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On the nonemptiness of approximate cores of large games [1]

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Abstract. We provide a new proof of the non-emptiness of approximate cores of games with many players of a finite number of types. Earlier papers in the literature proceed by showing that, for games with many players, equal-treatment cores of their "balanced cover games", which are non-empty, can be approximated by equal-treatment ε-cores of the games themselves. Our proof is novel in that we rely on a fixed point theorem.

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1 Introduction

The core is an anchoring concept in game theory going back, in its origins, to Edgeworth's contract curve, and the contributions of Debreu and Scarf (1963) and Aumann (1964). The core remains a central concept in economics and most recently, in market design; see, for example, Roth (2002). Even in games with many, but finite numbers of players, however, the core may be empty. The addition of a single player to a large game with a nonempty core may result in a game with an empty core. The problem of the emptiness of the core is especially salient in economies with public goods subject to congestion and exclusion (local public goods) or in economies with clubs. Even in pure exchange economies, the nonemptiness of the core can depend on whether commodities are infinitely divisible. It is, however, a remarkable fact that, as established by Wooders (1983), in games with many players satisfying apparently mild conditions approximate cores are nonempty.

In this paper, inspired by the payoff dependent balancedness notion[2] of Herings and Predtetchinski (2004) and Bonnisseau and Iehle (2007), we demonstrate nonemptiness of approximate cores for sequences of games with arbitrary distributions of players. Recall that much of the literature on approximate cores of NTU games, beginning with Wooders (1983) and most recently Kovalenkov and Wooders (2001, 2003) and Wooders (2008), establishes nonemptiness of approximate cores of large games by showing that payoffs in the cores of derived "balanced cover" games can be approximated by feasible payoffs of the original games. Quite surprisingly, a modification of a key construct from the literature on payoff dependent balancedness, a correspondence from limiting feasible payoffs to distributions of players types[3] achieving them, enables us to establish that for large games limiting payoffs vary continuously with the distribution of player types. With such a correspondence in hand, we can bypass approximation of the original games by balanced cover games and simply appeal to a fixed point argument rather than to approximating balanced games.

More specifically, for sequence of games with growing numbers of players of each of a finite number of types and arbitrary distributions of player types we introduce a set of limiting equal treatment payoffs, denoted by Γ, and

[2] Payoff dependent balancedness generalizes the well-known notion of Scarf balancedness for NTU games.

[3] In interpretation a distribution of player types reflects a player set *á la* Aubin (1979), where players have different participation rates (see also Florenzano (1990)).

a correspondence from payoffs in Γ to distributions of players types able to achieve them. A limiting equal treatment payoff is approximately feasible for some group, possibly large, described by the distribution of player types in the group. We require essentially three conditions for our results:

1. Superadditivity: Any coalition S of players can realize at least the payoffs achievable by cooperation only within coalitions in a partition of S;

2. Small group effectiveness (SGE): All or almost all gains to coalition formation can be realized by coalitions bounded in size (an apparently mild condition); and

3. Quasi-transferable utility (QTU): It is possible to make small transfers from one player to another player, not necessarily at a one-to-one rate; payoff sets are uniformly bounded away from having level segments (sometimes called "non-leveledness").

Our result differs from Wooders (1983) primarily in that we allow sequences of games with player sets converging to distributions of player types with possibly non-rational components, and with distributions having positive measures of players of each type. This is enabled by the assumptions of SGE and, in a secondary way, QTU[4]. Recall that Wooders (2008, Theorem 2) uses these same conditions as employed in this paper to demonstrate that, for games with a compact metric space of player types, given $\varepsilon > 0$ there is an integer $\eta_0(\varepsilon)$ such that all games with more than $\eta_0(\varepsilon)$ players have nonempty equal-treatment ε-cores. As demonstrated in a concluding section of this paper, our result is a Corollary of Wooders earlier result; our contribution is our new proof.

Although both this paper and Predtetchinski (2005) and Allouch and Predtetchinski (2008) use the payoff dependent balancedness notion, their approaches differ in many aspects from that of the current paper. First, as in Wooders (1983), the current paper deals with a sequence games defined in characteristic form with possibly ever-increasing equal-treatment payoff sets. Our framework can accommodate a general class of exchange economies including ones with (local) public goods and clubs. In contrast, Predtetchinski

[4] We could have, for example, required convexity instead of QTU. Or we could have used a less restrictive notion of approximate cores, ignoring small percentages of players of some types (cf., Wooders 2008, Theorem 1).

