for top-moving alternatives requires that $f(P_1,P_2,\dots,P_2) = c$, and hence $c = a$ and $a = r_1(P_2)$, which is inconsistent with $b\,P_2\,a$ in (3.4). But since $c \neq r_1(P_2)$ and $b \in \{r_1(P_1), r_2(P_1)\}$ implies $b\,P_2\,c$, we obtain the contradiction that f can be top-2 manipulated at (3.5); thus, g must be top-2 strategy-proof.

[6] A top-2 domain Γ is *minimally rich* if for every $a \in \mathscr{A}$ there exists some $P \in \Gamma$ such that $r_1(P) = a$.

[7] In connection with proofs of the GS theorem or variants of it, reduction results that reduces the number of voters from N to 2 can be found in the following papers: Svensson (1999) and Sen (2001), who both assume an unrestricted domain of complete preferences, but inspection of their proofs reveals that these in fact also are valid for minimally rich domains; Barberà and Peleg (1990), who assume an infinite number of alternatives and continuous preferences; Aswal et al. (2003), which is the first paper where the preference domain explicitly is assumed to be minimally rich; and Kim and Roush (1989), which so far is the most general result for complete preferences and a finite number of alternatives as it replaces the unanimity assumption by the weaker requirement that the image of f contains at least three elements. Completeness of preferences is a crucial assumption in all of these papers, which means that our result is not implied by any of them.

Since g is top-2 strategy-proof and obviously satisfies unanimity (since f satisfies unanimity), g must be dictatorial as a consequence of the assumption of the proposition. Suppose first that individual 1 is the dictator for g. If individual 1 is not a dictator for f, then $f(P_1,P_2,\ldots,P_{N+1}) = a$ and $a \neq r_1(P_1)$ for some profile $(P_1,P_2,\ldots,P_{N+1}) \in \Gamma^{N+1}$. Since Γ is minimally rich, there exists $P_2' \in \Gamma$ with $r_1(P_2') = a$, and monotonicity for top-moving alternatives implies that $g(P_1,P_2') = f(P_1,P_2',\ldots,P_2') = a$, which contradicts that individual 1 is a dictator for g. Thus, individual 1 must also be a dictator for f.

Consider now the case when individual 2 is the dictator for g. Pick some $P_1 \in \Gamma$ and define $h_{P_1} : \Gamma^N \to \mathscr{A}$ by $h_{P_1}(P_2,\ldots,P_{N+1}) = f(P_1,P_2,\ldots,P_{N+1})$. Note that h_{P_1} must be top-2 strategy-proof since f is top-2 strategy-proof. Furthermore, since $h_{P_1}(P_2,P_2,\ldots,P_2) = g(P_1,P_2) = r_1(P_2)$, we find, using monotonicity for top-moving alternatives, that h_{P_1} satisfies unanimity. From the assumption of the proposition follows thus that h_{P_1} is dictatorial, and it remains to show that the dictator for h_{P_1} does not depend on the particular choice of P_1. Assume therefore that $P_1, P_1' \in \Gamma$ and individual i is the dictator for h_{P_1}, but individual j ($j \neq i$) is the dictator for $h_{P_1'}$. Choose $P_i, P_j \in \Gamma$ such that $r_1(P_i) \neq r_1(P_1)$, but $r_1(P_j) = r_1(P_1)$. Then $f(P_1,P_2,\ldots,P_{N+1}) = r_1(P_i) \neq r_1(P_1)$ and $f(P_1',P_2,\ldots,P_{N+1}) = r_1(P_j) = r_1(P_1)$, which contradicts the top-2 strategy-proofness of f. Hence, the dictator for h_{P_1} and $h_{P_1'}$ must be the same individual, which then also must be a dictator for f. The proposition is proved. □

We turn now to the second step of the proof and show by a chain of three lemmas that Theorem 3.1 holds for $N = 2$. We assume therefore throughout the following that Γ is a linked top-2 domain and let $a_1, a_2, a_3, \ldots$ be a list of the alternatives in $\mathscr{A}$ that satisfies (3.2). We will also make use of the following two notations: Denote by Γ_i the set of all $P \in \Gamma$ whose top alternative is a_i, i.e., $\Gamma_i = \{P \in \Gamma;\, r_1(P) = a_i\}$, and use $\left[\begin{smallmatrix} a_i \\ a_j \end{smallmatrix}\right]$ to denote some preference $P \in \Gamma$ for which $r_1(P) = a_i$ and $r_2(P) = a_j$. We consider now first, as a seed for an induction argument, the case when both individuals' top alternative is one of a_1, a_2, or a_3.

Lemma 3.1 *If $f : \Gamma^2 \to \mathscr{A}$ is a top-2 strategy-proof SCF that respects unanimity, then either $f(P_1,P_2) = r_1(P_1)$ for all $P_1,P_2 \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, or $f(P_1,P_2) = r_1(P_2)$ for all $P_1,P_2 \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.*

Proof Note first that Γ, since Γ is linked, must contain preferences $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} a_2 \\ a_1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} a_1 \\ a_3 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} a_3 \\ a_1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} a_2 \\ a_3 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} a_3 \\ a_2 \end{smallmatrix}\right]$.[8] Consider a profile $\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_2 \\ a_1 \end{smallmatrix}\right]\right)$. By (3.3), we have $f\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_2 \\ a_1 \end{smallmatrix}\right]\right) \in \{a_1, a_2\}$, and suppose for the continuation that

$$f\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_2 \\ a_1 \end{smallmatrix}\right]\right) = a_1. \tag{3.6}$$

Then $f\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], P_2\right) = a_1$ for all $P_2 \in \Gamma_2$ because $f\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], P_2\right) \in \{a_1, a_2\}$ by (3.3) and top-2 strategy-proofness requires $f\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], P_2\right) \neq a_2$. Monotonicity for top-moving alternatives implies now that $f(P_1,P_2) = a_1$ for all $P_1 \in \Gamma_1$ and $P_2 \in \Gamma_2$. In particular, $f\left(\left[\begin{smallmatrix} a_1 \\ a_3 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_2 \\ a_3 \end{smallmatrix}\right]\right) = a_1$, so by top-2 strategy-proofness $f\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_3 \\ a_1 \end{smallmatrix}\right]\right) \neq a_3$, and (3.3) implies

[8] Note that these preferences need not to be unique, but the arguments in this and the following two proofs will be independent of the particular choices of these preferences.

then $f\left(\begin{bmatrix}a_1\\a_3\end{bmatrix},\begin{bmatrix}a_3\\a_1\end{bmatrix}\right)=a_1$. From this, we can by the same argument as above conclude that $f(P_1,P_2)=r_1(P_1)$ for all $P_1\in\Gamma_1$ and $P_2\in\Gamma_3$. Noting that if $P_1,P_2\in\Gamma_1$, then $f(P_1,P_2)=r_1(P_1)$ by unanimity, we have thus so far shown that $f(P_1,P_2)=r_1(P_1)$ whenever $P_1\in\Gamma_1$ and $P_2\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$.

