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Endogenous Correlated Network Dynamics

Frank Page
Indiana University

Rui Gong
Indiana University

Myrna Wooders
Vanderbilt University

Abstract

We model the structure and strategy of social interactions prevailing at any point in time as a directed network and address the following question: given the rules of network and coalition formation, preferences of individuals over networks, strategic behavior of coalitions in forming networks, and the trembles of nature, what network and coalitional dynamics are likely to emergence and persist. We formulate the problemas a dynamic, stochastic game and v equilibrium (in network and coalition formation strategies), (ii) together with the trembles of nature, this correlated stationary equilibrium determines an equilibrium Markov process of network and coalition formation, and (iii) this endogenous Markov process possesses a finite set of ergodic measures, and generates a finite, disjoint collection of nonempty subsets of networks and coalitions, each constituting a basin of attraction. Moreover, we extend to the setting of endogenous Markov dynamics the notions of pairwise stability (Jackson-Wolinsky, 1996) and the path dominance core (Page Wooders, 2009a). We show that in order for any network-coalition pair to emerge and persist, it is necessary that the pair reside in one of finitely many basins of attraction. The results we obtain here build on Page and Wooders (2009a)and the seminal contributions of Jackson and Watts (2002), Konishi and Ray (2003), and Dutta, Ghosal, and Ray (2005).

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Contact: Frank Page - fpage.supernetworks@gmail.com, Rui Gong - ruigong@umail.iu.edu, Myrna Wooders - myrna.wooders@gmail.com.
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Rui Gong
Department of Economics
Indiana University
Bloomington, IN 47405
USA
ruigong@umail.iu.edu

Frank Page[1]
Department of Economics
Indiana University
Bloomington, IN 47405
USA
fpage@indiana.edu

Myrna Wooders
Department of Economics
Vanderbilt University
Nashville, TN 37235
USA
myrna.wooders@vanderbilt.edu

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[1]Also, Research Associate, Systemic Risk Centre, London School of Economics, London WC2A 2AE, UK.
[2]The authors are grateful to the Centre d'Economie de la Sorbonne (CES) at the University of Paris 1 for financial support and to Cuong Le Van, Bernard Cornet, and Jean-Marc Bonnisseau for their support and hospitality during our many visits to Paris, as well as to the Systemic Risk Centre (SRC) at LSE and to Jean-Pierre Zigrand and Jon Danielsson for their support and hospitality during our visits to the SRC. The authors are also grateful to Ehud Kalai, Matthew Jackson, A. S. Nowak, Rabah Amir, Bhaskar Dutta, Philip Reny, Gabrielle Demange, J.-J. Herings, Luca Merlino, and Jacomo Corbo for many helpful discussions and comments during the writing of this paper. Finally, the authors gratefully acknowledge financial support from the Douglas Grey Fund for Research in Economics at Vanderbilt and from the Systemic Risk Centre under ESRC grant number ES/K002309/1. GPW_ECND_Nov 15 2015.tex

Abstract

We model the structure and strategy of social interactions prevailing at any point in time as a directed network and we address the following open question in the theory of social and economic network formation: given the rules of network and coalition formation, preferences of individuals over networks, strategic behavior of coalitions in forming networks, and the trembles of nature, what network and coalitional dynamics are likely to emergence and persist. Our main contributions are to formulate the problem of network and coalition formation as a dynamic, stochastic game and to show that: (i) the game possesses a correlated stationary Markov equilibrium (in network and coalition formation strategies), (ii) together with the trembles of nature, this correlated stationary equilibrium determines an equilibrium Markov process of network and coalition formation, and (iii) this endogenous Markov process possesses a finite set of ergodic measures, and generates a finite, disjoint collection of nonempty subsets of networks and coalitions, each constituting a basin of attraction. Moreover, we extend to the setting of endogenous Markov dynamics the notions of pairwise stability (Jackson-Wolinsky, 1996) and the path dominance core (Page-Wooders, 2009a). We show that in order for any network-coalition pair to emerge and persist, it is necessary that the pair reside in one of finitely many basins of attraction. The results we obtain here for endogenous network dynamics and stochastic basins of attraction are the dynamic analogs of our earlier results on endogenous network formation and strategic basins of attraction in static, abstract games of network formation (Page and Wooders, 2009a), and build on the seminal contributions of Jackson and Watts (2002), Konishi and Ray (2003), and Dutta, Ghosal, and Ray (2005).

KEYWORDS: endogenous network dynamics, dynamic stochastic games of network formation, stationary Markov correlated equilibrium, equilibrium Markov process of network formation, basins of attraction, Harris decomposition, ergodic probability measures, dynamic path dominance core, dynamic pairwise stability.

JEL Classifications: A14, C71, C72

1 Introduction

1.1 Overview

In all social and economic interactions, individuals or coalitions choose not only with whom to interact but how to interact, and over time both the structure (the "with whom") and the strategy ("the how") of interactions change. Our objectives here are to model the structure *and* strategy of interactions prevailing at any point in time as a directed network and to shed new light on the co-evolution of network structure and strategic behavior by addressing the following open question in the theory of social and economic network formation: given rules of network formation, preferences of individuals over networks, strategic behavior of coalitions in forming networks, and trembles of nature, what *network and coalitional dynamics* are likely to emergence and persist. Thus, we propose to study the emergence of endogenous network and coalitional dynamics resulting from the interactions between network structure, strategic behavior, and the randomness in nature through time.

The main contributions of the paper are (i) to formulate the problem of network formation as a dynamic, stochastic game, and (ii) to show: (a) that this game possesses a correlated stationary Markov equilibrium in network and coalition formation strategies, (b) that together with the trembles of nature, this correlated stationary equilibrium in strategies determines an equilibrium Markov process of network and coalition formation that respects the rules of network formation and the preferences of individuals and (c) that although uncountably many networks may form, this equilibrium process generates a *finite*, disjoint collection of nonempty subsets of networks and coalitions, each constituting a basin of attraction, and possesses a *finite*, nonempty set of ergodic measures.

In prior work on the co-evolution of network structure and strategic behavior using static abstract games of network formation, Page and Wooders (2009a), have shown that, given the rules of network formation and the preferences of individuals, these games possess *strategic basins of attraction* and these contain all networks that are likely to emerge and persist as the game unfolds. Moreover, Page and Wooders have shown that when any one of these strategic basins contains only one network, then that network (i.e., the network contained in the singleton basin) is stable against all coalitional network deviation strategies - and thus the game has a nonempty *path dominance core*. Finally, Page and Wooders have shown that depending on how the rules of network formation and the dominance relation over networks are specialized (via additional assumptions), any network contained in the path dominance core is pairwise stable (Jackson-Wolinsky, 1996), strongly stable (Jackson-van den Nouweland, 2005), Nash (Bala-Goyal, 2000), or consistent (Chwe, 1994).

Here it is shown that there are many parallels between the static abstract game formulation and the Page-Wooders results for static games and the results obtained here for the Markovian dynamic game formulation. This is suggested already by the seminal paper by Jackson and Watts (2002) on the evolution of networks. Jackson and Watts present to my knowledge the first theory of stochastic dynamic network formation over a finite set of linking networks governed by a Markov chain generated by the myopic strategic behavior of players (following the Jackson-Wolinsky rules of network formation) and the trembles of nature. Their model builds on the earlier, nonstochastic model of dynamic network formation due to Watts (2001) - as far as we know, the first model of network dynamics (see also Skyrms and Pemantle, 2000). By considering a sequence of perturbed, irreducible and aperiodic Markov chains (i.e., each chain with a unique invariant measure) converging to the original Markov chain, they show that any pairwise stable network is necessarily contained in the support of an invariant measure - that is, in the support of a probability measure that places all its mass on sets of networks likely to form in the long run. We show here that similar conclusions can be reached for directed networks with uncountably many arc types governed by a Markov process *generated endogenously* by the farsighted strategic behavior of players (following arbitrary network formation rules) and the trembles of nature.

In a general Markov game setting, with farsighted players, what precisely does it mean for a network to be pairwise stable - or stable in any sense? For example, if the state space of networks is large, then the endogenous Markov process of network formation is likely to have many invariant measures - and in fact many ergodic probability measures (i.e., measures that place all their probability mass on a single absorbing set). Which absorbing set contains networks stable in the sense of pairwise stability, or strong stability, or Nash stability? These are some of the questions we answer here in our study of endogenous network dynamics.

We show that the endogenous Markov process of network and coalition formation with uncountably many possible networks possesses only *finitely many* ergodic measures and generates only *finitely many* basins of attraction. This endogenous finiteness property of basins in equilibrium has serious implications for empirical work on networks. In particular, since nature does not afford the empirical observer multiple observations across states but rather only multiple observations across time, the fact that only finitely many long run equilibrium sets are possible, and more importantly, the fact that on these sets (i.e., on these basins of attraction) state averages are equal to time averages gives meaning and significance to time series observations which seek to infer the long run equilibrium network. Moreover, to the extent that networks can truly represent various social and economic interactions, our understanding of how and why the network formation process moves toward or away from any particular basin can potentially shed new light on the persistence or transience of many social and economic conditions. For example, how and why does a particular path of entrepreneurial and scientific interactions carry an economy beyond a tipping point and onto a path of economic growth driven by a particular industry - and why might it fail to do so? How and why does a particular path of product line-nonlinear pricing schedule configurations lead a strategically competitive industry to become more concentrated - or fade? These are some of the applied questions which hopefully can be addressed using a model of endogenous network dynamics.

1.2 Endogenous Network Dynamics

The approach taken here to endogenous dynamics is motivated by the observation that the stochastic process governing network and coalition formation through time is determined not only by nature's randomness (or nature's trembles) through time - as envisioned in random graph theoretic approaches - *but also* by the strategic behavior of individuals and coalitions through time in attempting to influence the networks and coalitions that emerge under the prevailing rules of network formation and the trembles of nature. Thus, we develop a theory of endogenous network and coalitional dynamics that brings together elements of random graph theory and game theory in a discounted stochastic game model of network and coalition formation. As pointed out by Neyman (2003), the dynamics of the states of a stochastic game form a Markov chain whenever the players' strategies are stationary, and as we will make clear here in a dynamic game of network formation, the same is true whenever the players' strategies are correlated stationary Markov strategies. While dynamic stochastic games have been used elsewhere in economics (see, for example, Amir, 1991, 1996; Amir and Lambson, 2003; and Chakrabarti, 2008; Duffie, Geanakoplos, Mas-Colell, and McLennan, 1994; Mertens and Parthasarathy 1987, 1991; Herings and Peeters, 2004; Nowak, 2003, 2007), their application to the analysis of the evolution of social and economic networks is new.[1]

The analysis has two parts. In part (1) a dynamic game model of network *and* coalition formation is constructed, and then shown to have a correlated stationary Markov equilibrium. In part (2), the stability properties of the endogenous Markov process of network and coalition formation induced by this correlated stationary Markov equilibrium are analyzed in detail.

The existence result presented in part (1) is based on the work by Nowak and Raghavan (1992) and Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) on the existence of correlated stationary Markov equilibria.[2] Here the dynamic game of network and coalition formation is formulated in

[1] Also, see the edited volume on stochastic games and applications by Neyman and Sorin (2003).