(2005) and Allouch and Predtetchinski (2008) treat a pure exchange economy, where equal-treatment payoff sets are identical under replications of the total player set. As a result, in our approach both feasible payoffs and core concepts are defined approximately, in contrast to Predtetchinski (2005) and Allouch and Predtetchinski (2008) where feasible payoffs and core concepts are exact. Moreover, the crucial argument in our paper, based on small group effectiveness, is to show that payoffs achieved in the limit by a distribution of player types vary continuously with the distribution of player types. However, in Allouch and Predtetchinski (2008) such a continuity argument is inferred directly from the upper semi-continuity of utility functions over feasible allocations. Finally, in our approach we seek a fixed point for an arbitrary limiting distribution of player types, (both rational and non-rational), unlike Allouch and Predtetchinski (2008) where the distribution of players type is fixed and rational.

The paper is organized as follows. In Section 2, we present the basic features of NTU games. In Section 3, we present our main result on the nonemptiness of approximate cores of a sequence of games with a finite number of types of players.

2 NTU games

We follow Scarf's (1967) classic paper in our definition of a game. An *NTU game* (a non-transferable utility game in coalitional function form) is a pair (N, V) where N is a finite set (the set of *players*) and V is a set-valued function that assigns to each nonempty subset S of N (a *group* or *coalition*) a nonempty subset $V(S)$ of $\mathbb{R}^N$, called a *payoff possibilities set* or simply a *payoff set*, with the following properties:

$V(S)$ is a closed subset of $\mathbb{R}^N$;

$0 \in intV(S)$;

$V(S)$ is comprehensive (that is, $x \in V(S)$ if and only if there is some $y \in V(S) \cap (\mathbb{R}^S_+ \times \mathbb{R}^{N \setminus S})$ such that $y \geq x$).

$V(S) \cap (\mathbb{R}^S_+ \times 0_{N \setminus S})$ is bounded above.

A *payoff vector for a game* (N, V) is a vector x in $\mathbb{R}^N$. A payoff vector x is *feasible* for N if $x \in V(N)$. Let x be a payoff vector. We assume that, if there exists a partition $\{S^k\}$ of N into groups with the property that $x \in V(S^k)$ for each k, then $x \in V(N)$ (superadditivity). This implies that any payoff vector that can be realized by groups in a partition of the total player set is feasible for the entire game.

A payoff vector x is in the ε-*core* of a game (N, V) if it is feasible for N and if, for every subset S of N, $x + \varepsilon 1_S \notin \text{int } V(S)$.[5] Informally, a feasible payoff vector x is in the ε-core if no set of players can improve upon x by more than ε for each player in the set.

For NTU games our definition of substitutes requires that if i and j are substitutes then they make the same contribution to any group they might join and, if they both belong to one group and a payoff vector x is feasible for the group, then x' is also feasible for the group, where x' is derived from x by interchanging the payoffs of i and j.[6] More formally, consider an NTU game (N, V). Two players $i, j \in N$ are *substitutes* if

1. For any $S \subset N$ such that $i, j \notin S$ if $x \in V(S \cup \{i\})$ then $x' \in V(S \cup \{j\})$ where x' is defined by $x'_j = x_i$ and $x'_\ell = x_\ell$ for all $\ell \in S$.
2. For any $S \subset N$ such that $i, j \in S$ if $x \in V(S)$ then $x' \in V(S)$ where x' is defined by $x'_j = x_i$, $x'_i = x_j$ and $x'_\ell = x_\ell$ for all $\ell \in S$, $\ell \neq i, j$.

3 The limiting utility possibilities set for NTU games with a finite number of types of players

We investigate sequences of games with finite sets of player types $t = 1, \ldots, T$. A typical player set is denoted by

$$N = \{(t, q) \mid t = 1, \ldots, T \text{ and } q = 1, \ldots, r^t\},$$

and the profile of N, denoted by pro(N), defined as follows:

$$\text{pro}(N) = (\text{pro}_1(N), \ldots, \text{pro}_T(N)),$$

where

$$\text{pro}_t(N) = |\{q : (t, q) \in N\}| = r^t.$$

[5]It would be possible to include the requirement that x is Pareto-optimal in the sense that there does not exist another feasible payoff y for N with $y \geq x$, $y \neq x$. We do not do so, however, since it does not seem consistent with the notion of an approximate core.

[6]The notion of substitute players in NTU games is introduced in Wooders (1983).

The set of players $\{q : (t, q) \in N\}$ consists of players of type t.

We assume that there is a correspondence V mapping all player sets N into R^N. With every set of players N we associate an NTU game (N, V). In the game (N, V) we assume that all players of the same type are substitutes.