Consider now a profile $\left(\begin{bmatrix}a_2\\a_1\end{bmatrix},\begin{bmatrix}a_3\\a_2\end{bmatrix}\right)$. By (3.3), we have $f\left(\begin{bmatrix}a_2\\a_1\end{bmatrix},\begin{bmatrix}a_3\\a_2\end{bmatrix}\right)\in\{a_2,a_3\}$, but since $f\left(\begin{bmatrix}a_1\\a_2\end{bmatrix},\begin{bmatrix}a_3\\a_2\end{bmatrix}\right)=a_1$, top-2 strategy-proofness requires $f\left(\begin{bmatrix}a_2\\a_1\end{bmatrix},\begin{bmatrix}a_3\\a_2\end{bmatrix}\right)=a_2$, and monotonicity for top-moving alternatives leads then to $f\left(\begin{bmatrix}a_2\\a_3\end{bmatrix},\begin{bmatrix}a_3\\a_2\end{bmatrix}\right)=a_2$. By the same arguments as we applied to (3.6) in the previous paragraph, we can from this conclude that $f(P_1,P_2)=r_1(P_1)$ for all $P_1\in\Gamma_2$ and $P_2\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$. Similarly, starting with a profile $\left(\begin{bmatrix}a_3\\a_1\end{bmatrix},\begin{bmatrix}a_2\\a_3\end{bmatrix}\right)$ instead of $\left(\begin{bmatrix}a_2\\a_1\end{bmatrix},\begin{bmatrix}a_3\\a_2\end{bmatrix}\right)$, we get also that $f(P_1,P_2)=r_1(P_1)$ whenever $P_1\in\Gamma_3$ and $P_2\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$. Thus, $f(P_1,P_2)=r_1(P_1)$ for all $P_1,P_2\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$. Note finally that if we had assumed $f\left(\begin{bmatrix}a_1\\a_2\end{bmatrix},\begin{bmatrix}a_2\\a_1\end{bmatrix}\right)=a_2$ in (3.6), then we would have arrived at $f(P_1,P_2)=r_1(P_2)$ for all $P_1,P_2\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$. □

Assume now for the continuation that Lemma 3.1 holds for individual 1, that is, $f(P_1,P_2)=r_1(P_1)$ for all $P_1,P_2\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$.

Lemma 3.2 *If $f:\Gamma^2\to\mathscr{A}$ is a top-2 strategy-proof SCF that respects unanimity, and $f(P_1,P_2)=r_1(P_1)$ for all $P_1\in\bigcup_{i=1}^k\Gamma_i$ and $P_2\in\bigcup_{i=1}^k\Gamma_i$, then $f(P_1,P_2)=r_1(P_1)$ holds also for all $P_1\in\bigcup_{i=1}^k\Gamma_i$ and $P_2\in\Gamma_{k+1}$.*

Proof We claim first that if $a_\ell\in\{a_1,\dots,a_k\}$, then a_ℓ and a_{k+1} can be linked in $\{a_1,\dots,a_k\}$ in the sense that there exist $a_{\ell_1},\dots,a_{\ell_p}\in\{a_1,\dots,a_k\}$ such that $a_\ell\sim a_{\ell_1}$, $a_{\ell_1}\sim a_{\ell_2}$, …, $a_{\ell_{p-1}}\sim a_{\ell_p}$ and $a_{\ell_p}\sim a_{k+1}$. This can be seen in the following way: If $a_\ell\neq a_1$, then there exists an alternative preceding a_ℓ in $a_1,\dots,a_k$ that is connected with a_ℓ, and this alternative is also connected with a preceding alternative, and so on. Thus, it must be possible to link a_ℓ with a_1, and similarly a_{k+1} with a_1. Joining these two chains of connections shows that a_ℓ and a_{k+1} can be linked. The lemma will now be proved by induction over the minimal number of connections needed to link $r_1(P_1)$ with a_{k+1} in $\{a_1,\dots,a_k\}$. Consider therefore first the case when $r_1(P_1)$ and a_{k+1} are connected. Set $r_1(P_1)=a_i$, and let $a_j\neq a_i$ be another alternative in $\{a_1,\dots,a_k\}$ that is connected with a_{k+1}. If $P_2\in\Gamma_{k+1}$, then $f\left(\begin{bmatrix}a_i\\a_{k+1}\end{bmatrix},P_2\right)\in\{a_i,a_{k+1}\}$ by (3.3). But $f\left(\begin{bmatrix}a_i\\a_{k+1}\end{bmatrix},\begin{bmatrix}a_j\\a_{k+1}\end{bmatrix}\right)=a_i$ by assumption, so top-2 strategy-proofness requires $f\left(\begin{bmatrix}a_i\\a_{k+1}\end{bmatrix},P_2\right)=a_i$, and monotonicity for top-moving alternatives implies then that $f(P_1,P_2)=r_1(P_1)$ for all $P_1\in\Gamma_i$ and $P_2\in\Gamma_{k+1}$.

Suppose now that the lemma is true whenever $r_1(P_1)$ can be linked to a_{k+1} with at most n connections, and consider $a_\ell\in\{a_1,\dots,a_k\}$ such that $n+1$ connections are required to link a_ℓ to a_{k+1}. Let $a_{\ell'}\in\{a_1,\dots,a_k\}$ be such that $a_{\ell'}\sim a_\ell$ and $a_{\ell'}$ can be linked to a_{k+1} with at most n connections. Then $f\left(\begin{bmatrix}a_{\ell'}\\a_\ell\end{bmatrix},P_2\right)=a_{\ell'}$, so $f\left(\begin{bmatrix}a_\ell\\a_{\ell'}\end{bmatrix},P_2\right)\in\{a_\ell,a_{\ell'}\}$ by top-2 strategy-proofness. But since $f\left(\begin{bmatrix}a_{\ell'}\\a_\ell\end{bmatrix},\begin{bmatrix}a_\ell\\a_{\ell'}\end{bmatrix}\right)=a_{\ell'}$ by the assumption of the lemma, top-2 strategy-proofness for individual 2 requires $f\left(\begin{bmatrix}a_\ell\\a_{\ell'}\end{bmatrix},P_2\right)\neq a_\ell$. Hence $f\left(\begin{bmatrix}a_\ell\\a_{\ell'}\end{bmatrix},P_2\right)=a_\ell$, and from monotonicity for top-moving alternatives follows then that $f(P_1,P_2)=r_1(P_1)$ for all $P_1\in\Gamma_\ell$ and $P_2\in\Gamma_{k+1}$. □

Lemma 3.3 *If $f : \Gamma^2 \to \mathscr{A}$ is a top-2 strategy-proof SCF that respects unanimity, and $f(P_1,P_2) = r_1(P_1)$ for all $P_1 \in \bigcup_{i=1}^{k} \Gamma_i$ and $P_2 \in \bigcup_{i=1}^{k+1} \Gamma_i$, then $f(P_1,P_2) = r_1(P_1)$ holds also for all $P_1 \in \Gamma_{k+1}$ and $P_2 \in \bigcup_{i=1}^{k+1} \Gamma_i$.*

Proof Let a_{k+1} be connected with $a_i, a_j \in \{a_1,\dots,a_k\}$, $a_i \neq a_j$, and consider a profile $\left(\begin{bmatrix} a_{k+1} \\ a_i \end{bmatrix}, P_2\right)$ for some $P_2 \in \bigcup_{i=1}^{k+1} \Gamma_i$. By assumption, $f\left(\begin{bmatrix} a_i \\ a_{k+1} \end{bmatrix}, P_2\right) = a_i$, so top-2 strategy-proofness requires $f\left(\begin{bmatrix} a_{k+1} \\ a_i \end{bmatrix}, P_2\right) \in \{a_i, a_{k+1}\}$. Assume, in order to derive a contradiction, that $f\left(\begin{bmatrix} a_{k+1} \\ a_i \end{bmatrix}, P_2\right) = a_i$. Monotonicity for top-moving alternatives gives then $f\left(\begin{bmatrix} a_{k+1} \\ a_i \end{bmatrix}, \begin{bmatrix} a_i \\ a_{k+1} \end{bmatrix}\right) = a_i$, which implies that also $f\left(\begin{bmatrix} a_{k+1} \\ a_j \end{bmatrix}, \begin{bmatrix} a_i \\ a_{k+1} \end{bmatrix}\right) = a_i$ because $f\left(\begin{bmatrix} a_{k+1} \\ a_j \end{bmatrix}, \begin{bmatrix} a_i \\ a_{k+1} \end{bmatrix}\right) \in \{a_i, a_{k+1}\}$ by (3.3) and top-2 strategy-proofness excludes a_{k+1}. But then individual 1 can top-2 manipulate f since $f\left(\begin{bmatrix} a_j \\ a_{k+1} \end{bmatrix}, \begin{bmatrix} a_i \\ a_{k+1} \end{bmatrix}\right) = a_j$ by assumption. Thus, $f\left(\begin{bmatrix} a_{k+1} \\ a_i \end{bmatrix}, P_2\right) = a_{k+1}$, and from monotonicity for top-moving alternatives follows then that $f(P_1,P_2) = a_{k+1}$ for all $P_1 \in \Gamma_{k+1}$ and $P_2 \in \bigcup_{i=1}^{k+1} \Gamma_i$. □

The previous three lemmas prove the case $N = 2$ in Theorem 3.1 because Lemma 3.1 shows that $f(P_1,P_2) = r_1(P_1)$ for all $P_1, P_2 \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, and alternatly using Lemma 3.2 and Lemma 3.3 we can successively extend the set of preferences for which $f(P_1,P_2) = r_1(P_1)$ holds; noting that $\Gamma = \bigcup_{i=1}^{M} \Gamma_i$, we finally get that $f(P_1,P_2) = r_1(P_1)$ for all $P_1, P_2 \in \Gamma$, which means that individual 1 must be a dictator for f. The proof of Theorem 3.1 is complete.