[2] Here we will focus on correlated stationary Markov equilibria. While the existence of Nash equilibria in stationary Markov strategies for discounted stochastic games with *finite or countable* state spaces has long been established (e.g., see Federgruen, 1978), the existence of such equilibria for discounted stochastic games with uncountable state spaces has been an open question since such games were first studied by Himmelberg, Parthasarathy, Raghavan, and Van Vleck (1976) - at least until Levy (2013). Levy has shown, via a counter example, that stationary Markov equilibria do not always exist for the discounted stochastic game models with uncountably many states. On the positive side, Page (2015) has established an existence result for stationary Markov Nash equilibria for a large sub-class of discounted stochastic

a compact metric space of directed networks, possibly containing uncountably many networks, and the existence of correlated stationary Markov equilibrium in players' network and coalition formation strategies is established. In our m-player discounted stochastic game of network and coalition formation we show that the farsighted strategic behavior of players in attempting to influence the path of network and coalition formation generates $m+1$ stationary Markov processes of network and coalition formation, one of which - depending on the current state - will be chosen by a public randomization device as the governing law of motion in each period. Thus, one of the main contributions of the paper is to provide a possible theoretical foundation in strategic behavior for the random graph theoretic approach to social and economic network formation found in the literature.

The assumptions of our discounted stochastic game model of network formation are similar to those required to establish the existence of correlated stationary equilibria in discounted stochastic games (e.g., Nowak and Raghavan 1992) and subgame perfect equilibria in discounted stochastic games (e.g., Mertens and Parthasarathy 1987, Salon 1998, and Maitra and Sudderth 2007). Our model has six primitives consisting of the following: (i) a feasible set of directed networks representing all possible configurations of social or economic interactions, (ii) a feasible set of coalitions allowed to form under the rules of network formation for the purpose of proposing alternative networks, (iii) a state space consisting of feasible network-coalition pairs, (iv) a set of players and player constraint correspondences specifying for each player and in each state the set of feasible alternative networks that a player can propose under the rules of network formation as a member of the current or status quo coalition - and as a nonmember, (v) a set of player discount rates and payoff functions defined on the graph of players' constraint correspondence, and (vi) a stochastic law of motion. This stochastic law of motion represents nature and specifies the probability with which each possible new status quo network-coalition (i.e., new state) might emerge as a function of the status quo network-coalition pair (i.e., the current state) and the profile of player-proposed new status quo networks (i.e., the current action profile). Using these primitives, we construct a discounted stochastic game model of network formation and show that it possesses a correlated stationary Markov equilibrium in network proposal strategies. More importantly, we are able to conclude via classical results due to Blackwell (1965) (also see Himmelberg, Parthasarathy, and vanVleck (1976)), Nowak and Raghavan (1992), and Duffie, Geanakoplos, Mas-Colell, and McLennan (1994)) that this correlated equilibrium over Markov stationary strategies is optimal against player defections to other network proposal strategies (including, as a consequence of Blackwell's 1965 Theorem, history-dependent proposal strategies) - thus showing that our decision to focus on correlation over stationary strategies (i.e., strategies that depend only on the status quo network-coalition pair) is well-founded.

In part (2), we analyze the stability properties of the endogenous Markov process of network and coalition formation. In particular, we show that the Markov transition kernel induced from the correlated stationary equilibrium in the network and coalition formation game not only satisfies the drift condition for Markov stability but also satisfies the uniform countable additivity condition for stability. We then show, via results due to Tweedie (2001), that the state space (of network-coalition pairs) underlying our dynamic game of network formation can be uniquely partitioned into a transient set together with finitely many basins of attraction, and more importantly, that no matter what the starting point is, the process of network and coalition formation reaches in finite time with probability 1 a basin of attraction. From this it is easy to conclude that the equilibrium Markov process of network and coalition formation possesses only *finitely many* ergodic measures (one for each basin of attraction).[3] Also, in part (2) we will introduce the notions of dynamic stability and consistency and using these notions extend the definitions of pairwise stability (Jackson and path dominance core to the dynamic Markov setting developed here. We will then show that networks that are stable with respect to either of these notions must necessarily reside in the basins of attraction generated by the endogenous network dynamic. In coming to these conclusions, it will become clear that in dynamic

games with uncountably many states. Nonetheless, here we will consider only correlated stationary Markov equilibria.

[3]Our stability results for equilibrium Markov processes of network and coalition formation, while classical in form and appearance, are completely new. Unlike in the classical setting where the state space is finite or countable, the state space here is uncountable, consisting of uncountably many networks and coalitions pairs.

Markov games of network and coalition formation, stability has two masters: strategic behavior and the laws of nature - with the laws of nature being dominate.

1.3 Related Literature

To our knowledge, the first paper to study endogenous dynamics in a related model is the paper by Konishi and Ray (2003) on dynamic coalition formation. The primitives of their model consist of (i) a finite set of outcomes (possibly a finite set of networks), (ii) a set of coalitional constraint correspondences specifying for each coalition and each status quo outcome, the set of new outcomes a coalition might bring about if allowed to do so, and (iii) a discount rate and set of player payoff functions defined on the set of all outcomes. Konishi and Ray show that their model possesses a stochastic law of motion governing movement from one outcome to another and a consistent valuation function such that (a) if a move from one outcome to another takes place with positive probability, then for some coalition this move makes sense in that no coalition member is made worse off by the move and no further move makes all coalition members better off, and (b) if for a given outcome there is another outcome making all members of some coalition better off and no further outcome makes this coalition even better off, then a move to another outcome takes place with probability 1 (i.e., the probability of standing still at the given outcome is zero). Stated loosely, then, Konishi and Ray show that for their model there exists a law of motion which generates coalitionally improving moves from one outcome to another (i.e., in our case it would be from one network to another).

Our model differs from the model of Konishi and Ray in several respects. First, in our model movements from one network (outcome) to another are largely determined by the *strategic behavior* of individuals. In our model, equilibrium strategic behavior, together with natures trembles, are central to determining equilibrium network dynamics.

Second, whereas Konishi and Ray, for technical reasons, restrict attention to a finite set of outcomes (in our model, a finite set of networks), we allow for uncountably many networks - this to allow for consideration of networks with a large number of nodes or networks with uncountably many arc types. This is more than a technical nicety. In order to capture the myriad and potentially complex nature of interactions between players (say for example in a stock market or in a contracting game with multiple principals and multiple agents) we must allow there to be uncountably many possible types of interactions. In our model the set of potential interactions are represented by a set of arc types (in fact, by a compact metric space of arc types) with each arc type (or arc label) representing a particular type of interaction (or connection) between nodes in a directed network. Thus, because we allow for uncountably many arc types in describing the interactions between nodes, in our model there are uncountably many possible networks (or outcomes, in the language of Konishi and Ray). Moreover, in order to model large networks (i.e., networks with many nodes), in our model we can allow there to be infinitely many nodes - although here we focus exclusively on the finite nodes case. Third, while Konishi and Ray restrict attention at the outset to Markov laws of motion, we will show that our strategically determined equilibrium Markov process of network and coalition formation is robust against all possible alternative dynamics, even those induced by history-dependent types of strategic behavior. Thus, at least for the class of Konishi-Ray types of models, we will show that Markov laws of motion are stable and robust with respect to other forms of history-dependent laws of motion.[4]

Finally, we take rules of network formation as given primitives of the model. We show that the interactions of strategic behavior, network structure, and the trembles of nature generate an equilibrium process of network and coalition formation and change consistent with these rules. We will also show that this process possesses a nonempty set of ergodic measures and generates basins of attraction. There are no rules of coalition formation – rules specifying how the process moves from one state to another in Konishi-Ray; instead they focus on transitions consistent with improvement properties for coalitions.

[4]By a Markov law of motion we mean a stochastic law of motion where probabilistic movements from one outcome or network to another depend only on the current outcome rather than on some history of outcomes.

In contrast to Konishi-Ray, Dutta, Ghosal, and Ray (2005) consider strategic behavior in a dynamic game of network formation over a finite set of undirected linking networks (rather than directed networks) under a particular set of network formation rules. They show existence of a Nash equilibrium and identify conditions under which efficiency can be sustained in equilibrium - thus, continuing in a dynamic setting the seminal work of Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997) on equilibrium and efficiency. Here our focus is on equilibrium and stability rather than equilibrium and efficiency and our analysis is carried out in a dynamic, stochastic game model of network and coalition formation, admitting all forms of network formation rules, over an uncountable set of directed networks. Dutta et. al. (2005) restrict attention to Markov network formation strategies and show that there is an equilibrium in this class. In contrast, we show for the class of all strategies that there is an equilibrium in correlated stationary Markov strategies; and therefore, by Blackwell's classical result (Blackwell, 1965, Theorem 6f) we conclude that this type of equilibrium is robust against defections by individual players to any other types of strategies. Moreover, as mentioned above, we show that in general, the resulting equilibrium Markov process of network and coalitional formation possesses finitely many ergodic measures and generates finitely many network and coalitional basins of attraction.

We view the starting point of our research to be the pioneering work of Jackson and Watts (2002) already discussed briefly above. Our model of endogenous network and coalitional dynamics extends their work on stochastic network dynamics in several respects. First, in our model players behave farsightedly in attempting to influence the path of network and coalition formation - farsighted in the sense of dynamic programing (e.g., Dutta, Ghosal, and Ray (2005))[5]. Moreover, in our model the game is played over a (possibly) uncountable collection of directed networks under general rules of network formation which include not only the Jackson-Wolinsky rules, but also other more complex rules. In our model the law of motion is such that the trembles of nature are Markovian rather than i.i.d. as in Jackson and Watt, and are functions of the current state and the current profile of network and coalition proposals by players. Extending the notion of pairwise stability to a dynamic setting, one of the benchmarks for our research is to show that in a Markov model of network and coalition formation, if a network is dynamically pairwise stable, then in order to persist, it must be contained in one of finitely many basins of attraction, and therefore, contained in the support of an ergodic probability measure.

2 Primitives

2.1 Directed Connections and Directed Networks

The basic ingredients of our model are as follows:

[A-1] (*nodes, arcs, and players*)
$N =$ *a finite set of nodes, with typical elements* i *and* j*, equipped with the discrete metric* ρ_N*,*[6]
$A =$ *a compact metric space of arc types, with typical element* a*, equipped with metric* ρ_A*,*
$D =$ *a finite set of players, with typical element* d*,*
$P(D) =$ *the collection of all nonempty subsets or coalitions of players, with typical element* S*.*

Arcs represent potential types of connections between nodes, and depending on the application, nodes can represent economic agents (players) or economic objects such as markets or firms. We will make a distinction between nodes and players - and thus, we will not assume automatically that the set of nodes N and the set of players D are one and the same.

[5] See Chwe (1994), Page, Wooders, and Kamat (2005), and Page and Wooders (2005) for notions of farsighted behavior in static, abstract games.

[6] Under the discrete metric the distance between two nodes i and j in N is given by

$$\rho_N(i,j) := \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

We begin by defining the notion of a directed connection.

Definition 1 (*Directed Connections*)
Given node set N and arc set A, a directed connection is an ordered pair $(a,(i,j)) \in A \times (N \times N)$ consisting of an arc type a and an order pair of nodes, i and j, indicating that nodes i and j are connected by a type a arc from node i to node j. The set of all possible directed connections is given by

$$K := A \times (N \times N). \tag{1}$$

Given our assumptions [A-1], the set of all possible directed connections, K, is a compact metric space with product metric

$$\rho_K\left((a,(i,j)),(a',(i',j'))\right) := \rho_A(a,a') + \rho_N(i,i') + \rho_N(j,j'). \tag{2}$$

A directed network is defined as follows:

Definition 2 (*Directed Networks*)
Given node set N and arc set A, a directed network, G, is a nonempty, ρ_K-closed subset of the set $K = A \times (N \times N)$ of directed connections. The collection of all directed networks is denoted by $P_f(K)$.

Thus, a network $G \in P_f(K)$ is a nonempty, closed set of connections specifying the various ways the nodes in N are connected by the arcs in A in network G.

Under our definition of a directed network, we allow an arc to go from a given node back to that given node (i.e., *loops* are allowed).[7] Also, under our definition an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, our definition does not allow a particular arc a to go *from* a node i *to* a node i' multiple times.