Let $\lambda \in \mathbb{R}$, $0 < \lambda < 1$ be given. The correspondence V satisfies the λ-QTU property if it is possible to decrease the payoff of one player by ε while increasing the payoff to another player by $\lambda\varepsilon$.[7] This ensures that payoff sets are uniformly bounded away from being level.[8] That is, V satisfies the λ-QTU property if, for any set of players N, for any $x \in V(N)$, given any $\varepsilon > 0$ it holds that $x' \in V(N)$ where, for some $(t', q'), (t'', q'') \in N$

$$x'_{tq} = \begin{cases} x_{tq} & \text{if } (t, q) \neq (t', q'), (t'', q'') \\ x_{tq} - \varepsilon & \text{if } (t, q) = (t', q') \\ x_{tq} + \lambda\varepsilon & \text{if } (t, q) = (t'', q'') \end{cases} .$$

For each player set N, we define the subset of payoff vectors that represent equal treatment payoffs for the game (N, V) :

$$V^{\text{etp}}(N) \overset{\text{def}}{=} \{v \in \mathbb{R}^T \mid \Pi_{t=1}^T (\Pi \text{v}_t)^{r_t} \in V(N)\},$$

If $v \in V^{\text{etp}}(N)$ we say that v *represents* an (equal treatment) payoff in $V(N)$. Note that since it always contains the $\mathbf{0}$ payoff the set of equal treatment payoffs is non-empty. When the meaning is clear, we will simply say that v is an equal treatment payoff for the game.

The correspondence V satisfies *small group effectiveness* if for every $\varepsilon > 0$ there is a positive integer $\tau(\varepsilon)$ such that each group N has a partition $\mathcal{P}(N) = (N_k)_{k=1}^K$ with the properties that $|N_k| \leq \tau(\varepsilon)$ for each k, and

$$V^{\text{etp}}(N) \subset \bigcap_{k=1}^{K} V^{\text{etp}}(N_k) + \varepsilon\mathbf{1},$$

[7] It would suffice to make require λ-QTU'ness only for payoffs with the equal treatment property, but this would require more complex notation.

[8] Non-levelness of payoff sets has played a role in the theory of large games since Wooders (1983). It has also appeared in a number of economic models, for example, Mas-Colell (1977) on private goods exchange economies and Wooders (1980) on economies with clubs or local public goods. Uniform non-levelness, as in this paper, appears in Kaneko and Wooders (1994) and in Wooders (2008), where it is called *compensation.*

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^T$. This property ensures that almost all (within ε, for ε arbitrarily small) gains to group formation can be realized by a partition of the total player set into groups uniformly bounded in size (given ε).[9]

During the proof of the following Theorem, we will use the following notation: Denote by $|\cdot|$ the sum-metric in $\mathbb{R}^T$; that is, for $s \in \mathbb{R}^T$ we have $|s| = \sum_{t=1}^{T} |s_t|$. Let Δ denote the simplex in $\mathbb{R}^T$: $\Delta = \{s \in \mathbb{R}^T_+ \mid |x| = 1\}$ and intΔ denote its interior. For each point s in Δ let supp(s) denote the set $\{t \in T \mid s_t > 0\}$, called the *support of* s.

THEOREM. Assume V satisfies *small group effectiveness* and the λ-QTU property. Let $\{(N^n, V)\}$ be a sequence of games such that $|N^n| \to \infty$ and

$$\lim_{n \to +\infty} \frac{\text{pro}(N^n)}{|N^n|} = s^* \in \text{int}\Delta.$$

Then there exists $v^* \in \mathbb{R}^T$ satisfying the property: for every $\varepsilon > 0$ there is an integer r_ε such that for each $n \geq r_\varepsilon$, $(v^* - \varepsilon\mathbf{1})$ is in the ε-core of (N^n, V).

Proof of the Theorem.

Define a subset Γ of $\mathbb{R}^T$ as follows:

$$\Gamma \stackrel{\text{def}}{=} \left\{ v \in \mathbb{R}^T \;\middle|\; \begin{array}{c} \text{There exists } s \in \Delta \cap \mathbb{Q}^T \text{ such that} \\ \text{for each } \; \varepsilon > 0 \text{ there exists a group } S_\varepsilon \text{ satisfying} \\ \frac{\text{pro}(S_\varepsilon)}{|S_\varepsilon|} = s \text{ and } \; (v - \varepsilon\mathbf{1}) \in V^{\text{etp}}(S_\varepsilon) \end{array} \right\}.$$

The set Γ represents equal treatment payoffs that are feasible or approximately feasible for some group, possibly large, described by the fixed distribution of player types in the group. When $(v - \varepsilon\mathbf{1}) \in V^{\text{etp}}(S_\varepsilon)$ we say that s *approximately achieves* v. Note that given $v \in \Gamma$ it may be that there does not exist a group S that can fully achieve v, that is, there need not exist a group S such that $v \in V^{\text{etp}}(S)$. Note also that if $(v - \varepsilon\mathbf{1}) \in V^{\text{etp}}(S_\varepsilon)$ then $(v - \varepsilon\mathbf{1}) \in V^{\text{etp}}(S'_\varepsilon)$ for every group S'_ε containing a positive integer multiple of players of each type as S_ε, that is, for every group S such that $\text{pro}(S'_\varepsilon) = k \; \text{pro}(S_\varepsilon)$ for any positive integer k.[10]

[9] This property, for NTU games, originates in Wooders (2008).