4 The degree of manipulation

We give now an alternative interpretation to Theorem 3.1. In the original GS theorem, the interesting implication is of course not that dictatorial SCFs are strategy-proof, but that all democratically acceptable SCFs are manipulable. This conclusion is, however, a purly qualitative result, ensuring the existence of a manipulable profile, but it does not provide any information on how profitable misrepresentation can be and which alternatives an individual can obtain be means of manipulation. This shortcoming of the GS theorem has already been noted by Campbell and Kelly (2002b, p. 152):

> For instance, suppose there are 100 feasible alternatives, and person 1's forty-ninth ranking alternative is selected when he submits his true preference ordering, but by misrepresentating that preference scheme he can precipitate the selection of his forty-eighth ranking alternative. This social choice rule does not pass the Gibbard-Satterthwaite test, because it is assumed that any individual will exploit any opportunity for gain, however slight the improvement. However, this might be one instance in which the gain from misrepresentation is swamped by the cost of acquiring enough information about reported preferences of others to ensure that the misrepresentation is profitable.

This objection captures the important insight that an individual's propensity to misrepresent his preference reasonably depends on which alternative he can obtain instead of the sincere outcome. But the scenario of an SCF which allows an individual to get at most his forty-eighth alternative by means of manipulation turns however out to be contrafactual, because as an immediate consequence of Theorem 3.1, we obtain the following strengthened version of the GS theorem.

Theorem 4.1 (The Degree of Manipulation in the GS Theorem) *If $\mathscr{A}$ is a countable set of at least three alternatives, then every non-dictatorial SCF $f : \Sigma^N \to \mathscr{A}$ that respects unanimity is top-2 manipulable.*

Theorem 4.1 shows that every non-dictatorial SCF that respects unanimity provides strong incentives for strategic reasoning because it can be manipulated in a non-negligible way with some individual obtaining either his first or second alternative.

It is now natural to ask whether Theorem 4.1 can be strengthened further, i.e., whether every non-dictatorial SCF $f : \Sigma^N \to \mathscr{A}$ that respects unanimity can be top-1 manipulated. The answer to this question is however negative, and a counterexample is provided by the SCF in Example 2.1, which is easily checked to be top-1 strategy-proof.

5 Strategy-proof social choice of fixed-sized subsets

In this section, we generalize the notion of strategy-proofness to SCFs which are not single-valued, but whose outcomes are fixed-sized subsets of the set of alternatives, and we investigate whether the main results for single-valued strategy-proof SCFs also apply for this class of multi-valued SCFs.

Previous research on multi-valued strategy-proof SCFs has essentially followed two main lines. The first line consists, without claim of completeness, of the contributions by Pattanaik (1973), Gärdenfors (1976), Barberà (1977), Kelly (1977), Feldman (1979), Duggan and Schwartz (2000), Barberà et al. (2001), Benoît (2002), and Ching and Zhou (2002). In these papers, society's actual intention is to choose a single alternative from a set $\mathscr{A}$, but at some profiles multi-valued outcomes are admitted, for example in order to preserve anonymity or neutrality, or as an attempt to obtain strategy-proofness; the final outcome is in these cases either chosen at a later stage (most often by a lottery), or the set of chosen alternatives is accepted as a compromise solution. The papers differ mainly in their ways they extend preferences over alternatives to preferences over subsets of alternatives, and hence also in their notions of manipulability and strategy-proofness. Their results concerning the possibility of non-dictatorial strategy-proof social choice are however almost unanimously negative, and thereby, the GS theorem is generalized in the sense that its conclusion is shown to be valid even in a larger class of SCFs, which extends the class of single-valued SCFs considered in the original theorem. The social choice of fixed-sized subsets to be considered in this section differs obviously from these papers on multi-valuedness in that society actually intends to choose a subset of the set of alternatives. The generalization of the GS theorem in Theorem 5.1 below is thus not obtained by an extension of the set of admissible SCFs, but it is proved for a new class of SCFs, of which the class single-valued SCFs is a special case (obtained, of course, when the size of the subset to be chosen equals one).

The second line of research on multi-valued strategy-proof social choice has its origin in the paper "Voting be Committees" by Barberà et al. (1991), and has been followed up inter alia by Serizawa (1995), Le Breton and Sen (1999), Aswal et al. (2003), Barberà et al. (2005), and Svensson and Torstensson (2008). In the original model in Barberà et al. (1991), society faces a finite set $\mathscr{A}$ of alternatives, which are not considered to be mutually exclusive, and society is going to elect some subset of $\mathscr{A}$ without restrictions on the size of this subset. This model is applicable, for instance, when a club decides on which applicants to accept for membership, or when a parliament decides on which bills to pass. The main difference between this model and the model in this section is that in Barberà et al. (1991) there is no fundamental conflict between the election of different alternatives and society decides therefore for each alternative separately whether it is socially desirable in an absolute sense, whereas in our model, alternatives compete for the limited number of places in the subset to be chosen, and the election of an alternative means therefore that it is socially desirable in the relative sense that it is preferred to other alternatives which are not elected. As a consequence of this difference, the structure which is assumed for voters' preferences is fundamentally different in the two models, which also leads to different results: while the model of Barberà et al. (1991) admits a large class of non-dictatorial strategy-proof SCFs, non-dictatorial strategy-proof social choice of fixed-sized subsets turns out to be impossible in general.

The social choice of fixed-sized subsets represents a third kind of multi-valuedness, which is common in any democracy, the foremost example of course being elections to national parliaments, but it is present whenever an association, society, or club elects a fixed number of members to a committee. It appears further in matching problems such as the college-admission problem, studied for example in Gale and Shapley (1962) and Roth (1985). Even though votings with fixed-sized subsets as outcomes thus are frequent phenomena, they have until recently not been studied systematically in strategy-proof social choice theory, and the first paper on the strategy-proof social choice of fixed-sized subsets is, to our knowledge, due to Özyurt and Sanver (2008). The main result in that paper is quite close to Theorem 5.1 below, and we postpone therefore a comparison between Özyurt and Sanver (2008) and the approach chosen in this section.

The social choice of fixed-sized subsets will here be studied in the following formal framework: A finite society of N individuals, indexed by the set $I = \{1, \ldots, N\}$, has to choose exactly k elements from a countable set $\mathscr{A}$ containing M alternatives, where M possibly equals infinity. We assume $k < M$ and denote the set of all subsets of $\mathscr{A}$ of size k by $\mathscr{A}_k$, i.e., $\mathscr{A}_k = \{A \subset \mathscr{A}; \sharp A = k\}$. Society makes its choice using a social choice function of the form

$$f : \Omega^N \to \mathscr{A}_k, \tag{5.1}$$

where $\Omega \subset \Sigma$ denotes some domain of complete, transitive, and asymmetric preferences over $\mathscr{A}$. Note that the SCF in (5.1) thus aggregates preferences over the single alternatives in $\mathscr{A}$ to a subset of $\mathscr{A}$.