The following notation is useful in describing networks. Given directed network $G \in P_f(K)$, let

$$\left.\begin{array}{c} G(a) := \{(i,j) \in N \times N : (a,(i,j)) \in G\}, \\ \text{and} \\ G(ij) := \{a \in A : (a,(i,j)) \in G\}. \end{array}\right\} \tag{3}$$

Thus, in network G,

$G(a)$ is the *set of node pairs* connected by arc a,
and
$G(ij)$ is the *set of arcs* from node i to node j.

If for some arc $a \in A$, $G(a)$ is empty, then arc a is not used in network G. Also, if for some node $i \in N$, $G(ij)$ and $G(ji)$ are empty for all $j \neq i$, then node i is isolated.

We will also find the following notation useful. Given directed network $G \in P_f(K)$, let

$$\left.\begin{array}{c} G^+(i) := \{j \in N : (a,(i,j)) \in G \text{ for some } a \in A\}, \\ \text{and} \\ G^-(i) := \{j \in N : (a,(j,i)) \in G \text{ for some } a \in A\}. \end{array}\right\} \tag{4}$$

Thus, in network G,

$G^+(i)$ is the *set of nodes* j such that there is *at least one* arc from i to j,
and
$G^-(i)$ is the *set of nodes* j such that there is *at least one* arc from j to i.

[7] By allowing loops we are able to represent a network having no connections between distinct nodes as a network consisting entirely of loops at each node.

Thus, $G^+(i)$ is the set of nodes, "you can get to" and $G^-(i)$ is the set of nodes "you can come from" at node i in network G. Note that in a directed network with multiple connections between nodes, the cardinality of $G^+(i)$, denoted by $|G^+(i)|$, is *not* the out degree of note i.[8] Nor is $|G^-(i)|$ the in degree of node i. The out degree of node i in network G is given by $\sum_{j\in N} |G(ij)|$. Similarly, the in degree of node i in network G is given by $\sum_{j\in N} |G(ji)|$.

2.2 The Space of Directed Networks

In order to analyze the co-evolution of strategic behavior, network structure and equilibrium dynamics, we must find a topology for the space of directed networks that is simultaneously *coarse* enough to guarantee compactness and *fine* enough to discriminate between differences across networks that are due to differences in the ways nodes are connected (via differing arc types) and differences across networks that are due to the complete absence of connections. We resolve this topological dilemma by equipping the space of directed networks, $P_f(K)$, with the Hausdorff metric h. Because the set of directed connections, $K := A \times (N \times N)$, is a compact metric space, the space of directed networks, $P_f(K)$ equipped with the Hausdorff metric is automatically compact (see Section 7 below, also see Section B.11 in Hildenbrand 1974, or Sections 3.16-3.18 in Aliprantis and Border 2006). Moreover, given the nature of the discrete metric on the set of nodes, it is easy to show that if the Hausdorff distance between any pair of networks G and G' is less than $\varepsilon \in (0,1)$, then the networks can differ only in the ways a given set of node pairs are connected - and not in the set of node pairs that are connected. In particular, if for networks G and G', $h(G,G') < \varepsilon < 1$, then

$$(a,(i,i')) \in G \text{ if and only if } (a',(i,i')) \in G'$$

for arcs a and a' with $\rho_A(a,a') < \varepsilon$.

To illustrate the sensitivity of the Hausdorff metric topology to absence or presence of connections across networks, consider the following example. Suppose that the set of nodes is given by $N := \{i_1, i_2, i_3\}$, while the set of arcs types is given by $A = [0,1]$. We can think of arc types $a \in [0,1]$ as representing intensity levels or flow levels from one node to another or as expressing the probabilities with which one node interacts with another.[9] Consider the three networks, G_1, G_2, and G_3 depicted

[8] Recall that $|G(ij)| = 0$ if and only if $G(ij) = \emptyset$.

[9] In the context of linking networks, this class of networks (i.e., networks with constrained, variable link strength) has recently been used to investigate the endogenous formation of efficient and reliable communications networks by Bloch and Dutta (2009). See Page and Wooders (2009b) for a further discussion of differences between linking networks with variable length strength and directed networks with heterogeneous arc types.

in Figure 1.

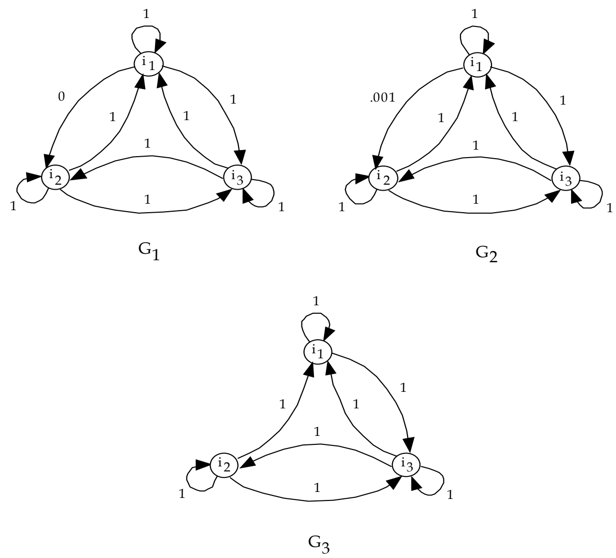

Figure 1

Note that the three networks differ only in the nature of the connection from node i_1 to node i_2. In network G_1 this connection is inactive (i.e., has a zero intensity level), that is, $(0,(i_1,i_2)) \in G_1$. In network G_2 the connection from i_1 to i_2 is weak, that is, $(.001,(i_1,i_2)) \in G_2$. However, in network G_3, there is no connection at all from i_1 to i_2. Under the network metric h (see 51), networks G_1 and G_2 are close, while networks G_1 and G_3 as well as networks G_2 and G_3 are far apart. In particular, $h(G_1,G_2) = .001$, while

$$h(G_1,G_3) = 2$$
and
$$h(G_2,G_3) = 2 - .001.$$

In the analysis to follow, one of our main objectives will be to better understand the emergence and stability properties of equilibrium network dynamics generated by the endogenous interplay between network structure and strategic behavior in the formation of networks over time. In order to achieve this objective, we must allow for the emergence of networks where some connections are absent altogether (i.e., where some node pairs are not connected in any direction by any arc types, as in network G_3 in Figure 1). The Hausdorff metric topology on the space of networks is particularly well suited for the type of analysis required to achieve this objective.[10]

[10] Another way to see this: rather than think of a network G as a nonempty, closed subset of the Cartesian product of arcs and node pairs,

$$G \subset K := A \times (N \times N),$$

think of network G as a set-valued function, f_G,

$$f_G : dom f_G \rightarrow P_f(A)$$

from the subset $dom f_G \subseteq N \times N$ of node pairs connected in G into the space $P_f(A)$ of nonempty, closed subsets of the set of arcs A. If network G is incomplete (i.e., has some node pairs *without* connections) then the domain of definition, $dom f_G$, of function f_G will be a *proper* subset of the set of all node pairs $N \times N$. Now consider the space of all such functions (i.e., the space of all networks). Because domains can vary across functions, f_G, (i.e., because domains are not fixed and constant across functions) it is very difficult to define a topology on such a function space (called a space of partial functions). One way around the variable domain problem is to equip function space with a graph topology (e.g., see Naimpally 1966 or Beer 1993). This is precisely the role played by the Hausdorff metric topology in the space of networks, $P_f(K)$, where each network is represented by a nonempty, closed subset of the space of connections K and where the set of node pairs involved in connections can vary across networks. The Hausdorff metric topology in $P_f(K)$ is a graph topology, and as is the case with graph topologies in spaces of partial functions, it solves the variable connections problem by making the variability of connections part of the topology (i.e. part of the way we measure the distance between networks).

In many applications it is useful to restrict attention to a particular feasible h-closed subset of networks. We will denote by $\mathbb{G}$ the feasible set of networks.

2.3 Players and Feasible Coalitions

The path taken by a network through time depends in large measure on the actions taken by groups of players in attempting to influence how the network changes across time. Thus, coalitions play a central role in determining the path taken by network formation. Recall that D denotes the set of players (a set not necessarily equal to the set of nodes) with typical element d and $P(D)$ denotes the collection of all coalitions (i.e., nonempty subsets of D) with typical element denoted by S. We will assume that the set of players D has cardinality m (i.e., $|D| = m$).

In many applications it is useful to restrict attention to a particular feasible subset of coalitions. Often restrictions on the feasible set of coalitions are the result of the rules of network formation. We will let

$$\mathcal{C} \subset P(D)$$

denote the feasible set of coalitions, and we will equip $\mathcal{C}$ with the discrete metric $\rho_{\mathcal{C}}$ (i.e., $\rho_{\mathcal{C}}(S', S) = 0$ if $S' = S$, and $\rho_{\mathcal{C}}(S', S) = 1$ if $S' \neq S$).

2.4 The State Space

We shall take as the state space the set $\Omega := \mathbb{G} \times \mathcal{C}$ of all feasible network-coalition pairs. Each state $\omega \in \Omega$ has the following interpretation: if $\omega = (G, S)$ is the current state, then G is the current status quo network of social interactions and it is coalition S's turn to move in the game of network formation. We will refer to the coalition whose turn it is to move as the *status quo coalition*.

The state space $\mathbb{G}\times\mathcal{C}$ is a compact metric space under the product metric ρ_Ω given by

$$\rho_\Omega((G', S'), (G, S)) := h(G, G') + \rho_{\mathcal{C}}(S', S).$$

Letting $B(\Omega) := B(\mathbb{G} \times \mathcal{C})$ be the Borel σ-field generated by the metric ρ_Ω, we will equip the state space $(\mathbb{G}\times\mathcal{C}, B(\mathbb{G} \times \mathcal{C}))$ with the product probability measure

$$\mu = \nu \times \eta \tag{5}$$

where the probability measure η on feasible coalitions is such that $\eta(S) > 0$ for all $S \in \mathcal{C}$ and where the probability measure ν on feasible networks is such that the set of, at most, countably many disjoint atoms[11] is given by

$$\{\mathbb{A}_{\alpha 1}, \mathbb{A}_{\alpha 2}, \ldots\} = \{\mathbb{A}_{\alpha k}\}_{k=1}^{\infty} \subset \mathbb{G}. \tag{6}$$

Thus, we have as our state space, the probability space

$$(\Omega, B(\Omega), \mu) = (\mathbb{G}\times\mathcal{C}, B(\mathbb{G} \times \mathcal{C}), \nu \times \eta), \tag{7}$$

a compact metric space with metric $\rho_\Omega = h + \rho_{\mathcal{C}}$ and typical element $\omega = (G, S)$.

[11] A set of networks $\mathbb{A}_{\alpha k} \in B(\mathbb{G})$ is an atom of the probability space $(\mathbb{G}, B(\mathbb{G}), \nu)$ if $\nu(\mathbb{A}_{\alpha k}) > 0$ and for all subsets $\mathbb{B} \subseteq \mathbb{A}_{\alpha k}$, $\mathbb{B} \in B(\mathbb{G})$, $\nu(\mathbb{B}) = \nu(\mathbb{A}_{\alpha k})$ or $\nu(\mathbb{B}) = 0$. The set of networks $\mathbb{G}$ contains at most countably many disjoint atoms, $\{\mathbb{A}_{\alpha k}\}_{k=1}^{\infty}$, and $\mathbb{G}$ can be written as

$$\mathbb{G} = \mathbb{NA} \cup [\cup_{k=1}^{\infty} \mathbb{A}_{\alpha k}],$$

where the set $\mathbb{NA}$ contains no atoms. We say that the probability space $(\mathbb{G}, B(\mathbb{G}), \nu)$ is atomless or nonatomic if it contains no atoms.