[10] This is an easy consequence of superadditivity. See Wooders (1983) for details.

Given $v \in \Gamma$, there are multiple groups with different distributions that can all *approximately* achieve v. Thus, we define the correspondence $\Pi : \Gamma \rightrightarrows \Delta$ as follows:

$$\Pi(v) \stackrel{\text{def}}{=} \left\{ s \in \Delta \cap \mathbb{Q}^T \;\middle|\; \begin{array}{c} \text{For each } \;\varepsilon > 0 \\ \text{there exists a group } S_\varepsilon \text{ satisfying} \\ \frac{\text{pro}(S_\varepsilon)}{|S_\varepsilon|} = s \text{ and } (v - \varepsilon \mathbf{1}) \in V^{\text{etp}}(S_\varepsilon) \end{array} \right\}.$$

The set $\Pi(v)$ consists of those distributions s of player types that approximately support v, those distributions of player types that are required to exist in the definition of Γ. Note that the nonemptiness of the set $\Pi(v)$ follows immediately from the definitions of Γ and Π.

The graph of the correspondence Π is denoted by $G(\Pi)$ and defined by

$$G(\Pi) = \{(v, s) \in \Gamma \times (\Delta \cap \mathbb{Q}^T) \mid s \in \Pi(v)\}.$$

Obviously, given that the domain of the graph $G(\Pi)$ is $\Gamma \times (\Delta \cap \mathbb{Q}^T)$, there are some converging sequences $\{(v^n, s^n)\}_n$ with each element in the sequence contained in the graph but the limits of the sequences are not. The following proposition is crucial to be able to use a fixed point argument, since it will allow us to show that the closure of the graph of $G(\Pi)$ equals the graph of the closure of the correspondence Π.

Proposition 1. Let $(v, s) \in \mathbb{R}^T \times \Delta$. Let $\{(v^n, s^n)\}_n$ be a sequence in $\Gamma \times (\Delta \cap \mathbb{Q}^T)$ converging to (v, s) such that $s^n \in \Pi(v^n)$ for each n. That is, $(v^n, s^n) \in G(\Pi)$ for each n.

(1). $s \in \text{cl}(\Pi(v))$.

(2). For any sequence of groups $\{N^n\}_n$ satisfying $\lim_{n\to+\infty} \frac{\text{pro}(N^n)}{|N^n|} = s$ (with possibly $\frac{\text{pro}(N^n)}{|N^n|} \neq s^n$) and $|N^n| \to \infty$, for every $\varepsilon > 0$ there exists r_ε such that for each $n \geq r_\varepsilon$, it holds that $(v - \varepsilon \mathbf{1}) \in V^{\text{etp}}(N^n)$.

(3). *If $s \in int\Delta$ there exists $B_s \in R^*_+$ such that $v_t \leq B_s$, for each $t = 1, \ldots, T$.*

Before the proof of Proposition 1 below, we provide an example that illustrates some of the issues.

Proof of Proposition 1.

(1). Let us fix $\varepsilon > 0$. Since $s^n \in \Pi(v^n)$, there exists a group S^n_ε such that

$$\frac{\text{pro}(S^n_\varepsilon)}{|S^n_\varepsilon|} = s^n \text{ and } (v^n - \frac{\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(S^n_\varepsilon).$$

If the sequence $|S^n_\varepsilon|$ is bounded, then passing to a subsequence if necessary, we may assume that there is a group S such that $pro(S^n_\varepsilon) = pro(S)$ for each n. Then, obviously it holds that

$$\frac{\text{pro}(S)}{|S|} = s \text{ and } (v - \frac{\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(S).$$

Thus, $s \in \Pi(v)$.