Simple practically used SCFs of the form (5.1) are for example a generalized majority rule where each voter casts one vote and the k alternatives with most votes are elected, or a generalized Borda count method where voters report complete linear rankings over the M alternatives in $\mathscr{A}$ and the k alternatives with the lowest rank sums are elected. Both procedures require of course appropriate deterministic tie-breaking rules in order to be well-defined SCFs. Another, quite different example of an SCF for the social choice of fixed-sized subsets, which inter alia is used to elect the 659 members of the House of Commons in the United Kingdom, is obtained when both I and $\mathscr{A}$ are partitioned into k subsets, and every subset of individuals is associated with exactly one subset of alternatives and elects from this one alternative using some single-valued SCF.

Before it is possible to investigate whether the conclusion of the GS theorem also is valid for SCFs of the form (5.1), we note that since the arguments in (5.1) are preferences over $\mathscr{A}$ while the outcomes are subsets of $\mathscr{A}$, the standard definitions of the notions of strategy-proofness, respect of unanimity, and dictatorship as they are stated in Section 2, and which are used in the original GS theorem, are not directly applicable to the SCF in (5.1), and therefore, we must first find appropriate generalizations of these notions, which is done in the following.

Firstly, a notion of *manipulability* for an SCF $f : \Omega^N \to \mathscr{A}_k$ requires of course an assumption on how an individual who holds a preference $P \in \Omega$ ranks different outcomes in $\mathscr{A}_k$. It seems reasonable, in particular in connection with strategic reasoning in a voting, that if an individual considers an outcome $A \in \mathscr{A}_k$ and some alternative in A is replaced by another alternative, which is better from the individual's perspective, then the individual will prefer the new outcome to A. We assume therefore that a preference $P \in \Omega$ induces a preference $\hat{P}$ on $\mathscr{A}_k$ in accordance with the following definition, which in a slightly different form has been used earlier in Roth (1985) and which also can be found in Barberà et al. (2004).

Definition 5.1 (Induced preference) A transitive and asymmetric preference $\hat{P}$ on $\mathscr{A}_k$ is said to be *induced* by $P \in \Omega$ if for all $A \in \mathscr{A}_k$, $x \in A$, and $y \in \mathscr{A} \setminus A$,

$$A \hat{P} \big((A \setminus \{x\}) \cup \{y\} \big) \quad \Leftrightarrow \quad xPy. \tag{5.2}$$

The set of all preferences over $\mathscr{A}_k$ that are induced by some $P \in \Omega$ is denoted by Ω_k.

Note that not all outcomes in $\mathscr{A}_k$ can be ranked with (5.2), and we do neither require $\hat{P}$ to be complete, which is motivated by the following observation: If for example $\mathscr{A} = \{a,b,c,d\}$ and $aPbPcPd$, then $\{a,d\}$ and $\{b,c\}$ can obviously not be ranked using (5.2) and an individual might be unwilling to express any clear ranking in such a situation, and it is therefore reasonable to allow induced preferences over $\mathscr{A}_k$ to be incomplete.[9] After having specified the structure of preferences over the outcomes in $\mathscr{A}_k$, it is now straightforward to define notions of manipulability and strategy-proofness for an SCF of the form (5.1).

Definition 5.2 (Manipulability and strategy-proofness) The SCF $f : \Omega^N \to \mathscr{A}_k$ is *manipulable* if there exist $P_i, P_i' \in \Omega$ and $P_{-i} \in \Omega^{N-1}$ such that $f(P_i', P_{-i}) \hat{P}_i f(P_i, P_{-i})$ for all $\hat{P}_i \in \Omega_k$ that are induced by P_i. If f is not manipulable, f is *strategy-proof.*

[9] It might be tempting to model an individual's indecisiveness between two subsets as indifference, but this approach is in general not compatible with a transitive indifference relation. To see this, let for example $\mathscr{A} = \{a,b,c,d,e\}$ and suppose that P is such that $aPbPcPdPe$. Let further $\hat{P}$ be a preference on $\mathscr{A}_2$ which satisfies (5.2) and which in addition is indifferent between any two outcomes in A_2 that cannot be ranked by Definition 5.1. Then $\hat{P}$ is obviously indifferent between $\{a,d\}$ and $\{b,c\}$ and between $\{b,c\}$ and $\{a,e\}$, but since $\{a,d\} \hat{P} \{a,e\}$, this indifference relation is not transitive.

Secondly, in order to generalize the notion of *respect of unanimity*, we note first that there are (at least) two thinkable ways to define under which circumstances unanimity prevails in the society. On the one hand, since society has to choose an element from $\mathscr{A}_k$, a profile $(P_1,\ldots,P_N) \in \Omega^N$ can be considered to exhibit unanimity if there is some $A \in \mathscr{A}_k$ such that every $P_i \in (P_1,\ldots,P_N)$ prefers every alternative in A to any alternative in $\mathscr{A} \setminus A$. Alternatively, unanimity can also be defined in a smaller sense by which unanimity only prevails if society, in addition to the preceding requirement, also agrees on the internal order of the alternatives in A, i.e., if for all $j \in \{1,\ldots,k\}$, $r_j(P_i)$ is the same for all $i \in I$. In both cases, respect of unanimity should of course mean that if a profile $(P_1,\ldots,P_N)$ agrees unanimously (in one of the two sense) on some $A \in \mathscr{A}_k$, then $f(P_1,\ldots,P_N) = A$. It turns out that if f is strategy-proof, then respect of the smaller notion of unanimity implies also respect of unanimity in the wider sense[10], and therefore, we will use the following apparently less restrictive condition for respect of unanimity.

Definition 5.3 (Respect of unanimity) The SCF $f : \Omega^N \to \mathscr{A}_k$ *respects unanimity* if the following implication holds for all profiles $(P_1,\ldots,P_N) \in \Omega^N$:

$$\exists\, a_1,\ldots,a_k \in \mathscr{A} \text{ s.t. } r_j(P_i) = a_j \;\forall i \in I, \forall j \in \{1,\ldots,k\} \quad \Rightarrow \quad f(P_1,\ldots,P_N) = A.$$

Thirdly, and finally, the notion of *dictatorship* can be generalized more unambiguously, because an individual prefers, as a consequence of (5.2), the subset containing his k highest ranked alternatives to any other subset in $\mathscr{A}_k$, and therefore, dictatorship is defined as follows.

Definition 5.4 (Dictatorial SCF) The SCF $f : \Omega^N \to \mathscr{A}_k$ is *dictatorial* if there exists some $i \in I$ such that $f(P_1,\ldots,P_N) = \{r_1(P_i), r_2(P_i),\ldots,r_k(P_i)\}$ for all profiles $(P_1,\ldots,P_N) \in \Omega^N$.

After these reformulations, it is now possible to ask whether the conclusion of the GS theorem also holds for SCFs of the form (5.1), or equivalently, whether the domain Ω_k is dictatorial. The answer to this question depends of course on the structure of the preference domain Ω, and below, we consider the two special cases when Ω equals Σ, that is, the unrestricted domain of complete, transitive and asymmetric preferences over $\mathscr{A}$, respectively when Ω is a domain of single-peaked preferences It will turn out that if $\Omega = \Sigma$, then we obtain in analogy with the GS theorem an impossibility result, but if Ω is single-peaked, then it is possible to find non-dictatorial strategy-proof social choice functions. One might therefore expect the equivalence

$$\Omega \text{ is dictatorial} \quad \Leftrightarrow \quad \Omega_k \text{ is dictatorial.} \tag{5.3}$$

Unfortunately, the relationship between Ω and Ω_k turns out to be more complicated, and in the appendix, we provide two examples, which show that none of the two implications in (5.3) holds true in general.

[10] To see this, let $f : \Omega^N \to \mathscr{A}_k$ be a strategy-proof SCF that satisfies unanimity in the smaller sense, and suppose that $(P_1,\ldots,P_N) \in \Omega^N$ is such that there is some $A \in \mathscr{A}_k$ with $A = \{r_1(P_i),\ldots,r_k(P_i)\}$ for all $i \in I$. By respect of unanimity, $f(P_1,P_1,\ldots,P_1) = A$. If we now successively replace the second argument in $f(P_1,P_1,\ldots,P_1)$ by P_2, the third by P_3, and so on, the value of f must still be A, because otherwise some $i \in I$ would be able to manipulate f, and hence, $f(P_1,P_2,\ldots,P_N) = A$.