2.5 Feasible Actions and the Feasible Action Correspondence

In each move of the game, each player takes an action in an effort to optimally influence the path of network change governed by the law of motion. In our game, each player's action takes the form of a network recommendation or network proposal. In particular, given current state $\omega = (G, S) \in \Omega$, each player $d \in D$ has available a nonempty subset of network proposals $\Phi_d(\omega) \subseteq \mathbb{G}$ that can be put forth by player d for consideration by nature. However, only players who are members of the status quo coalition S (i.e., the coalition whose turn it is to move) are allowed to propose *substantive* changes. If $G' \in \Phi_d(G, S)$ is proposed by player $d \in S$ (and therefore, by a member of the status quo player coalition S), this means that if player $d's$ proposal is chosen by nature (i.e., by the law of motion), then under the rules of network formation, player d acting in concert with some or all the members of coalition S, has the power and ability to implement the proposed network (i.e., change the status quo network G to network G'). Moreover, because players who are not members of the status quo coalition are not allowed to propose substantive changes, these players (i.e., players $d \notin S$) can only propose that the status quo network be maintained. Thus, players' feasible action correspondences, $\Phi_d(\cdot)$, are the formal expressions of the rules of network formation (see Page and Wooders, 2009a, for a discussion of rules of network formation in static games).

A state-action profile pair (ω, G_D) is contained in the graph of $\Phi(\cdot)$, denoted by $Gr\Phi(\cdot)$, if $G_D \in \Phi(\omega)$. We will assume the following concerning feasible action correspondences, $\Phi_d(\cdot)$.

[A-2] (*feasible action correspondence*)
(1) For all states $\omega = (G, S)$, $\Phi_d(G, S) \subseteq G$ is h-closed with

$$\left.\begin{array}{c} \text{(a) } G \in \Phi_d(G, S), \\ \text{and} \\ \text{(b) } \{G\} = \Phi_d(G, S) \text{ for all } d \notin S. \end{array}\right\} \tag{8}$$

(2) $\Phi_d(\cdot)$ has a measurable graph[12]*, that is, $Gr\Phi_d(\cdot) \in B(\Omega) \times B(\mathbb{G})$.*
(3) $\Phi_d(\cdot)$ has a $\rho_{\Omega\times\mathbb{G}}$-graph, that is, $Gr\Phi_d(\cdot) \in B(\Omega) \times B(\mathbb{G})$.[13]

Under [A-2](1)-(2) the feasible proposal profile correspondence

$$\omega \longrightarrow \Phi(\omega) := \prod_{d \in D} \Phi_d(\omega) \tag{9}$$

is measurable with nonempty, h-closed values in $\mathbb{G}^m$, and under [A-2](1)-(3), $\Phi(\cdot)$ has a closed graph.

2.6 Payoff Functions

Players decide which feasible networks to propose, in part, based on their payoff functions. We shall assume that

[12]We say that $\Phi_d(\cdot)$ is measurable if

$$\Phi_d^{-1}(\mathbb{E}) := \{\omega \in \Omega : \Phi_d(\omega) \cap \mathbb{E} \notin \emptyset\} \in B(\Omega)$$

for $\mathbb{E} \subset \mathbb{G}$ open (sometimes called weak or lower measurability). Because $\mathbb{G}$ is compact, the following statements are equivalent:
(1) $\Phi_d(\cdot)$ is measurable.
(2) $\Phi_d^{-1}(\mathbb{F}) \in B(\Omega)$ for $\mathbb{F} \subset \mathbb{G}$ closed.
(3) $Gr\Phi_d(\cdot) \in B(\Omega) \times B(\mathbb{G})$. (see Aliprantis and Border 2006, Nowak 1984, or Wagner 1977).

[13]Here, $\rho_{\Omega\times\mathbb{G}} := \rho_\Omega + h$ where $\rho_\Omega := h + \rho_{\mathcal{F}}$. $Gr\Phi_d(\cdot)$ is $\rho_{\Omega\times\mathbb{G}}$-closed if for any converging sequence, $\{(\omega^n, G^n)\}$, contained in $Gr\Phi_d(\cdot)$ with

$$(\omega^n, G^n) \underset{\rho_{\Omega\times\mathbb{G}}}{\longrightarrow} (\omega^*, G^*),$$

$(\omega^*, G^*) \in Gr\Phi_d(\cdot)$. Because $\Omega \times \mathbb{G}$ is compact metric, the fact that $\Phi_d(\cdot)$ has a closed graph is equivalent to $\Phi_d(\cdot)$ being upper semicontinuous.

[A-3] (*payoff functions*)
Each player $d \in D$ has a payoff function defined on states and proposal profiles,

$$r_d(\cdot, \cdot) : \Omega \times \mathbb{G}^m \longrightarrow [-M, M], \tag{10}$$

such that
(1) $r_d(\cdot, \cdot)$ is jointly measurable on $Gr\Phi(\cdot)$; and
(2) $r_d(\omega, \cdot)$ is continuous in proposal profiles, G_D, on $\Phi(\omega)$ for all $\omega \in \Omega$.

Thus, if the current state is $\omega = (G, S)$ (i.e., if the status quo network is G and it is coalition S's turn to move) and if players propose networks $G_D \in \Phi(\omega)$, then player $d's$ payoff is given by

$$r_d(\omega, G_D) := r_d(\omega, (G_d, G_{-d})).$$

2.7 The Law of Motion

2.7.1 Definition and Assumptions

Given the current state, $\omega \in \Omega$, if the network proposal profile is given by $G_D \in \Phi(\omega)$, then *nature chooses* the next state (i.e., the next network-coalition pair) according to the probability measure,

$$q(\cdot|\omega, G_D) \in \mathcal{P}(\Omega). \tag{11}$$

The function,

$$(\omega, G_D) \longrightarrow q(\cdot|\omega, G_D),$$

relating current states and proposal profiles to the probability measures governing the generation of states is called the *law of motion*, a mapping defined on the graph of $\Phi(\cdot)$ with values in the space of probability measures on the state space $(\Omega, B(\Omega))$. We have the following list of assumptions concerning the law of motion:

[A-4] (*the law of motion*)
(1) For each $E \in B(\Omega)$, the function $q(E|\cdot, \cdot)$ is jointly measurable on $Gr\Phi(\cdot)$, and for each $(\omega, G_D) \in Gr\Phi(\cdot)$ the probability measure $q(\cdot|\omega, G_D)$ is absolutely continuous with respect the probability measure $\mu = \nu \times \eta$ defined on $(\Omega, B(\Omega))$ (i.e., $q(\cdot|\omega, G_D) \ll \mu$ for all $(\omega, G_D) \in Gr\Phi(\cdot)$).
(2) The collection of probability densities

$$H_\mu := \{z(\cdot|\omega, G_D) : (\omega, G_D) \in Gr\Phi(\cdot)\} \tag{12}$$

of $q(\cdot|\omega, G_D)$ with respect μ is such that a.e.$[\mu]$ in ω' and for all states ω

$$G_D \longrightarrow z(\omega'|\omega, G_D) \text{ is } h\text{-continuous in } G_D \text{ on } \Phi(\omega).$$

(3) The law of motion,

$$(\underbrace{(G, S)}_{\omega}, G_D) \longrightarrow q(\cdot|\underbrace{(G, S)}_{\omega}, G_D)$$

is such that for any sequence of state-action profile pairs, $\{(\omega^n, G_D^n)\}_n$, converging to (ω^*, G_D^*) under the sum metric $\rho_{\Omega \times \mathbb{G}^m} := h + \rho_F + h_{\mathbb{G}^m}$ on the product of the state space $\Omega = (\mathbb{G} \times \mathcal{C})$ and the action profile space $\mathbb{G}^m$,

$$q(F|\omega^n, G_D^n) \longrightarrow q(F|\omega^*, G_D^*) \tag{13}$$

for all nonempty ρ_Ω-closed subsets F of Ω.

Already by assumption [A-4](2), we have for fixed ω and for any $F \in P_f(\Omega)$ and $G_D^n \underset{h_{\mathbb{G}^m}}{\longrightarrow} G_D^*$, $q(F|\omega, G_D^n) \longrightarrow q(F|\omega, G_D^*)$. Thus, we have strengthened closed setwise convergence of $q(\cdot|\omega, \cdot)$ on $\mathbb{G}^m$ for each ω to closed setwise convergence of $q(\cdot|\cdot, \cdot)$ on $\Omega \times \mathbb{G}^m$.

In all that follows we will require that assumptions [A-1] and [A-3] hold. For existence, we will require that assumptions [A-2](1) and (2) and assumptions [A-4](1) and (2) hold. For stability we will require that assumptions [A-2](1) and (3) and assumptions [A-4](1) and (3) hold. Call our set of assumptions for existence, assumptions [A], and call our set of assumptions for stability, assumptions [A]*.

2.7.2 Observations Concerning Stochastic Continuity

The continuity of the function $z(\omega'|\omega, \cdot)$ in G_D on $\Phi(\omega)$, *a.e.*$[\mu]$ in ω', implies via Scheffee's Theorem (see Billingsley, 1986, Theorem 16.11) that

$$\left.\begin{array}{l} \sup_{E \in B(\Omega)} |q(E|\omega, G_D^n) - q(E|\omega, G_D^*)| \\ \\ \leq \int_\Omega |z(\omega'|\omega, G_D^n) - z(\omega'|\omega, G_D^*)| \, d\mu(\omega') \longrightarrow 0. \end{array}\right\} \tag{14}$$

for any sequence of network proposal profiles $\{G_D^n\}_n$ in $\Phi(\omega)$ converging to $G_D^* \in \Phi(\omega)$. Sometimes this is written, $a_D^n \underset{A}{\longrightarrow} a_D^*$ implies that

$$\|q(\cdot|\omega, G_D^n) - q(\cdot|\omega, G_D^*)\|_\infty \longrightarrow 0.$$

Our stochastic continuity assumptions, [A-4](2), is stronger than the usual narrow (or weak continuity) assumption. Under weak continuity, we would have for fixed ω and any sequence $\{(a_D^n)\}_n$ in $\Phi(\omega)$ with

$$a_D^n \stackrel{n}{\longrightarrow} a_D^* \in \Phi(\omega),$$

and any closed $F \in B(\Omega)$,

$$\limsup_n q(F|\omega, a_D^n) \leq q(F|\omega, a_D^*)$$
or equivalently,
$$\int_\Omega c(\omega')q(d\omega'|\omega, a_D^n) \stackrel{n}{\longrightarrow} \int_\Omega c(\omega')q(d\omega'|\omega, a_D^*),$$

for any bounded, continuous function $c(\cdot)$. With our stochastic continuity assumption (on densities), we have strengthened weak continuity so that for any such sequence,

$$\lim_n q(F|\omega, a_D^n) = q(F|\omega, a_D^*)$$

or equivalently (by Delbaen's Lemma (1974)),

$$\int_\Omega v(\omega')q(d\omega'|\omega, a_D^n) \stackrel{n}{\longrightarrow} \int_\Omega v(\omega')q(d\omega'|\omega, a_D^*),$$

for any bounded, *measurable* function $v(\cdot)$.