If the sequence $|S^n_\varepsilon|$ is unbounded then, since V satisfies small group effectiveness, there is an integer $\tau(\varepsilon)$ and a partition $\mathcal{P}(S^n_\varepsilon) = (S^n_{\varepsilon,k})_{k=1}^{K(\varepsilon)}$ of S^n_ε such that

$$V^{\text{etp}}(S^n_\varepsilon) \subset \bigcap_{k=1}^{K(\varepsilon)} V^{\text{etp}}(S^n_{\varepsilon,k}) + \frac{\varepsilon}{3}\mathbf{1},$$

with the property $|S^n_{\varepsilon,k}| \le \tau(\varepsilon)$, for each $S^n_{\varepsilon,k} \in \mathcal{P}(S^n_\varepsilon)$. Since there is only a finite number of possible profiles for any group S satisfying $|S| \le \tau(\varepsilon)$, we can define $M(\varepsilon)$ as the cardinality of these profiles and let $p_1, \ldots, p_m, \ldots, p_{M(\varepsilon)}$ denote these profiles. Thus, one can write

$$\text{pro}(S^n_\varepsilon) = \sum_{m=1}^{M(\varepsilon)} \alpha^n_m p_m.$$

for some non-negative real numbers α^n_m. Since the sequence $\{s^n\}_n$ converges to s, without loss of generality we can assume that $(\frac{\alpha^n_m}{|S^n_\varepsilon|})$ converges to a real number β_m, for each $m = 1, \ldots, M(\varepsilon)$. Let

$$\mathcal{M}^*_\varepsilon = \{m \mid \beta_m > 0\}, \text{ and}$$

$$\mathcal{P}^*(S^n_\varepsilon) \stackrel{\text{def}}{=} \{S^n_{\varepsilon,k} \in \mathcal{P}(S^n_\varepsilon) \mid \text{there exists } m \in \mathcal{M}^*_\varepsilon \text{ such that } \text{pro}(S^n_{\varepsilon,k}) = p_m\}.$$

Since $\mathcal{M}^*_\varepsilon$ is finite there exists n^* such that for all $n \ge n^*$ and $m \in \mathcal{M}^*_\varepsilon$ we have $\alpha^n_m > 0$. Thus, for every coalition $S \in \mathcal{P}^*(S^n_\varepsilon)$ it holds that for all $n \ge n^*$

$$(v^n - \frac{\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(S) + \frac{\varepsilon}{3}\mathbf{1}.$$

Rearranging terms, it follows that

$$(v^n - \frac{2\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(S).$$

Since $V^{\text{etp}}(S)$ is a closed set it holds that

$$(v - \frac{2\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(S). \tag{1}$$

Let

$$S_\varepsilon^{n,*} \stackrel{\text{def}}{=} \bigcup_{S_{\varepsilon,k}^n \in \mathcal{P}^*(S_\varepsilon^n)} S_{\varepsilon,k}^n.$$

Then, by superadditivity, it holds that

$$(v - \frac{2\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(S_\varepsilon^{n,*}).$$

Given that

$$\lim_{n\to\infty} \frac{|S_\varepsilon^n \setminus S_\varepsilon^{n,*}|}{|S_\varepsilon^n|} = 0 \tag{2}$$

Therefore, by the λ-QTU property and the fact that for a large enough n, one can "subsidize" the reminders $(S_\varepsilon^n \setminus S_\varepsilon^{n,*})$ so that

$$(v - \varepsilon\mathbf{1}) \in V^{\text{etp}}(S_\varepsilon^n).$$

That is, we can make small transfers from the set of players in $S_\varepsilon^{n,*}$ to the players in $S_\varepsilon^n \setminus S_\varepsilon^{n,*}$ so that no group of players can improve on $v - \varepsilon$ by more then ε.[11] Hence,

$$\frac{\text{pro}(S_\varepsilon^n)}{|S_\varepsilon^n|} = s^n \in \Pi(v),$$

which implies that $s \in \text{cl}(\Pi(v))$.

(2). Consider an arbitrary sequence of groups $\{N^n\}_n$ satisfying $\lim_{n\to+\infty} \frac{\text{pro}(N^n)}{|N^n|} = s$ (with possibly $\frac{\text{pro}(N^n)}{|N^n|} \neq s^n$) and $|N^n| \to \infty$. Then taking ε (and other

[11] This idea has been used in a number of papers so we spare the reader the details; see, for example, Wooders (2008) and Allouch and Wooders (2008). As a simple illustration, suppose that any two players can earn \$1.00 and that all other groups can only achieve the payoffs attainable by splitting into two-person groups. Suppose that there is a large, but odd, number of players, say 1,000,001. Then it is obvious that by imposing a small "tax" on 1,000,000,players and subsidize one player so that no player has a strong (greater than ε) incentive for try to form am improving coalition.

definitions) as given in the proof of (1) above, let N^n_ε denote the projection of N^n on the convex cone $\mathcal{C}$ spanned by $(p_m)_{m\in\mathcal{M}^*_\varepsilon}$ (by the definition of the projection on a convex set N^n_ε exists and is unique). Thus, one can write

$$N^n = N^n_\varepsilon + N^n \setminus N^n_\varepsilon, \text{ where } N^n_\varepsilon \in \mathcal{C}.$$