The case of unrestricted preferences

For the case $\Omega = \Sigma$, we have the following generalization of the GS theorem.

Theorem 5.1 (The GS Theorem for the Social Choice of Fixed-sized Subsets) *Let $\mathscr{A}$ be a finite set of $M \geq 3$ alternatives, and suppose that the SCF $f : \Sigma^N \to \mathscr{A}_k$, where $1 \leq k \leq M-1$, respects unanimity. Then f is strategy-proof if and only if f is dictatorial.*

The following proof shows that the structure imposed on preferences by Definition 5.2 implies that the domain Σ_k is a linked top-2 domain, after which the dictatorial result follows almost at once from Theorem 3.1.

Proof If f is dictatorial, then f is obviously strategy-proof. For the *only if*-part, suppose now that f is strategy-proof. Denote by $\overline{\Sigma}_k \subset \Sigma_k$ the set of all $\hat{P} \in \Sigma_k$ that are minimal in the sense that they do not contain more rankings than required by (5.2), and note that if $\hat{P} \in \overline{\Sigma}_k$ is induced by $P \in \Sigma$, then P can be reconstructed from $\hat{P}$ via (5.2). Thus, we can define an SCF $\hat{f} : \overline{\Sigma}_k \to \mathscr{A}_k$ by $\hat{f}(\hat{P}_1, \ldots, \hat{P}_N) \stackrel{\text{def}}{=} f(P_1, \ldots, P_N)$, where $P_i \in \Sigma$ induces $\hat{P}_i$ for every $i \in I$, and we prove now in four steps that $\hat{f}$ must be dictatorial, which then immediately implies that also f is dictatorial.

Step 1: If $P \in \Sigma$, then $\hat{P} \in \overline{\Sigma}_k$ is a top-2 preference with $r_1(\hat{P}) = \{r_1(P), \ldots, r_k(P)\}$ and $r_2(\hat{P}) = \{r_1(P), \ldots, r_{k-1}(P), r_{k+1}(P)\}$.

First, set $A_1 = \{r_1(P), \ldots, r_k(P)\} \in \mathscr{A}_k$, and note that if $A \in \mathscr{A}_k \setminus \{A_1\}$, then every alternative in $A_1 \setminus A$ is preferred to every alternative in $A \setminus A_1$ by P, thus $A_1 \hat{P} A$, and $\hat{P}$ is a top-1 preference with $r_1(\hat{P}) = A_1$. Next, set $A_2 = (A_1 \setminus \{r_k(P)\}) \cup \{r_{k+1}(P)\}$, and pick some $A \in \mathscr{A}_k \setminus \{A_1, A_2\}$. If $r_k(P) \notin A$, then every $a \in A_2 \setminus A$ is preferred to every $a' \in A \setminus A_2$, whence $A_2 \hat{P} A$ in this case. On the other hand, if $r_k(P) \in A$, then $r_i(P) \notin A$ for some $i \in \{1, \ldots, k-1\}$ since $A \neq A_1$, and the set $A' = (A \setminus \{r_k(P)\}) \cup \{r_i(P)\} \in \mathscr{A}_k$ satisfies $A' \hat{P} A$. We have either $A' = A_2$ or, since $r_k(P) \notin A'$, $A_2 \hat{P} A'$, and in both cases follows, either directly or by transitivity, that $A_2 \hat{P} A$. Thus, $\hat{P}$ is a top-2 preference and $r_2(\hat{P}) = A_2$.

Step 2: If $A, A' \in \mathscr{A}_k$ and $\sharp(A \cap A') = k-1$, then $A \sim A'$ in $\overline{\Sigma}_k$.[11]

Let $P, P' \in \Sigma$ be such that $\{r_1(P), \ldots, r_{k-1}(P)\} = \{r_1(P'), \ldots, r_{k-1}(P')\} = A \cap A'$, $\{r_k(P)\} = \{r_{k+1}(P')\} = A \setminus (A \cap A')$, and $\{r_{k+1}(P)\} = \{r_k(P')\} = A' \setminus (A \cap A)$. Then by Step 1, we have $r_1(\hat{P}) = A$, $r_2(\hat{P}) = A'$, $r_1(\hat{P}') = A'$, and $r_2(\hat{P}') = A$, which means that $A \sim A'$.

Step 3: $\overline{\Sigma}_k$ is a linked top-2 domain.

It remains to show that $\overline{\Sigma}_k$ satisfies (3.2). Let $a_1, a_2, a_3, \ldots$ be a list of the alternatives in $\mathscr{A}$, and define a linear order $\prec$ on $\mathscr{A}_k$ by $A \prec A'$ if and only if $A, A' \in \mathscr{A}_k$, $A \neq A'$, and $\operatorname{maxind}(A \setminus A') < \operatorname{maxind}(A' \setminus A)$, where $\operatorname{maxind}(A)$ denotes the largest index among the alternatives in $A \subset \mathscr{A}$ according to the list $a_1, a_2, a_3, \ldots$. Given $\prec$, the

[11] It is worth noting that also the converse implication is true, but since it is not needed here, the simple proof of this fact is omitted.

elements in $\mathscr{A}_k$ can be ordered in a list $A_1, A_2, A_3, \ldots$ in the sense that $A_i \prec A_j$ if and only if $i < j$, and $\bigcup_{i=1}^{M} \{A_i\} = \mathscr{A}_k$, and we show that this list satisfies (3.2). The first $k+1$ elements in the list are $A_i = \left(\bigcup_{j=1}^{k+1} \{a_j\}\right) \setminus \{a_{k+2-i}\}$, $1 \leq i \leq k+1$, and it follows from Step 2 that $A_1 \sim A_2$, and for $i = 3, \ldots, k+1$, $A_i \sim A_1$ and $A_i \sim A_2$. If $i > k+1$, then $\operatorname{maxind}(A_i) \geq k+2$, and there exist therefore $a_\ell, a_{\ell'} \in \mathscr{A}$, of course depending on i, such that $a_\ell, a_{\ell'} \notin A_i$, $a_\ell \neq a_{\ell'}$, and both $\ell < \operatorname{maxind}(A_i)$ and $\ell' < \operatorname{maxind}(A_i)$. Then the two sets $A = \left(A_i \setminus \{a_{\operatorname{maxind}(A_i)}\}\right) \cup \{a_\ell\}$ and $A' = \left(A_i \setminus \{a_{\operatorname{maxind}(A_i)}\}\right) \cup \{a_{\ell'}\}$ satisfy $A, A' \in \mathscr{A}_k$ and $\sharp(A_i \cap A) = \sharp(A_i \cap A') = k-1$, so by Step 2, $A_i \sim A$ and $A_i \sim A'$. Moreover, $A \prec A_i$ and $A' \prec A_i$, and hence, $\overline{\Sigma}_k$ is linked.

Step 4: $\hat{f}$ is dictatorial.

Since f is strategy-proof and respects unanimity, $\hat{f}$ is strategy-proof and respects unanimity (in the usual sense), so $\hat{f}$ must be dictatorial by Theorem 3.1. □

In connection with Theorem 5.1, we want to make the following three remarks: Note firstly that the original GS theorem of course turns out as the special chase of Theorem 5.1 which corresponds to $k = 1$.

Secondly, from the preceding proof it is obvious that Theorem 5.1 also is valid under the assumption that the preference $\hat{P}$ on $\mathscr{A}_k$ which is induced by $P \in \Omega$ is such that $r_1(\hat{P}) = \{r_1(P), \ldots, r_k(P)\}$ and $r_2(\hat{P}) = \{r_1(P), \ldots, r_{k-1}, r_{k+1}\}$, which is weaker than (5.2). But we think that (5.2) reflects the way in which people reason when they are voting strategically, and therefore we chose (5.2) as axiomatic starting point in our analysis of strategy-proof SCFs of the form (5.1).