2.8 Strategies

2.8.1 Stationary Markov Strategies

A *Markov strategy* for player d is a measurable function, $\omega \longrightarrow \sigma_d(\cdot|\omega)$, which specifies in each state ω the probability measure, $\sigma_d(\cdot|\omega)$, governing player $d's$ choice of a network proposal G from feasible set $\Phi_d(\omega)$. Under Markov strategy $\sigma_d(\cdot|\cdot)$, in each state ω player $d's$ probability measure $\sigma_d(\cdot|\omega) \in \mathcal{P}(\mathbb{G})$ concentrates all of its probability mass on the set $\Phi_d(\omega)$ of feasible network proposals available to player

d in state ω. Denote this set of probability measures by $\mathcal{P}(\Phi_d(\omega))$. Thus, the function $\omega \longrightarrow \sigma_d(\cdot|\omega)$ is a Markov strategy if and only if the function $\sigma_d(\cdot|\cdot)$ is measurable and $\sigma_d(\cdot|\omega) \in \mathcal{P}(\Phi_d(\omega))$ for all ω.[14] Under Markov behavioral strategy $\sigma_d(\cdot|\cdot)$ in state ω, the probability with which player d proposes a feasible network $G \in \Phi_d(\omega)$ contained in measurable subset of networks $\mathbb{E} \in B(\mathbb{G})$ is given by $\sigma_d(\mathbb{E}|\omega)$. Note that if $\mathbb{E} \cap \Phi_d(\omega) = \emptyset$, then $\sigma_d(\mathbb{E}|\omega) = 0$.

We will denote by

$$R_d := \Sigma(\mathcal{P}(\Phi_d(\cdot))), \tag{15}$$

the set of all measurable selections from the mapping $\mathcal{P}(\Phi_d(\cdot))$, and therefore, the set of all Markov behavioral strategies. By Theorem 3 in Himmelberg and Van Vleck (1975), each player's feasible probability measure correspondence, $\mathcal{P}(\Phi_d(\cdot))$, is measurable (upper semicontinuous) if and only if the feasible action correspondence, $\Phi_d(\cdot)$, is measurable (upper semicontinuous). The measurability of the feasible probability correspondences, $\mathcal{P}(\Phi_d(\cdot))$, implies via the Kuratowski and Ryll-Nardzewski Theorem (see 18.13 in Aliprantis and Border, 2006), that the set of Markov strategies R_d is nonempty.

We will denote by

$$R_D := \prod_d R_d := \prod_d \Sigma(\mathcal{P}(\Phi_d(\cdot))), \tag{16}$$

the set of all profiles (or m-tuples) of Markov strategies.

Definitions 3 (*Stationary Markov Strategies*)
A stationary Markov strategy for player $d \in D$ is a constant sequence of measurable functions $(\sigma_d(\cdot), \sigma_d(\cdot), \ldots) \in R_d^\infty$, where the function, $\sigma_d(\cdot) \in R_d$, is a Markov strategy.
A stationary Markov strategy profile for players is a constant sequence of profiles $(\sigma_D(\cdot), \sigma_D(\cdot), \ldots) \in R_D^\infty$, where the function, $\sigma_D(\cdot) \in R_D$, is an m-tuple of Markov strategies.

2.8.2 Correlated Stationary Markov Strategies

A *correlated Markov strategy* consists of $m+1$ measurable functions

$$\lambda^i(\cdot) : \Omega \longrightarrow [0,1]$$

such that $\sum_{i=0}^m \lambda^i(\omega) = 1$ for all ω and $m+1$ Markov strategy profiles,

$$\sigma_D^i(\cdot) = (\sigma_d^i(\cdot))_{d \in D} \in R_D.$$

A correlated Markov strategy is given by $(\lambda^i(\cdot), \sigma_D^i(\cdot))_{i=0}^m$, and we will denote such a strategy by

$$\sigma_D^\lambda(\cdot) = \sum_{i=0}^m \lambda^i(\cdot)\sigma_D^i(\cdot), \tag{17}$$

where for each state ω, $\sigma_D^i(\omega)$ is the product probability measure on $\Phi(\omega)$ given by

$$\sigma_D^i(\omega) := \sigma_1^i(\cdot|\omega) \times \cdots \times \sigma_m^i(\cdot|\omega). \tag{18}$$

[14]Sometimes we will write $\sigma_d(\cdot)$ rather than $\sigma_d(\cdot|\cdot)$. We say that $\sigma_d(\cdot)$ is (lower or weakly) measurable if for all open subsets $E \in B(\mathcal{P}(\mathbb{G}))$,

$$\sigma_d^{-1}(E) := \{\omega \in \Omega | \sigma_d(\omega) \in E\} \in B(\Omega),$$

where $B(\mathcal{P}(\mathbb{G}))$ is the Borel σ-field in the space of probability measures $\mathcal{P}(\mathbb{G})$ generated by the compact and metrizable narrow topology (i.e., the topology of weak convergence of measures). Because the space of probability measures $\mathcal{P}(\mathbb{G})$ is a compact metric space, lower measurability is equivalent to

$$\sigma_d^{-1}(F) := \{\omega \in \Omega | \sigma_d(\omega) \in F\} \in B(\Omega),$$

for all closed subsets $F \in B(\mathcal{P}(\mathbb{G}))$.

Observe that for $t \in [0,1]$, and $\sigma_d^1(\cdot)$ and $\sigma_d^2(\cdot)$ in R_d and $\sigma_{-d}(\cdot)$ in $R_{D\setminus\{d\}}$, we have for all ω

$$t(\sigma_d^1(\omega), \sigma_{-d}(\omega)) + (1-t)(\sigma_d^2(\omega), \sigma_{-d}(\omega))$$

$$= (t\sigma_d^1(\omega) + (1-t)\sigma_d^2(\omega), \sigma_{-d}(\omega))$$

$$= t\sigma_d^1(\omega) + (1-t)\sigma_d^2(\omega) \times \sigma_{-d}(\omega).$$

Definitions 4 (*Correlated Stationary Markov Strategies***)**
A correlated stationary Markov strategy is a constant sequence of measurable functions $(\sigma_D^\lambda(\cdot), \sigma_D^\lambda(\cdot), \ldots)$*, where each function,* $\sigma_D^\lambda(\cdot)$*, is given by*

$$\sigma_D^\lambda(\cdot) = \sum_{i=0}^{m} \lambda^i(\cdot)\sigma_D^i(\cdot),$$

where

$$\sigma_D^i(\cdot) := \sigma_1^i(\cdot) \times \cdots \times \sigma_m^i(\cdot),$$

and $\sigma_d^i(\cdot) \in R_d$ *for each player* d*.*

2.8.3 History-Dependent Strategies

A *history-dependent strategy* ξ_d^n for player $d \in D$ in period n is a history-dependent measurable function defined on the state space Ω taking values in the set of probability measures defined on networks, $\mathcal{P}(\mathbb{G})$. Under history-dependent strategy ξ_d^n in period n given the history of states and proposal m-tuples (i.e., the $(n-1)$-sequence of state and action m-tuple pairs)

$$H^{n-1} := (\omega^1, G_D^1, \omega^2, G_D^2, \ldots, \omega^{n-1}, G_D^{n-1}),$$

and given the current (period n) state $\omega^n \in \Omega$, the probabilities with which player d will propose various feasible networks is given by the probability measure

$$\xi_d^n(H^{n-1}, \omega^n) \in \mathcal{P}(\Phi_d(\omega^n)) \subseteq \mathcal{P}(\mathbb{G}). \tag{19}$$

Let $\mathcal{H}^{n-1}$ denote set of all $(n-1)$-histories and let

$$L_d^n := \Sigma_{\mathcal{H}^{n-1}}(\mathcal{P}(\Phi_d(\cdot))) \tag{20}$$

denote the set of all measurable functions, $(H^{n-1}, \omega^n) \longrightarrow \xi_d^n(H^{n-1}, \omega^n) \in \mathcal{P}(\mathbb{G})$ such that $\xi_d^n(H^{n-1}, \omega^n) \in \mathcal{P}(\Phi_d(\omega^n))$ for all $\omega^n \in \Omega$. We will denote by

$$L_D^n := \prod_d L_d^n$$

the set of period n, history-dependent strategy profiles.

Definition 5 (*History-Dependent Strategies***)**
A history-dependent strategy for player $d \in D$ *is a sequence of measurable functions*

$$\xi_d(\cdot) = (\xi_d^1(\cdot), \xi_d^2(\cdot), \ldots) \in L_d^\infty := \prod_{n=1}^{\infty} L_d^n,$$

where for each n *the function,* $\xi_d^n(\cdot) \in L_d^n$*, is a history-dependent strategy.*
A history-dependent strategy profile for players is a sequence of measurable functions

$$\xi_D(\cdot) = (\xi_D^1(\cdot), \xi_D^2(\cdot), \ldots) \in L_D^\infty := \prod_{n=1}^{\infty} L_D^n,$$

where for each n *the function,* $\xi_D^n(\cdot) \in L_D^n$*, is a history-dependent strategy profile for period* n*.*

2.9 Expected Payoffs Under Correlated Markov Strategies

For any profile (or m-tuple) of feasible probability measures $\sigma_D \in \prod_d \mathcal{P}(\Phi_d(\omega))$, player $d's$ immediate expected payoff in state ω is

$$r_d(\omega, \sigma_D) = \int_{\mathbb{G}^m} r_d(\omega, G_D)\sigma_D(dG_D), \tag{21}$$

where $G_D := (G_d)_{d\in D} \in \Phi(\omega) := \Pi_{d\in D}\Phi_d(\omega)$, and where

$$\sigma_D(dG_D) := \sigma_1(dG_1) \times \cdots \times \sigma_m(dG_m).$$

Under correlated Markov strategy $\sigma_D^\lambda(\cdot)$, the function $r_d(\cdot, \sigma_D^\lambda(\cdot))$ is $B(\Omega)$-measurable and player $d's$ immediate expected payoff in state $\omega \in \Omega$ is

$$\left.\begin{aligned} & r_d(\omega, \sigma_D^\lambda(\omega)) = \textstyle\int_{\mathbb{G}^m} r_d(\omega, G_D)\sigma_D^\lambda(dG_D|\omega) \\ & = \textstyle\int_{\mathbb{G}^m} r_d(\omega, G_D)\sum_{i=0}^m \lambda^i(\omega)\sigma_D^i(dG_D|\omega) \\ & = \textstyle\int_{\mathbb{G}^m} r_d(\omega, G_D)\sum_{i=0}^m \lambda^i(\omega)(\sigma_1^i(dG_1|\omega) \times \cdots \times \sigma_m^i(dG_m|\omega)) \\ & = \textstyle\sum_{i=0}^m \lambda^i(\omega)\left[\int_{\mathbb{G}^m} r_d(\omega, G_D)(\sigma_1^i(dG_1|\omega) \times \cdots \times \sigma_m^i(dG_m|\omega))\right]. \end{aligned}\right\} \tag{22}$$

If in state ω, stationary Markov strategy profile $\sigma_D^i(\cdot|\omega)$ is chosen by the public randomization device $\lambda^i(\omega)$, $i = 0, 1, 2, \ldots, m$, and if network proposal profile G_D is chosen by the product measure $\sigma_D^i(dG_D|\omega)$ induced by probability measure profile $\sigma_D^i(\cdot|\omega)$, then given the law of motion $q(\cdot|\cdot,\cdot)$ nature chooses the next state (i.e., the next network-coalition pair) according to the probability measure $q(\cdot|\omega, G_D)$.