Since

$$\lim_{n\to+\infty} \frac{\text{pro}(N^n)}{|N^n|} = s = \lim_{n\to\infty} \frac{\text{pro}(S^{n,*}_\varepsilon)}{|S^{n,*}_\varepsilon|} \in \mathcal{C},$$

it follows that

$$\lim_{n\to\infty} \frac{|N^n \setminus N^n_\varepsilon|}{|N^n|} = 0.$$

Note that one can write $\text{pro}(N^n_\varepsilon) = \sum_{m\in\mathcal{M}^*_\varepsilon} \xi^n_m p_m$, for some real numbers $\xi^n_m \in \mathbb{R}_+$. Let $N^{n,*}_\varepsilon = \sum_{m\in\mathcal{M}^*_\varepsilon} [\xi^n_m] p_m$, where $[u]$ denotes the integer part of u. Then,

$$\lim_{n\to\infty} \frac{|N^n \setminus N^{n,*}_\varepsilon|}{|N^n|} = \lim_{n\to\infty} \frac{|N^n \setminus N^n_\varepsilon| + |N^n_\varepsilon \setminus N^{n,*}_\varepsilon|}{|N^n|} \leq 0 + \lim_{n\to\infty} \frac{|\mathcal{M}^*_\varepsilon|}{|N^n|}\tau(\varepsilon) = 0.$$

Moreover, given that from (1)

$$(v - \frac{2\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(S).$$

if $\text{pro}(S) = p_m$ for some $m \in \mathcal{M}^*_\varepsilon$ it follows that for a large enough n

$$(v - \frac{2\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(N^{n,*}_\varepsilon).$$

Therefore, by the λ-QTU property, one could subsidize the reminders $(N^n \setminus N^{n,*}_\varepsilon)$ so that

$$(v - \varepsilon\mathbf{1}) \in V^{\text{etp}}(N^n).$$

3. Given $s \in \text{int}\Delta$ define

$$B_s = \max_t \max_{\{v' \in V^{\text{etp}}(S) | S \in \mathcal{P}^*(S^n_\varepsilon),\ t\in S\}} (v'_t + \frac{2\varepsilon}{3}).$$

Because $s \in \text{int}\Delta$, for each type t there exists $S \in \mathcal{P}^*(S^n_\varepsilon)$ such that $t \in S$, it holds that $B_s \in \mathbb{R}_+$, that is, B_s is finite. The remainder of the proof follows from the fact that

$$(v - \frac{2\varepsilon}{3}\mathbf{1}) \in V^{\text{etp}}(S^{n,*}_\varepsilon) = V^{\text{etp}}(\bigcup_{S^n_{\varepsilon,k} \in \mathcal{P}^*(S^n_\varepsilon)} S^n_{\varepsilon,k}).$$

■

Now, we define the correspondence $\widetilde{\Pi} : \Gamma \rightrightarrows \Delta$ as follows: for each $v \in \Gamma$

$$\widetilde{\Pi}(v) \overset{\text{def}}{=} \text{cl}(\Pi(v)).$$

Proposition 2. The graph of $\widetilde{\Pi}$ is closed.

Proof of Proposition 2. Let $G(\widetilde{\Pi})$ denote the graph of the correspondence $\widetilde{\Pi}$. Let $\{(v^n, s^n)\}_n$ be a sequence of points in $G(\widetilde{\Pi})$ converging to a point (v, s) in $\mathbb{R}^T \times \Delta$. For each n, since $s^n \in \widetilde{\Pi}(v^n) = \text{cl}(\Pi(v^n))$ it holds that there exists $t^n \in \Pi(v^n)$ such that $|t^n - s^n| \leq \frac{1}{n}$. Hence t^n converges to s. From (1). of Proposition 1 it holds that $s \in \widetilde{\Pi}(v) = \text{cl}(\Pi(v))$ and hence (v, s) belongs to $G(\widetilde{\Pi})$.■

Proposition 3. For each $v \in \Gamma$, the set $\widetilde{\Pi}(v)$ is nonempty and convex.