Thirdly, one way to escape from the impossibility result of the original GS theorem is to modify the efficiency condition of respect of unanimity by restricting the range of the SCF to two alternatives. It is worth noting that in the context of the social choice of fixed-sized subsets, non-dictatorial strategy-proof SCFs can be obtained by less severe modifications of the efficiency condition, which is illustrated by the following example.

Example 5.1 Suppose that a society has to choose two of the alternatives in the set $\mathscr{A} = \{a_1, a_2, a_3, a_4\}$, and to this end, voters are divided into two groups, of which the first one chooses one of $\{a_1, a_2\}$ using the majority rule, while the second group chooses one of $\{a_3, a_4\}$, also using majority. The SCF defined in this way is obviously strategy-proof because a voter belonging to the first group, for example, must take the second group's choice as given, and the best thing he can do when choosing between a_1 and a_2 is therefore to vote sincerely. But this SCF must be considered inefficient because the two subsets $\{a_1, a_2\}$ and $\{a_3, a_4\}$ can, even if they would be unanimously preferred to any other subset in $\mathscr{A}_2$, never be the outcome of this SCF. Strategy-proofness is however obtained with less concessions concerning efficiency than possible in the single-valued case because the range of this SCF contains still four of the six possible subsets in $\mathscr{A}_2$.

We conclude this subsection by comparing the results in this section with Özyurt and Sanver (2008), which to our knowledge is the first systematic study on the strategy-proof social choice of fixed-sized subsets. Theorem 3.1 in Özyurt and Sanver (2008) is quite close to our Theorem 5.1, but the two results differ however in the following respects: Firstly, the functional form of the social choice of fixed-sized subsets is modelled differently. In Özyurt and Sanver (2008), voters report complete and transitive preferences over $\mathscr{A}_k$ as arguments in the SCFs, which they refer to as *resolute social choice correspondences*, whereas in our model, voters report preferences over $\mathscr{A}$. The first approach leads of course to a more unambiguous notion of manipulability, but the second one corresponds better with actually used SCFs for the social choice of fixed-sized subsets. Secondly, the two models differ in their assumptions on preferences. Özyurt and Sanver (2008) use the weak assumption that if two outcomes $A, A' \in \mathscr{A}_k$ differ in exactly one alternative, then there exist admissible preferences over $\mathscr{A}_k$ by which A and A' are ranked first and second respectively second and first, and the domain of preferences over $\mathscr{A}_k$ is then said to be obtained through a *reasonable extension map* from Σ. We have remarked above that Theorem 5.1 is actually also valid under this weaker assumption on preferences, but since we consider SCFs of the form $f : \Omega^N \to \mathscr{A}_k$, which are based on preferences over the single alternatives in $\mathscr{A}$, we chose a more elementary way to derive preferences over $\mathscr{A}_k$ from preferences over $\mathscr{A}$. Moreover, in our model, preferences over $\mathscr{A}_k$ are, in contrast to Özyurt and Sanver (2008), allowed to be incomplete, and we argued that it is appropriate to do so. Thirdly, while the model in Özyurt and Sanver (2008) assumes a finite number of alternatives, our result is proved for all countable $\mathscr{A}$. Fourthly, and finally, in our analysis we do not, in contrast to Özyurt and Sanver (2008), require that the domain of preferences over the alternatives in $\mathscr{A}$ is unrestricted, but in (5.1), the preference domain Ω is allowed to be some subset of the set of all transitive, complete and asymmetric preferences over $\mathscr{A}$. This more general framework leads naturally to the question how the conclusion of Theorem 5.1 depends on the structure of Ω; although we are not able to answer this question completely, we can prove in the following subsection that non-dictatorial strategy-proof social choice of fixed-sized subsets is possible in the important special case when Ω is a domain of single-peaked preferences, and we can also show (in the appendix) that the relationship between Ω and Ω_k is non-trivial in general.

The case of single-peaked preferences

We turn now to the case when Ω is a domain of single-peaked preferences. It is a well-known fact in strategy-proof social choice theory that if voters' preferences are single-peaked, then there exist single-valued SCFs that are non-dictatorial, strategy-proof and respect unanimity, the foremost example of course being the median rule (see, for instance, Sprumont (1995) or Barberà (2001)). We show in this subsection that if Ω is single-peaked, then Ω_k inherits sufficiently much of this structure so that it is possible to generalize the median rule to the social choice of fixed-sized subsets and prove that this generalization is strategy-proof.

Assume for this subsection, in addition to the previous formal framework for the social choice of fixed-sized subsets, that $\mathscr{A}$ is equipped with some linear order $\prec$, and that the preferences in the domain Ω are single-peaked with respect to $\prec$, which means that the two implications

$$r_1(P) \preccurlyeq a \prec a' \ \Rightarrow \ a\hat{P}a' \quad \text{and} \quad a' \prec a \preccurlyeq r_1(P) \ \Rightarrow \ a\hat{P}a' \tag{5.4}$$

hold for all $P \in \Omega$ and all $a, a' \in \mathscr{A}$.[12]

We start by identifying a subset of $\mathscr{A}_k$ which will be central for the following constructions.

Definition 5.5 (Connected subset) A subset $A \subset \mathscr{A}$ is *connected* if $a_i, a_j \in A$, $a \in \mathscr{A}$, and $a_i \prec a \prec a_j$ implies $a \in A$. The set of all connected subsets of $\mathscr{A}$ containing exactly k alternatives will be denoted by $\mathscr{A}_k^\circ$.[13]

Definition 5.5 is motivated by the following two fortunate coincidences. Firstly, the linear order $\prec$ on $\mathscr{A}$ induces in a natural way a linear order on $\mathscr{A}_k^\circ$.

Definition 5.6 (Induced linear order) Given a linear order $\prec$ on $\mathscr{A}$, the corresponding *induced linear order* $\prec_k$ on $\mathscr{A}_k$ is defined by $A_1 \prec_k A_2$ if and only if $A_1, A_2 \in \mathscr{A}_k^\circ$, $A_1 \neq A_2$, and $x \in A_1 \setminus A_2$ implies $x \prec y$ for all $y \in A_2$.

Note that $\prec_k$ indeed has the properties of a linear order, i.e., $\prec_k$ is complete, transitive, and asymmetric on $\mathscr{A}_k^\circ$.

Secondly, if Ω is single-peaked, then not all subsets in $\mathscr{A}_k$ can turn out to be top-ranked, but only those which belong to $\mathscr{A}_k^\circ$:

Lemma 5.1 *If the preference $\hat{P}$ on $\mathscr{A}_k$ is induced by some $P \in \Omega$, then $r_1(\hat{P}) \in \mathscr{A}_k^\circ$.*

Proof Suppose that $a_i, a_j \in r_1(\hat{P})$ and $a_i \prec a \prec a_j$, but $a \notin r_1(\hat{P})$. Then $a \neq r_1(P)$, and hence either $a_i \prec a \prec r_1(P)$ or $r_1(P) \prec a \prec a_j$. If $a_i \prec a \prec r_1(P)$, then aPa_i since P is single-peaked, and we obtain the contradiction $\big((r_1(\hat{P}) \setminus \{a_i\}) \cup \{a\}\big)\,\hat{P}\,\big(r_1(\hat{P})\big)$. A similar contradiction occurs if $r_1(P) \prec a \prec a_j$, and the lemma is proved. □

Furthermore, the structure of an induced preference $\hat{P} \in \Omega_k$ is on $\mathscr{A}_k^\circ$ in the following important respect the same as that of a single-peaked preference.[14]

Lemma 5.2 *If the preference $\hat{P}$ on $\mathscr{A}_k$ is induced by some $P \in \Omega$, then for all $A, A' \in \mathscr{A}_k^\circ$,*

$$r_1(\hat{P}) \preccurlyeq_k A \prec_k A' \ \Rightarrow \ A\hat{P}A' \quad \textit{and} \quad A' \prec_k A \preccurlyeq_k r_1(\hat{P}) \ \Rightarrow \ A\hat{P}A'. \tag{5.5}$$

[12] If P is represented by a utility function $u_P : \mathscr{A} \to \mathbb{R}$, then (5.4) implies that u_P has a unique maximum, which motivates the name of single-peaked preferences.