Let

$$r_d^n(\sigma_D^\lambda)(\omega) := \begin{cases} r_d(\omega, \sigma_D^\lambda(\omega)) & \text{for } n = 1 \\ \int_\Omega r_d(\omega', \sigma_D^\lambda(\omega'))q^{n-1}(\omega'|\omega, \sigma_D^\lambda(\omega)) & \text{for } n \geq 2, \end{cases} \tag{23}$$

denote the n^{th} period *expected* payoff to player d under Markov correlated strategy $\sigma_D^\lambda(\cdot)$ starting at state ω given law of motion $q(\cdot|\cdot,\cdot)$. Here, for $n \geq 2$, $q^n(\cdot|\omega, \sigma_D^\lambda(\omega))$ is defined recursively by

$$\left.\begin{aligned} & q^n(E|\omega, \sigma_D^\lambda(\omega)) \\ & = \textstyle\int_\Omega q^{n-1}(E|\omega', \sigma_D^\lambda(\omega'))q(\omega'|\omega, \sigma_D^\lambda(\omega)) \\ & = \textstyle\int_\Omega q(E|\omega', \sigma_D^\lambda(\omega'))q^{n-1}(\omega'|\omega, \sigma_D^\lambda(\omega)), \end{aligned}\right\} \tag{24}$$

where

$$q(E|\omega, \sigma_D^\lambda(\omega)) = \int_{\mathbb{G}^m} q(E|\omega, G_D)\sigma_D^\lambda(dG_D|\omega).$$

The discounted expected payoff to player d, with discount rate $\beta_d \in [0, 1)$, over an infinite time horizon under correlated Markov strategy $\sigma_D^\lambda(\cdot)$ starting at state ω is then given by

$$E_d r_d(\sigma_D^\lambda)(\omega) := \sum_{n=1}^\infty \beta_d^{n-1} E_d r_d^n(\sigma_D^\lambda)(\omega). \tag{25}$$

3 Correlated Stationary Markov Equilibrium

A discounted stochastic game over stationary Markov strategies is given by

$$\mathcal{G} := (\Omega, E_d r_d(\cdot)(\cdot), R_d)_{d \in D}.$$

Definition 6 (*Correlated Stationary Markov Equilibria*)
A correlated *Markov strategy*

$$\sigma_D^{\lambda *}(\cdot) = \sum_{i=0}^{m} \lambda^{i*}(\cdot)\sigma_D^{i*}(\cdot)$$

is a correlated stationary equilibrium of the discounted stochastic game $\mathcal{G}$, if no player d can unilaterally benefit by deviating from any of the Markov strategies $\sigma_d^{i*}(\cdot) \in R_d$ assigned to him (under correlated strategy $\sigma_D^{\lambda *}(\cdot)$) to any other Markov strategy or any history dependent strategy.

Thus, a correlated stationary Markov strategy $\sigma_D^{\lambda *}(\cdot)$ is a dynamic correlated equilibrium of the discounted stochastic game $\mathcal{G}$ if no player has an incentive to unilaterally change his part, $\sigma_d^{i*}(\cdot)$, of the correlated Markov strategy $\sigma_D^{\lambda *}(\cdot)$ to any other strategy.

Theorem 1 (*The Existence of Correlated Stationary Markov Equilibrium*)
Any discounted stochastic game of network and coalition formation,

$$\mathcal{G} := (\Omega, E_d r_d(\cdot)(\cdot), R_d)_{d \in D},$$

satisfying assumptions [A] has a correlated stationary Markov equilibrium,

$$\sigma_D^{\lambda *}(\cdot) = \sum_{i=0}^{m} \lambda^{i*}(\cdot)\sigma_D^{i*}(\cdot), \tag{26}$$

where each Markov strategy profile $\sigma_D^{i*}(\cdot)$ is such that for each state ω

$$\sigma_D^{i*}(\cdot|\omega) \in \mathcal{N}_{v^*}(\omega),$$

where $\mathcal{N}_{v^*}(\omega) \subset \prod_d \mathcal{P}(\Phi_d(\omega))$ is the set of Nash equilibria of the one-shot game $\mathcal{G}_{v^*}(\omega)$ given by

$$\mathcal{G}_{v^*}(\omega) := (\mathcal{P}(\Phi_d(\omega)), u_d(\omega, \cdot)(v_d^*))_{d \in D}, \tag{27}$$

with player payoff functions given by

$$\sigma_D \longrightarrow u_d(\omega, \sigma_D)(v_d^*) := (1-\beta_d) r_d(\omega, \sigma_D) + \beta_d \int_\Omega v_d^*(\omega') q(\omega'|\omega, \sigma_D). \tag{28}$$

Our approach to proving existence follows the broad outlines of the approach introduced by Nowak and Raghavan in their seminal 1992 paper. For the convenience of the reader we include a proof (see the appendix). The basic objective of the proof is to show that there exists a correlated stationary strategy

$$\sigma_D^{\lambda *}(\cdot) = \sum_{i=0}^{m} \lambda^{i*}(\cdot)\sigma_D^{i*}(\cdot),$$

with corresponding m-tuple of value functions, $w_d^*(\cdot) : \Omega \longrightarrow [-M, M]$ such that for each player $d \in D$ and for all states $\omega \in \Omega$,

$$\left.\begin{array}{c} (1)\ w_d^*(\omega) = u_d(\omega, \sigma_D^{\lambda *}(\omega))(w_d^*), \text{ where } w_d^* := \frac{v_d^*}{1-\beta_d}, \\ \\ \text{and} \\ \text{for } i = 0, 1, \ldots, m \\ \\ (2)\ u_d(\omega, (\sigma_d^{i*}(\omega), \sigma_{-d}^{i*}(\omega))(w_d^*) = \max_{\sigma_d \in \mathcal{P}(\Phi_d(\omega))} u_d(\omega, (\sigma_d, \sigma_{-d}^{i*}(\omega))(w_d^*), \end{array}\right\} \tag{29}$$

4 Equilibrium Markov Processes of Network and Coalition Formation

4.1 Equilibrium Transitions

Under the equilibrium correlated stationary Markov strategy, $\sigma_D^{\lambda*}(\cdot)$, the Markov process of network and coalition formation,

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty},$$

is governed by the equilibrium Markov transition,

$$p^*(E|\omega) = q(E|\omega, \sigma_D^{\lambda*}(\omega))$$

$$= \int_{\mathbb{G}^m} q(E|\omega, G'_D)\sigma_D^{\lambda*}(G'_D|\omega).$$

Thus,

$$\Pr\left\{W_{n+1}^* \in E|W_n^* = \omega\right\} = p^*(E|\omega)$$

and

$$\Pr\left\{W_n^* \in E|W_0^* = \omega\right\} = p^{*n}(E|\omega) = q^n(E|\omega, \sigma_D^{\lambda*}(\omega)),$$

where the n-step transition $p^{*n}(\cdot|\cdot)$ is defined recursively as follows: for all $\omega \in \Omega$ and $E \in B(\Omega)$,

$$p^{*n}(E|\omega) = \int_{\Omega} p^*(E|\omega')p^{*n-1}(d\omega'|\omega) = \int_{\Omega} p^{*n-1}(E|\omega')p^*(d\omega'|\omega), \tag{30}$$

for $n = 1, 2, \ldots$, and $p^{*0}(\cdot|\omega) = \delta_\omega(\cdot)$ is the Dirac measure at ω.

4.2 Absorbing Sets and Invariant and Ergodic Probability Measures

A set $E \in B(\Omega)$ (of network and coalition pairs) is called a p^*-absorbing set if $p^*(E|\omega) = 1$ for all network-coalition pairs $\omega \in E$. Let $\mathcal{L}^* \subseteq B(\Omega)$ denote the collection of all p^*-absorbing sets. A p^*-absorbing set $E \in \mathcal{L}^*$ is said to be *indecomposable* if it does not contain the union of two disjoint absorbing sets. Note that the set of all absorbing sets is closed under countable unions and intersections.

A probability measure $\gamma(\cdot)$ on the state space of feasible network-coalition pairs $(\Omega, B(\Omega))$ is invariant for Markov transition $p^*(\cdot|\cdot)$ (i.e., is p^*-invariant) if

$$\gamma(E) = \int_{\Omega} p^*(E|\omega)d\gamma(\omega) \text{ for all } E \in B(\Omega). \tag{31}$$

Thus, if probability measure $\gamma(\cdot)$ is p^*-invariant, then for any set of network-coalition pairs $E \in B(\Omega)$, if the current status quo network-coalition pair $\omega^n = (G_n, S_n)$ is chosen according to probability measure $\gamma(\cdot)$ - so that the probability that ω^n lies in E is just $\gamma(E)$ - then the probability that next period's network-coalition pair $\omega^{n+1} = (G_{n+1}, S_{n+1})$ lies in E is also $\gamma(E) = \int_{\Omega} p^*(E|\omega)d\gamma(\omega)$. Denote by $\mathcal{I}^*$ the collection of all p^*-invariant measure.

A p^*-invariant measure $\gamma(\cdot)$ is said to be p^*-ergodic if $\gamma(E) = 0$ or $\gamma(E) = 1$ for all $E \in \mathcal{L}^*$. Denote by $\mathcal{E}^*$ the collection of all p^*-ergodic measures. Because the p^*-ergodic probability measures are the extreme points of the (possibly empty) convex set $\mathcal{I}^*$ of p^*-invariant measures (see Theorem 19.25 in Aliprantis and Border 2006), each measure $\gamma(\cdot)$ in $\mathcal{I}^*$ can be written as a convex combination of the measures in $\mathcal{E}^*$.

4.3 Visitations and Hitting Times

The number of visitations by the process $\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$ to the set of network-coalition pairs $E \in B(\Omega)$, is given by

$$\eta_E^* := \sum_{n=1}^{\infty} I_E(W_n^*), \tag{32}$$

where $I_E(W_n^*) = 1$ if $W_n^* \in E$ and zero otherwise. Thus, the expected number of visitations to E starting from network-coalition pair $\omega = (G, S)$ is given by

$$G^*(\omega, E) := E_\omega^*[\eta_E^*] = \sum_{n=1}^{\infty} p^{*n}(E|\omega). \tag{33}$$

The probability that the network-coalition formation process $\{W_n^*\}_n$ visits E infinitely often (denoted by i.o.) is given by

$$\left.\begin{aligned} Q^*(\omega, E) &:= \Pr\{W_n^* \in E \text{ i.o.}|W_0^* = \omega\} = \Pr\{\eta_E^* = \infty|W_0^* = \omega\} \\ &= \Pr\{\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} (W_n^* \in E|W_0^* = \omega)\} \text{ for all } \omega \in \Omega. \end{aligned}\right\} \tag{34}$$

The *hitting time* for set E is given by

$$\tau_E^* := \inf\{n \geq 1 : W_n^* \in E\}. \tag{35}$$

Following Tweedie (2001),

$$L^*(\omega, E) := \Pr\{\tau_E^* < \infty|W_0^* = \omega\} = \Pr\{\cup_{n=1}^{\infty} (W_n^* \in E|W_0^* = \omega)\} \tag{36}$$

is the probability that the process $\{W_n^*\}_n$ hits (or reaches) in finite time the set of network-coalition pairs E starting from network-coalition pair $\omega \in \Omega$ given transition $p^*(\cdot|\cdot)$. By Proposition 9.1.1 in Meyn and Tweedie (2009), if for any $E \in B(\Omega)$, $L^*(\omega, E) = 1$ for all $\omega \in E$, then

$$L^*(\omega, E) = Q^*(\omega, E) \text{ for all } \omega \in \Omega. \tag{37}$$

4.4 Recurrence, Transience, and Irreducibility

The set of network-coalition pairs E is *recurrent* if

$$G^*(\omega, E) := E_\omega^*[\eta_E^*] = \sum_{n=1}^{\infty} p^{*n}(E|\omega) = +\infty.$$

By Proposition 8.1.3 in Meyn and Tweedie (2009), for any state $\omega \in \Omega$,

$$G^*(\omega, \{\omega\}) = +\infty \text{ if and only if } L^*(\omega, \{\omega\}) = 1.$$

A set of network-coalition pairs $T \in B(\Omega)$ is *transient* if (i) T is the disjoint union of countably many *uniformly transient sets* U_j, that is, sets $U_j \in B(\Omega)$ such that $T = \cup_j U_j$ and if (ii) for each set there is a finite constant M_j, such that for all network-coalition pairs $\omega \in U_j$,

$$E_\omega^*[\eta_{U_j}^*] = \sum_{n=1}^{\infty} p^{*n}(U_j|\omega) < M_j. \tag{38}$$

The set of network-coalition pairs E is said to be p^**-inessential* if

$$Q^*(\omega, E) = 0 \text{ for all } \omega \in \Omega. \tag{39}$$

Thus, a set of states E is inessential if the probability that the network-coalition formation process visits the set E infinitely often is zero stating from any state. If a set of states is inessential, then if the process visits the state at all, it leaves the state for good after finitely many moves. The union of countable many inessential states is called an *improperly* p^**-essential set*. Any other set is called *properly* p^**-essential*.