Proof of Proposition 3. Let $v \in \Gamma$. Obviously, the set $\widetilde{\Pi}(v)$ is nonempty since $\Pi(v)$ is nonempty. Moreover, from superadditivity it holds that

$$\text{co}_{\mathbb{Q}}(\Pi(v)) = \Pi(v).$$[12]

Since $\text{co}_{\mathbb{Q}}(\Pi(v))$ is dense in $\text{co}(\Pi(v))$ it holds that

$$\text{clco}(\Pi(v)) = \text{cl}(\text{co}_{\mathbb{Q}}(\Pi(v))) = \text{cl}(\Pi(v)) = \widetilde{\Pi}(v).$$

Thus, $\widetilde{\Pi}(v)$ is convex.■

The set Γ is a nonempty, closed, and comprehensive from below subset of $\mathbb{R}^T$. Note that the set Γ is a proper set of $\mathbb{R}^T$. Moreover, it holds from our assumptions that $0 \in \text{int}\Gamma$.

Define W as the set

$$W = \Gamma \cap [-\infty, B_{s^*} + 1]^T,$$

where B_{s^*} is defined as in (3). of Proposition 1. A point $v \in W$ belongs to the boundary of W if and only if either $v \in \partial\Gamma$ or $v^t = B_{s^*} + 1$ for some $t = 1, \ldots, T$.

[12] The set $\text{co}_{\mathbb{Q}}(A)$ is the set of all convex combinations of A with rational coefficients.

Proposition 4. There is a homeomorphism h from the space Δ to the space $\partial W \cap \mathbb{R}^T_+$ such that $h^t(s) = 0$ whenever $s \in \Delta$ and $t \in T \setminus \operatorname{supp}(s)$.

Proof of Proposition 4. Let $s \in \Delta$ be given. Let R be the ray emanating from the point $0 = (0, \ldots, 0)$ in the direction of s. Thus, every point r of R is of the form $r = \theta s$ for some non-negative real number θ. It is clear that, since W is closed, comprehensive from below, and bounded from above, R intersects the boundary of W at exactly one point.

Define the map h from Δ to $\partial W \cap \mathbb{R}^T_+$, by letting $h(s)$ be the unique point in the intersection of the ray R and the set ∂W. We now demonstrate that h has an inverse. Let g denote the map from $\partial W \cap \mathbb{R}^T_+$ to Δ given by the equation

$$g(v) = \frac{v}{|v|}.$$

The map g is well defined since the point 0 lies in the interior of W. It is easy to see that g is indeed the inverse of h, that is, $h \circ g$ and $g \circ h$ are equal to the respective identity maps.

Clearly, g is a continuous map. Furthermore, because its domain is compact and the codomain is Hausdorff, it carries closed sets to closed sets. Therefore, h is also a continuous map. This proves that h is a homeomorphism.

The rest of the proof relies on a version of the well-known Fan's coincidence theorem, as stated below. Given a nonempty and convex subset Y of $\mathbb{R}^N$ and a point y of Y, let $N(Y, y) = \{z \in \mathbb{R}^N \mid (y - y')^\top z \geq 0 \text{ for each } y' \in Y\}$ denote the normal cone of the set Y at the point y. A zero point of a correspondence $\Phi : Y \rightrightarrows \mathbb{R}^N$ is a point y of Y such that $\Phi(y)$ contains the zero.

Theorem. (Fan, 1972). Let Y be a nonempty compact and convex subset of $\mathbb{R}^N$. Let $\Phi : Y \rightrightarrows \mathbb{R}^N$ be a correspondence with nonempty convex values having a compact graph. Suppose that for each $y \in Y$ and for each $z \in N(Y, y)$ there exists a $\phi \in \Phi(y)$ such that $z^\top \phi \leq 0$. Then, Φ has a zero point.

Proposition 5. There exists $v^* \in \partial\Gamma$ such that $s^* \in \widetilde{\Pi}(v^*)$.

Proof of Proposition 5. Define the correspondence $\Phi : \Delta \rightrightarrows \mathbb{R}^T$ by letting $\Phi(s) = \widetilde{\Pi}(h(s)) - \{s^*\}$ for each $s \in \Delta$. Clearly, the correspondence Φ has nonempty and convex values. Its graph is closed, because h is continuous and the graph of $\widetilde{\Pi}$ is closed. Since Φ maps a compact set Δ into a compact set $\Delta - \{s^*\}$, its graph is, in fact, a compact set.

We now verify the boundary condition of Fan's coincidence theorem. Let $s \in \Delta$ be given. Let v denote the vector $h(s)$ and let H denote the (possibly empty) set $T \setminus \mathrm{supp}(s)$. Then, the normal cone of Δ at s is the set

$$N(\Delta, s) = \left\{ z \in \mathbb{R}^T \,\middle|\, z = a\mathbf{1} + \textstyle\sum_{t \in H} l_t \mathbf{e}_t,\ a \in \mathbb{R},\ l_t \leq 0 \right\},$$

where $(\mathbf{e}_1, \ldots, \mathbf{e}_T)$ is the standard canonical basis of $\mathbb{R}^T$. Let $z \in N(\Delta, s)$ be given. If s belongs to the relative interior of Δ so that $\mathrm{supp}(s) = T$, then every $z (\in N(\Delta, s))$ is proportional to the vector $\mathbf{1}$. In this case the equality $z^\top \phi = 0$ holds for each $\phi \in \Phi(s)$.