[13] This notion of connectedness is of course entirely unrelated with Definition 3.3.

[14] Even if (5.5) is completely analogue to (5.4), it would be precipitated to call $\hat{P}$ for single-peaked since $\hat{P}$ still can be incomplete (even on $\mathscr{A}_k^\circ$), and in this case, $\hat{P}$ cannot be represented by a single-peaked utility function. To see that $\hat{P}$ can be incomplete on $\mathscr{A}_k^\circ$, let $a_1 \prec a_2 \prec a_3 \prec a_4$ be a linearly ordered set of alternatives, and define P by $a_2Pa_3Pa_4Pa_1$; then P is single-peaked, but $\hat{P}$ induced by P on $\mathscr{A}_2$ cannot rank the two connected subsets $\{a_1, a_2\}$ and $\{a_3, a_4\}$.

Proof Assume first that $r_1(\hat{P}) \preccurlyeq_k A_1 \prec_k A_2$. Let $a_i, a_j \in \mathscr{A}$ be such that $a_i \in A_1 \setminus A_2$ and $a_j \in A_2 \setminus A_1$, and hence $a_i \prec a_j$. If $r_1(P) \preccurlyeq a_i$, then $a_i P a_j$ since P is single-peaked. If, on the other hand, $a_i \prec r_1(P)$, then $a_i \in r_1(\hat{P})$ because $r_1(\hat{P}) \preccurlyeq_k A_1$, and then $a_i P a_j$ since $a_j \notin r_1(\hat{P})$. Thus, we have $a_i P a_j$ in any case, and since $A_1 = \big(A_2 \setminus (A_2 \setminus A_1)\big) \cup (A_1 \setminus A_2)$, it follows that $A_1 \hat{P} A_2$. The second implication in (5.5) is proved in an analogous way. □

Lemma 5.1 makes it possible to define a generalized median rule, which then as a consequence of Lemma 5.2 will turn out to be strategy-proof. In order to generalize the concept of a median of a profile $(P_1, \ldots, P_N)$, let first, for a fixed $A \in \mathscr{A}_k^\circ$, the number of voters to the right respectively to the left of A in the profile $(P_1, \ldots, P_N) \in \Omega^N$ be defined by

$$r_A(P_1, \ldots, P_N) = \sharp\{P_i \in (P_1, \ldots, P_N);\, r_1(\hat{P})_i \succcurlyeq_k A\},$$

$$\text{respectively} \quad l_A(P_1, \ldots, P_N) = \sharp\{P_i \in (P_1, \ldots, P_N);\, r_1(\hat{P})_i \preccurlyeq_k A\}.$$

We say now that $A \in \mathscr{A}_k^\circ$ is a *median set* of the profile $(P_1, \ldots, P_N) \in \Omega^N$ if both $r_A(P_1, \ldots, P_N) \geq N/2$ and $l_A(P_1, \ldots, P_N) \geq N/2$. The last step needed to generalize the median rule is provided by the following lemma.

Lemma 5.3 *For every profile $(P_1, \ldots, P_N) \in \Omega^N$ there exists a median set, and if N is odd, then the median set is unique.*

Proof Suppose that $A_1 \prec_k \ldots \prec_k A_{M-k+1}$ is the linear ordering of the sets in $\mathscr{A}_k^\circ$, and define $n_j = \sharp\{P_i \in (P_1, \ldots, P_N);\, r_1(\hat{P}_i) = A_j\}$. Let m be the least index such that $l_{A_m}(P_1, \ldots, P_N) = \sum_{j=1}^{m} n_j \geq N/2$. Then $\sum_{j=1}^{m-1} n_j < N/2$, so $r_{A_m}(P_1, \ldots, P_N) = \sum_{j=m}^{M-k+1} n_j = N - \sum_{j=1}^{m-1} n_j \geq N/2$, and hence, A_m is a median set for $(P_1, \ldots, P_N)$.

Let now N be odd, and suppose that A_m and $A_{m'}$ with $m < m'$ are two different median sets for $(P_1, \ldots, P_N)$. Since N is odd, we have then $l_{A_m}(P_1, \ldots, P_N) = \sum_{j=1}^{m} n_j \geq (N+1)/2$ and $r_{A_{m'}}(P_1, \ldots, P_N) = \sum_{j=m'}^{M-k+1} n_j \geq (N+1)/2$. But then $N = \sum_{j=1}^{M-k+1} n_j \geq \sum_{j=1}^{m} n_j + \sum_{j=m'}^{M-k+1} n_j \geq N+1$, so the median set must be unique. □

For odd N, we define now the *median set rule* to be that SCF $f : \Omega \to \mathscr{A}_k^\circ$ that assigns to each profile $(P_1, \ldots, P_N) \in \Omega^N$ its unique median set. If $k = 1$, the median set rule coincides of course with the usual median rule. The median set rule is non-dictatorial and respects unanimity, and, mostly as a consequence of (5.5), it is also strategy-proof.

Proposition 5.1 *The median set rule is strategy-proof.*

Proof Let $(P_1, \ldots, P_N) \in \Omega^N$ be a profile with median set $\bar{A}$. If $i \in I$ and $r_1(\hat{P}_i) = \bar{A}$, then individual i can obviously not gain from misrepresentation. Suppose therefore, without loss of generality, that $r_1(\hat{P}_i) \succ_k \bar{A}$. If $P_i' \in \Omega$ is such that $r_1(\hat{P}_i') \succ_k r_1(\hat{P}_i)$, then $r_{\bar{A}}(P_i', P_{-i}) = r_{\bar{A}}(P_i, P_{-i})$ and $l_{\bar{A}}(P_i', P_{-i}) = l_{\bar{A}}(P_i, P_{-i})$, and hence, $\bar{A}$ is also the median set of (P_i', P_{-i}). If, on the other hand, $r_1(\hat{P}_i') \preccurlyeq_k r_1(\hat{P}_i)$, then $r_{\bar{A}}(P_i', P_{-i}) \leq r_{\bar{A}}(P_i, P_{-i})$ and $l_{\bar{A}}(P_i', P_{-i}) \geq l_{\bar{A}}(P_i, P_{-i})$, and hence, if $\bar{A}'$ is the median set of (P_i', P_{-i}), then $\bar{A}' \preccurlyeq_k \bar{A}$. But since $r_1(\hat{P}_i) \succ \bar{A}$, Lemma 5.2 gives that either $\bar{A}' = \bar{A}$ or $\bar{A}\, \hat{P}_i \bar{A}'$, and hence, individual i is not able to manipulate the median set rule. □

If Ω is single-peaked and N is odd, then the median rule permits thus an escape from the negative conclusion of the GS theorem. But it is now easy to construct a strategy-proof SCF $f : \Omega^N \to \mathscr{A}_k$ also for even N. Consider, for instance, the SCF that assigns to $(P_1,\ldots,P_N) \in \Omega^N$ the median set of the profile $(P_1,\ldots,P_N,P_N) \in \Omega^{N+1}$, where P_N has been doubled; if N is even, then this SCF is well-defined, and it turns out to be strategy-proof by essentially the same arguments as those used in the proof of Proposition 5.1.

A Proving the non-equivalence in (5.3)

In this appendix, we show that there exist both non-dictatorial domains Ω such that Ω_k is dictatorial, and dictatorial domains Ω such that Ω_k is non-dictatorial, thereby proving that none of the two implications in (5.3) holds in general.