Finally, the network-coalition formation process $\{W_n^*\}_n$ governed by $p^*(\cdot|\cdot)$ is said to be *ψ-irreducible* if for some probability measure $\psi(\cdot)$ on $B(\Omega)$,[15]

$$\psi(E) > 0 \text{ implies } L^*(\omega, E) > 0 \text{ for all } \omega \in \Omega.$$

Thus if the process $\{W_n^*\}_n$ governed by $p^*(\cdot|\cdot)$ is ψ-irreducible, then it hits all the "important" sets of network-coalition pairs (i.e., the sets $E \in B(\Omega)$ such that $\psi(E) > 0$) with positive probability starting from any network-coalition pair in the state space $\Omega = \mathbb{G} \times \mathcal{C}$.

The network-coalition formation process $\{W_n^*\}_n$ governed by $p^*(\cdot|\cdot)$ is said to be *ψ-recurrent* if,

$$\psi(E) > 0 \text{ implies } Q^*(\omega, E) = 1 \text{ for all } \omega \in \Omega.$$

5 Stability of Equilibrium Markov Processes of Network and Coalition Formation

In addition to modeling the emergence of endogenous network dynamics from the co-evolution of strategic behavior and network structure, one of our main objectives is to study the dynamic stability properties of the resulting equilibrium process of network and coalition formation. A key component of our analysis is the notion of a dynamic basin of attraction. Intuitively, a set of network-coalition pairs H is a basin of attraction if the network and coalition formation process $\{W_n^*\}_n$ reaches H in finite time with probability 1 and once there, stays there. The question we wish to answer is this: does the process of network and coalition formation $\{W_n^*\}_n$ that emerges from the equilibrium interplay of strategic behavior, network structure, and the trembles of nature generate basins of attraction. We begin by considering the classical notion of a Maximal Harris set of network and coalition pairs.

5.1 Dynamic Basins of Attraction: Maximal Harris Sets

A set of network-coalition pairs $H \in B(\Omega)$ is called a *maximal Harris set* if there exists some probability measure $\varphi(\cdot)$ on $B(\Omega)$ such that $\varphi(H) > 0$,

$$\varphi(A) > 0 \text{ implies } L^*(\omega, A) = 1 \text{ for all } \omega \in H,$$
$$\text{and}$$
$$L^*(\omega, H) = 1 \text{ implies that } \omega \in H.$$

Note that a maximal Harris set is a *maximal absorbing set* and is indecomposable. Moreover, if H and H' are distinct Maximal Harris sets, then they are disjoint. Finally, note that if the network-coalition formation process reaches a particular Harris set then it remains there for all future periods. By Proposition 9.1.1 in Meyn and Tweedie (2009), because we have $L^*(\omega, H) = 1$ for all $\omega \in H$,

$$L^*(\omega, E) = Q^*(\omega, E) = 1 \text{ for all } \omega \in H.$$

Thus, if the set of network-coalition pairs H is maximal Harris, then process $\{W_n^*\}_n$ restricted to H is φ-irreducible and Harris recurrent - where Harris recurrence means that $Q^*(\omega, E) = 1$ for all $\omega \in H$.

The fact that a maximal Harris set is a maximal absorbing set makes it a good candidate for a basin of attraction. But in order to fully qualify as a basin of attraction we must show that - or identify conditions under which - the process reaches such a set in finite time with probability 1.

[15] Here, the probability measure $\psi(\cdot)$ is a maximal irreducibility measure (see Section 4.2.2 in Meyn and Tweedie (second edition, 2009).

5.2 The Fundamental Conditions for Stability: Drift and Global Uniform Countable Additivity

Given the equilibrium Markov transition $p^*(\cdot|\cdot)$ what can be said concerning stability? What conditions guarantee that the equilibrium process of network and coalition formation reaches a Harris set in finite time with probability 1. It turns out that the Tweedie Conditions (2001) do just that:

The Tweedie Conditions *(2001)*:
There exists a measurable set of network-coalition pairs $C \subseteq \Omega$, a nonnegative measurable function

$$V(\cdot) : \Omega \longrightarrow [0, \infty],$$

and a finite real number b such that
(1) (the drift condition) for all $\omega \in \Omega$

$$\int_{\Omega} V(\omega')dp^*(\omega'|\omega) \leq V(\omega) - 1 + bI_C(\omega),$$

and
(2) (uniform countable additivity) for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

$$\lim_{n \longrightarrow \infty} \sup_{\omega \in C} p^*(B_n|\omega) = 0.$$

We say that the Markov transition $p^*(\cdot|\cdot)$ satisfies *global uniform countable additivity* if for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

$$\lim_{n \longrightarrow \infty} \sup_{\omega \in \Omega} p^*(B_n|\omega) = 0, \tag{40}$$

and we will say that the Tweedie conditions are satisfied globally if both conditions (1) and (2) hold with $C = \Omega$.

Using results due to Meyn and Tweedie (2009), Tweedie (2001), and Costa and Dufour (2005), we will show below that if the equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the equilibrium process of network and coalition formation is *globally uniformly countably additive*, then the equilibrium process possesses some striking stability properties - analogous to those demonstrated in Page and Wooders (2009a) for static abstract games of network formation.

We have our main result on global uniform countable additivity.

Theorem 2 *(Global Uniform Countable Additivity)*
Suppose assumptions [A] hold. Then $p^*(\cdot|\cdot)$ is globally uniformly countably additive.*

Proof. : Let

$$\Delta_{\Omega \times \mathbb{G}^m}(\Omega) := \{q(\cdot|\omega, G_D) : (\omega, G_D) \in \Omega \times \mathbb{G}^m\}.$$

We will show that $\Delta_{\Omega \times \mathbb{G}^m}(\Omega)$ is sequentially compact in the $\sigma(rca(\Omega), \mathcal{B}^{\infty}_{\Omega})$ topology.[16]

By the compactness of $\Omega \times \mathbb{G}^m$, for any sequence $\{q(\cdot|\omega^n, G^n_D)\}_n \subset \Delta_{\Omega \times \mathbb{G}^m}(\Omega)$, there is a subsequence, $\{q(\cdot|\omega^{n_k}, G^{n_k}_D)\}_k$ such that $(\omega^{n_k}, G^{n_k}_D) \underset{\rho_{\Omega \times \mathbb{G}^m}}{\longrightarrow} (\omega^*, G^*_D)$ implying by assumption [A-2](6) that for all $E \in \mathcal{B}(\Omega)$

$$q(E|\omega^{n_k}, G^{n_k}_D) \longrightarrow q(E|\omega^*, G^*_D) \in \Delta_{\Omega \times \mathbb{G}^m}(\Omega).$$

Thus, for each $f \in \mathcal{B}^{\infty}_{\Omega}$, we have

$$\int_{\Omega} f(\omega')q(\omega'|\omega^{n_k}, G^{n_k}_D) \longrightarrow \int_{\Omega} f(\omega')q(\omega'|\omega^*, G^*_D).$$

[16] $rca(\Omega)$ is the Banach space of finite signed Borel measures on $(\Omega, B(\Omega))$ and $\mathcal{B}^{\infty}_{\Omega}$ is the Banach space of μ-equivalence classes of real-valued, bounded measurable functions on $(\Omega, B(\Omega))$.

Thus, $\Delta_{\Omega\times\mathbb{G}^m}(\Omega)$ is sequentially compact in the $\sigma(rca(\Omega),\mathcal{B}_{\Omega}^{\infty})$ topology. By Corollary 2.2 in Lassere (1998), $p^*(\cdot|\cdot)$ is globally uniformly countably additive. In particular, letting $\{B_k\}_k \subset \mathcal{B}(\Omega)$ be any decreasing sequence(i.e., $B_k \downarrow \emptyset$) and $\{f_k(\cdot)\}_k$ be the sequence of functions in $\mathcal{B}_{\Omega}^{\infty}$ where for each k,

$$f_k(\omega) := I_{B_k}(\omega) \in \mathcal{B}_{\Omega}^{\infty},$$

we have by Corollary 2.2 in Lassere (1998) that the sequential compactness of $\Delta_{\Omega\times\mathbb{G}^m}(\Omega)$ implies that

$$\lim_{k\longrightarrow\infty} \sup_{(\omega,G_D)\in\Omega\times\mathbb{G}^m} \int_{\Omega} f_k(\omega')q(\omega'|\omega,G_D) = \lim_{k\longrightarrow\infty} \sup_{(\omega,G_D)\in\Omega\times\mathbb{G}^m} q(B_k|\omega,G_D) = 0.$$

Thus, because

$$\sup_{(\omega,G_D)\in\Omega\times\mathbb{G}^m} q(B_k|\omega,G_D) \geq \sup_{\omega\in\Omega} q(B_k|\omega,\sigma^*(\omega)) \geq 0,$$

we have

$$\lim_{k\longrightarrow\infty} \sup_{\omega\in\Omega} q(B_k|\omega,\sigma^*(\omega)) = \lim_{k\longrightarrow\infty} \sup_{\omega\in\Omega} p^*(B_k|\omega) = 0.$$

□

By Theorem 2, under assumptions [A]* the equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the process of network and coalition formation is globally uniformly countably additive. Moreover, letting $C = \Omega$, $V(\omega) = 1$ for all $\omega \in \Omega$, and $b = 2$, the drift condition is also satisfied. Thus, by strengthening the stochastic continuity properties of the law of motion $q(\cdot|\cdot,\cdot)$ mildly beyond what is required to guarantee the existence of an equilibrium Markov transition, $p^*(\cdot|\cdot)$, we are able to conclude in Theorem 2 that the Tweedie conditions are satisfied globally (i.e., with $C = \Omega$).

6 Basins of Attraction, Invariance, and Ergodicity

We now have our main result concerning stochastic basins of attraction and the stability of the equilibrium network-coalition formation process

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

governed by $p^*(\cdot|\cdot)$.

Theorem 3 (*Basins of Attraction: The Finite Decomposition of the State Space*)
Under assumptions [A] the equilibrium network-coalition formation process*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot,\sigma_D^{\lambda *}(\cdot))$ *generates a decomposition of the state space of network-coalition pairs* $\Omega = \mathbb{G}\times\mathcal{C}$ *into a finite number of disjoint basins of attraction and a disjoint transient set. In particular, this decomposition is of the form*

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T, \tag{41}$$

where each H_i *is a basin of attraction (i.e., maximal Harris) and* T *is transient, and has the property that for every network-coalition pair* $\omega \in \Omega$

$$L^*(\omega, \cup_i H_i) = 1. \tag{42}$$

By Theorem 3 the equilibrium network-coalition formation process $\{W_n^*\}_n$ is such that starting at *any* network-coalition pair not contained in a basin of attraction (i.e., a maximal Harris set), the process will reach in finite time with probability 1, one of *finitely many* basins of attraction H_i, and once there will stay there. An analogous conclusion is reached in Page and Wooders (2009a) for static, abstract games of network formation over finitely many networks. There it is shown that no matter what rules of network formation prevail, given any profile of player preferences, the feasible set of networks contains a finite, disjoint collection of sets, each set representing a *strategic* basin of attraction in the sense that if the game is repeated - each time starting at the status quo network reached in the previous play of the game - the process of network formation generated by repeating this static game will reach a strategic basin of attraction in finitely many moves and once there will stay there.