Consider now the case where s lies on the relative boundary of Δ, so that the set H is nonempty. Then, $v_t = h(s)_t = 0$ for each $t \in H$. But this implies that the set $\Pi(v)$ (and consequently $\widetilde{\Pi}(v) = \mathrm{cl}(\Pi(v))$) contains the entire face $\Delta_H = \{s \in \Delta \,|\, s_t = 0 \text{ for each } t \in T \backslash H\}$ of the simplex Δ. In particular, $\widetilde{\Pi}(v)$ contains a k^* of Δ_H defined as follows

$$k^* = \begin{cases} k^*_t = s^*_t + \frac{\sum_{j \in T \backslash H} s^*_j}{|T \backslash H|} & \text{if } t \in H, \\ \\ 0 & \text{otherwise.} \end{cases}$$

The vector $\phi = k^* - s^*$ is therefore an element of $\Phi(s)$. Since $0 \leq \phi_t$ for each $t \in H$, the inequality $\phi^\top z \leq 0$ holds for each $z \in N(\Delta, s)$. By Fan's coincidence theorem, the correspondence Φ has a zero point. Letting v^* be equal to $h(s^*)$, we see that $v^* \in \partial W$ and $s^* \in \widetilde{\Pi}(v^*)$. Since B_{s^*} is the per-capita bound it follows that $v^*_t < B_{s^*} + 1$ for each $t = 1, \ldots, T$ and thus $v^* \in \partial \Gamma$.

Finally, Proposition 5, together with (2). in Proposition 1, implies that for every $\varepsilon > 0$ there exists r_ε such that for each $n \geq r_\varepsilon$, $(v^* - \varepsilon \mathbf{1})$ is in the ε-core of (N^n, V).■

4 A Corollary

To further relate our work to Wooders (2008) we provide the following Corollary. Using our notation, following is a version of Wooders (2008, Theorem 2).[13]

[13]Wooders (2008) allows a compact metric space of player atributes, so our mdel is a special case.

Wooders (2008, Theorem 2). Assume V satisfies *small group effectiveness* and λ-QTU. Given $\varepsilon > 0$ there is an integer $\eta_0(\varepsilon)$ such that any game (N, V) with $|N| > \eta_0$ has a non-empty (equal treatment) ε-core.

Corollary. Assume V satisfies *small group effectiveness* and λ-QTU. Then for any sequence of games such that $|N^n| \to \infty$ and

$$\lim_{n\to+\infty} \frac{\mathrm{pro}(N^n)}{|N^n|} = s^* \tag{3}$$

there exists $v^* \in \mathbb{R}^T$ satisfying the property: for every $\varepsilon > 0$ there exists r_ε such that for each $n \geq r_\varepsilon$, $(v^* - \varepsilon\mathbf{1})$ is in the ε-core of (N^n, V).

Proof of the Corollary. Let $\{(N^n, V)\}$ be a sequence satisfying (3). From Wooders (2008), for each term (N^n, V) in the sequence with $|N^n| > \eta_0(\frac{\varepsilon}{3})$ the game (N^n, V) has a non-empty $\frac{\varepsilon}{3}$-core (and contains an equal-treatment payoff). Let $r_{\frac{\varepsilon}{3}}$ be sufficiently large so that for every $n \geq r_\varepsilon$ it holds that $|N^n| > \eta_0(\frac{\varepsilon}{3})$. Let $v^n \in \mathbb{R}^T$ be in the $\frac{\varepsilon}{3}$-core of (N^n, V). Without loss of generality we can assume that $\lim_{n\to+\infty} v^n$ exists and equals v^*. Since $\lim_{n\to+\infty} v^n = v^*$ there is an integer $\eta_1(\frac{\varepsilon}{3}) \geq \eta_0(\frac{\varepsilon}{3})$ sufficiently large so that for all $n > \eta_1(\frac{\varepsilon}{3})$ it holds that $max_t|v_t^n - v_t^*| < \frac{\varepsilon}{3}$. This implies that $v_t^n - v_t^* \geq 0$ so $(v^* - \varepsilon\mathbf{1})$ is feasible for all sufficiently large games (N^n, V). Since v^n is in the $\frac{\varepsilon}{3}$-core for all n sufficiently large it follows that $(v^* - \varepsilon\mathbf{1})$ is in the ε-core of (N^n, V). ■

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