For the first example, let $\mathscr{A} = \{a_1,a_2,a_3,a_4\}$ be the set of available alternatives, and let the preference domain Ω consist of the following ten preferences:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
a_1	a_1	a_3	a_3	a_4	a_4	a_2	a_2	a_2	a_2
a_2	a_2	a_1	a_1	a_1	a_1	a_3	a_3	a_4	a_4
a_3	a_4	a_2	a_4	a_2	a_3	a_1	a_4	a_1	a_3
a_4	a_3	a_4	a_2	a_3	a_2	a_4	a_1	a_3	a_1

It turns out that Ω_1 is non-dictatorial, which can be verified by invoking Theorem 5.1 in Aswal et al. (2003): They defined a domain Ω to have the *unique seconds* property if there exist two alternatives $a_i, a_j \in \mathscr{A}$ such that $r_1(P) = a_i$ implies $r_2(P) = a_j$ for all $P \in \Omega$, and they proved that if Ω has the unique seconds property, then Ω is non-dictatorial. For the domain Ω above, $P \in \Omega$ and $r_1(P) = a_1$ implies $r_2(P) = a_2$ so Ω has the unique seconds property and is thus non-dictatorial. Consider now the domain of preferences on $\mathscr{A}_3$ that is induced by Ω. $\mathscr{A}_3$ consists of the four sets

$$A_1 = \{a_1,a_2,a_3\}, \quad A_2 = \{a_1,a_2,a_4\}, \quad A_3 = \{a_1,a_3,a_4\}, \quad A_4 = \{a_2,a_3,a_4\},$$

and we have

$$\begin{aligned}
r_1(\hat{P}_1) = r_2(\hat{P}_2) = A_1 \quad &\text{and} \quad r_2(\hat{P}_1) = r_1(\hat{P}_2) = A_2 \quad &\Rightarrow \quad A_1 \sim A_2,\\
r_1(\hat{P}_3) = r_2(\hat{P}_4) = A_1 \quad &\text{and} \quad r_2(\hat{P}_3) = r_1(\hat{P}_4) = A_3 \quad &\Rightarrow \quad A_1 \sim A_3,\\
r_1(\hat{P}_5) = r_2(\hat{P}_6) = A_2 \quad &\text{and} \quad r_2(\hat{P}_5) = r_1(\hat{P}_6) = A_3 \quad &\Rightarrow \quad A_2 \sim A_3,\\
r_1(\hat{P}_7) = r_2(\hat{P}_8) = A_1 \quad &\text{and} \quad r_2(\hat{P}_7) = r_1(\hat{P}_8) = A_4 \quad &\Rightarrow \quad A_1 \sim A_4,\\
r_1(\hat{P}_9) = r_2(\hat{P}_{10}) = A_2 \quad &\text{and} \quad r_2(\hat{P}_9) = r_1(\hat{P}_{10}) = A_4 \quad &\Rightarrow \quad A_2 \sim A_4.
\end{aligned}$$

This means that Ω_3 is linked, and hence, Ω_3 is dictatorial.

For the second example, let again $\mathscr{A} = \{a_1,a_2,a_3,a_4\}$, and define now Ω by

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
a_1	a_2	a_1	a_3	a_2	a_3	a_4	a_1	a_4	a_2	a_3
a_2	a_1	a_3	a_1	a_3	a_2	a_1	a_4	a_2	a_4	a_4
a_3	a_3	a_2	a_2	a_1	a_1	a_2	a_2	a_1	a_1	a_1
a_4	a_4	a_4	a_4	a_4	a_4	a_3	a_3	a_3	a_3	a_2

It is easily checked that $a_1 \sim a_2$, $a_1 \sim a_3$, $a_2 \sim a_3$, $a_1 \sim a_4$, and $a_2 \sim a_4$ in Ω, so Ω is linked and hence dictatorial. Consider now Ω_2, and note that there is only one $P \in \Omega$ such that $r_1(\hat{P}) = \{a_3,a_4\}$, namely P_{11}. Therefore, Ω_2 satisfies trivially the unique seconds property, and hence, Ω_2 is non-dictatorial.

References

1. Aswal N, Chatterji S, Sen A (2003) Dictatorial domains. Econ Theory 22:45–62.
2. Barberà S (1977) The manipulation of social choice mechanisms that do not leave "too much" to chance. Econometrica 45:1573–1588
3. Barberà S (2001) An introduction to strategy-proof social choice functions. Soc Choice Welf 18:619–653
4. Barberà S, Bossert W, Pattanaink P (2004) Ranking sets of objects. In: Barberà S, Hammond P, Seidl C (ed) Handbook of utility theory, Vol. 2, Kluwer academic publishers, Boston
5. Barberà S, Dutta B, Sen A (2001) Strategy-proof social choice correspondences. J Econ Theory 101:374–394
6. Barberà S, Massó J, Neme A (2005) Voting by committees under constraints. J Econ Theory 122:185–205
7. Barberà S, Peleg B (1990) Strategy-proof voting schemes with continuous preferences. Soc Choice Welf 7:31–38
8. Barberà S, Sonnenschein H, Zhou L (1991) Voting by committees. Econometrica 59:595–609
9. Benoît JP (2002) Strategic manipulation in voting games when lotteries and ties are permitted. J Econ Theory, 102:421–436
10. Campbell DE, Kelly J (2002a) Impossibility theorems in the Arrovian framework. In: Arrow K, Sen A, Suzumura K (ed) Handbook of social choice and welfare, Vol. 1, North-Holland, Amsterdam
11. Campbell DE, Kelly J (2002b) Preference revelation with a limited number of indifference classes. Span Econ Rev 4:151–158
12. Ching S, Zhou L (2002) Multi-valued strategy-proof social choice rules. Soc Choice Welf 19:569–580
13. Duggan J, Schwartz T (2000) Strategic manipulability without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized. Soc Choice Welf 17:85–93
14. Feldman A (1979) Nonmanipulable multi-valued social decision functions. Public Choice 34:177–188
15. Gale D, Shapley LS (1962) College admission and the stability of marriage. Am Math Mon 69:9–15
16. Gärdenfors P (1976) Manipulation of social choice functions. J Econ Theory 13:217–228
17. Gibbard A (1973) Manipulation of voting schemes: A general result. Econometrica 41:587–601
18. Kelly J (1977) Strategy-proofness and social choice functions without singlevaluedness. Econometrica 45:439–446
19. Kim KH, Roush FW (1989) Kelly's conjecture. Math Soc Sci 17:189–194
20. Le Breton M, Sen A (1999) Separable preferences, strategyproofness, and decomposability. Econometrica 67:605–628
21. Mas-Colell A, Whinston M, Green J (1995) Microeconomic Theory. Oxford University Press.
22. Muller E, Satterthwaite M (1977) The equivalence of strong positive association and strategy-proofness. J Econ Theory 14:412–418
23. Özyurt S, Sanver MR (2008) Strategy-proof resolute social choice correspondences. Soc Choice Welf 30:89–101
24. Pattanaik P (1972) On the stability of sincere voting situations. J Econ Theory 6:558–574
25. Roth AE (1985) The college admissions problem is not equivalent to the marriage problem. J Econ Theory 36:277–288
26. Satterthwaite M (1975) Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J Econ Theory 10:187–217
27. Sen A (1970) Collective Choice and Social Welfare. Holden-Day, San Francisco
28. Sen A (2001) Another direct proof of the Gibbard-Satterthwaite Theorem. Econ Lett 70:381–385
29. Serizawa S (1995) Power of voters and domain of preferences where voting by committees is strategy-proof. J Econ Theory 67:599–608
30. Sethuraman J, Piaw TC, Vohra RV (2003) Integer programming and Arrovian social welfare functions. Math Oper Res 28:309–326
31. Sprumont Y (1995) Strategyproof collective choice in economic and political environments. Can J Econ 28:68–107
32. Svensson LG (1999) The proof of the Gibbard-Satterthwaite theorem revisited. Working Paper Series 1999:1, Department of Economics, School of Economics and Management, University of Lund.
33. Svensson LG, Torstensson P (2008) Strategy-proof Allocation of Multiple Public Goods. Soc Choice Welf 30:181–196