Because in our model the *Tweedie conditions hold globally*, it follows from Theorem 2 in Tweedie (2001) that the entire state space Ω admits a finite decomposition,

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T,$$

consisting of a finite number of indecomposable, Maximal Harris sets, H_i, and a transient set T. The key step in establishing this finite decomposition is to show that because the equilibrium Markov transition,

$$\omega \longrightarrow q(\cdot|\omega, \sigma^{\lambda *}(\omega)),$$

is globally, uniformly countably additive, the state space contains at most a finite number of disjoint absorbing sets (see Tweedie 2001, Lemma 2). Moreover, by Theorem 2 in Tweedie (2001), this decomposition is such that $L^*(\omega, \cup_{i=1}^{N} H_i) = 1$ for all $\omega \in \Omega$. Thus, governed by the equilibrium Markov transition, $q(\cdot|\cdot, \sigma^{\lambda *}(\cdot))$, the process of network and coalition formation is such that no matter where the process begins (no matter what network-coalition pair is the starting point), it reaches in finite time with probability 1 one of finitely many basins of attraction, H_i, and once there, stays there. Thus, the proof of our Theorem 3 follows from Theorem 2 in Tweedie (2001) and the fact that the equilibrium Markov transition, $q(\cdot|\cdot, \sigma^{\lambda *}(\cdot))$, is globally uniformly countably additive.

Our next result establishes that the equilibrium Markov transition possesses a finite number of ergodic measures, one for each basin of attraction.

Theorem 4 (*Invariance and Ergodicity of the Process of Network and Coalition Formation*) *Suppose assumptions [A]* hold. Let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the equilibrium network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{\lambda *}(\cdot))$, *and let*

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T,$$

be the corresponding finite decomposition into basins of attraction.
The following statements are true:
(1) Corresponding to each basin of attraction H_i, *there is a unique* p^**-invariant probability measure* $\gamma_i(\cdot)$ *with* $\gamma_i(H_i) = 1$. *Moreover, for each network-coalition pair* $\omega = (G, S)$,

$$p^{*(n)}(E|\omega) := \frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \sum_{i=1}^{N} L^*(\omega, H_i)\gamma_i(E \cap H_i), \text{ for all } E \in B(\Omega). \tag{43}$$

where $p^{*k}(E|\omega)$ *is defined recursively, see (30).*
(2) The set of all ergodic probability measures is given by

$$\mathcal{E}^* = \{\gamma_i(\cdot)\}_{i=1}^{N}.$$

Moreover, a probability measure $\gamma(\cdot)$ on $(\Omega, B(\Omega))$ is p^*-invariant, i.e. $\gamma(\cdot) \in \mathcal{I}^*$, if and only if $\gamma(\cdot)$ is given by

$$\gamma(E) = \sum_{i}^{N} \gamma(H_i)\gamma_i(E \cap H_i), \text{ for all } E \in B(\Omega). \tag{44}$$

(3) $\mathcal{E}^*$ is a singleton (i.e., $\mathcal{E}^* = \{\gamma(\cdot)\}$) if and only if the network-coalition formation process $\{W_n^*\}_n$ is ψ-irreducible, in which case for each network-coalition pair $\omega = (G, S)$ and for every set of network-coalition pairs $E \in B(\Omega)$

$$\frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \gamma(E).$$

Proof. (1) Under our assumptions [A]* (see the proof of Theorem 2 above), $p^*(\cdot|\cdot)$ satisfies the Tweedie conditions globally. As a result, the first statement in part (1) is an immediate consequence of Lemma 5 in Tweedie (2001). The second statement also follows from the fact that in our model the Tweedie conditions hold globally and Theorem 1 in Tweedie (2001) (also, see Chapter 13 in Meyn and Tweedie 2009).

(2) Again because the Tweedie Conditions are satisfied globally, the first statement in part (2) follows from Lemma 2 in Tweedie (2001), Theorem 2.18 part (1) in Costa and Dufour (2005), Theorem 3.8 in Costa and Dufour, and the proof of Proposition 5.3 in Costa and Dufour. The second statement in part (2), that $\gamma(\cdot) \in \mathcal{I}^*$ implies (44), follows from the proof of Proposition 5.3 in Costa and Dufour (2005). The fact that (44) implies $\gamma(\cdot) \in \mathcal{I}^*$ follows from observation (but also, see Theorem 19.25 in Aliprantis and Border 2006 and Theorem 2 in Villareal 2004).

(3) Finally, because the Tweedie Conditions are satisfied globally, necessary and sufficient conditions for $\mathcal{E}^*$ to be a singleton, given in terms of ψ-irreducibility follow from Theorem 3 in Tweedie (2001). The convergence result in part (3) follows from the convergence result in part (1) of the Theorem and the fact that if there is only one basin of attraction H (i.e., one maximal Harris set), then by Theorem 3, $L^*(\omega, H) = 1$ for all $\omega \in \Omega$. □

Note that the probability measures in $\mathcal{E}^*$ are *orthogonal*, that is, for all i and i' in $\{1, 2, \ldots, N\}$ with $i \neq i'$,

$$\gamma_i(\Omega \backslash H_i) = \gamma_{i'}(H_i) = 0.$$

6.1 Ergodic Properties of Strategic Values

For each starting network-coalition pair $\omega = (G, S) \in \Omega$, $w_d^*(\omega)$ $(:= \frac{v_d^*(\omega)}{1-\beta_d})$ is the strategic value to player d of following his parts of the correlated stationary Markov strategy $\{\sigma_d^{i*}(\cdot)\}_{i=0}^m$, given that all other players follow their parts of the strategy $\{\sigma_{-d}^{i*}(\cdot)\}_{i=0}^m$. Because each Markov strategy profile $\sigma_D^{i*}(\cdot)$ is Nash (for $i = 0, 1, \ldots m$), we know that this is the best that player d can do relative to all other strategies, even those that are history dependent. Strategies $\sigma_D^{i*}(\cdot)$ together with the trembles of nature determine the equilibrium Markov process of network and coalition formation via the transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{\lambda*}(\cdot))$. The questions we wish to address in this section concern the properties of players' strategic values across time and states given the equilibrium process of network and coalition formation.

We begin by considering time averages. Let

$$p^{*(n)}w_d^*(\omega) := \frac{1}{n}\sum_{k=1}^{n}\int_\Omega w_d^*(\omega')p^{*k}(d\omega'|\omega) = \int_\Omega w_d^*(\omega')p^{*(n)}(d\omega'|\omega),$$

where recall,

$$w_d^*(\omega) = E_d r_d(\sigma_D^{\lambda *})(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1} E_d r_d^n(\sigma_D^{\lambda *})(\omega)$$

$$= r_d(\omega, \sigma_D^{\lambda *}(\omega)) + \beta_d \int_\Omega w_d^*(\omega')dq(\omega'|\omega, \sigma_D^{\lambda *}(\omega))$$

and

$$p^{*(n)}(E|\omega) := \frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) = \frac{1}{n}\sum_{k=1}^{n} \int_\Omega p^*(E|\omega')p^{*k-1}(d\omega'|\omega).$$

Here, $p^{*k}(E|\omega)$ is the probability that process reaches the set of network-coalition pairs E starting at network-coalition pair $\omega = (G, S)$ in k periods or moves if players follow the Markov strategies assigned via the correlated equilibrium strategy, $\sigma_D^{\lambda *}(\omega)$.

The function $p^{*(n)}w_d^*(\cdot)$ specifies for each starting network-coalition pair, player d's n-period time average expected strategic value (i.e., the average value of following his parts of the correlated stationary Markov strategy $\sigma_D^{\lambda *}(\cdot)$ for n moves). We can think of $\lim_n p^{*(n)}w_d^*(\cdot)$ therefore as specifying for each starting network-coalition pair, player d's time average expected value.

By part (1) of Theorem 4 above, we have *for all* $\omega \in \Omega$ and $E \in B(\Omega)$

$$p^{*(n)}(E|\omega) = \frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \sum_{i=1}^{N} L^*(\omega, H_i)\gamma_i(E \cap H_i) = \gamma^\omega(E), \tag{45}$$

where $\gamma^\omega(\cdot) \in \mathcal{I}^*$ for all $\omega \in \Omega$ and $\mathcal{E}^* = \{\gamma_i(\cdot) : i = 1, 2, \ldots, N\}$. Because $p^{*(n)}(\cdot|\omega)$ converges setwise for all ω, by Delbaen's Lemma (1974) we have *for all* $\omega \in \Omega$

$$p^{*(n)}w_d^*(\omega) \longrightarrow \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w_d^*(\omega')d\gamma_i(\omega'). \tag{46}$$

Thus, we obtain one of the fundamental principles of equilibrium dynamics: the equality of time averages and state averages.

Theorem 5 *(The Equality of Time Average Values and State Average Values)*
Under assumptions [A] the equilibrium network-coalition formation process*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{\lambda *}(\cdot))$ *is such that:*
(1) for each player d *starting at any network-coalition pair* $\omega = (G, S)$ *contained in a basin of attraction* H_i *the time average value of the correlated strategy* $\sigma_D^{\lambda *}(\cdot)$ *is equal to state average value of the correlated strategy, that is, for all basins of attraction* H_i *and for all initial states*
$\omega = (G, S) \in H_i$,

$$\underbrace{\lim_n p^{*(n)}w_d^*(\omega)}_{\text{time average}} = \underbrace{\int_{H_i} w_d^*(\omega')d\gamma_i(\omega')}_{\text{state average}}. \tag{47}$$

Moreover, for all initial states $\omega = (G, S) \in \Omega$,

$$\lim_n p^{*(n)}w_d^*(\omega) = \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w_d^*(\omega')d\gamma_i(\omega') \tag{48}$$

(2) For all invariant measures $\gamma(\cdot) \in \mathcal{I}^*$

$$\int_\Omega f_d^*(\omega')d\gamma(\omega') = \int_\Omega w_d^*(\omega')d\gamma(\omega'), \tag{49}$$

where

$$f_d^*(\omega) := \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w_d^*(\omega')d\gamma_i(\omega') \text{ for all } \omega \in \Omega. \tag{50}$$

Proof. (1) Part (1) is an immediate consequence of part (1) of Theorem 4, Delbaen's Lemma (1974), and the fact that for all basins H_i and all states $\omega \in H_i$, $L^*(\omega, H_i) = 1$.

(2) Let invariant probability measure $\gamma(\cdot) = \sum_{i=1}^{N} \gamma(H_i)\gamma_i(\cdot) \in \mathcal{I}^*$ be given. We have

$$\int_\Omega w_d^*(\omega')d\gamma(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_\Omega w_d^*(\omega')d\gamma_i(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_{H_i} w_d^*(\omega')d\gamma_i(\omega'),$$

and

$$\int_\Omega f_d^*(\omega')d\gamma(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_\Omega f_d^*(\omega')d\gamma_i(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_{H_i} f_d^*(\omega')d\gamma_i(\omega').$$

Letting $\int_{H_i} w_d^*(\omega')d\gamma_i(\omega') := w_d^*(H_i)$, we have

$$\int_{H_i} f_d^*(\omega')d\gamma_i(\omega') = \int_{H_i} \left[\sum_{i=1}^{N} L^*(\omega', H_i)w_d^*(H_i)\right] d\gamma_i(\omega').$$

Moreover, because for all $\omega' \in H_i$, $L^*(\omega', H_i) = 1$ and $L^*(\omega', H_{i'}) = 0$, for all $i' \neq i$,

$$\int_{H_i} \left[\sum_{i=1}^{N} L^*(\omega', H_i)w_d^*(H_i)\right] d\gamma_i(\omega') = w_d^*(H_i) = \int_{H_i} w_d^*(\omega')d\gamma_i(\omega').$$

Thus we have for each i

$$\int_{H_i} f_d^*(\omega')d\gamma_i(\omega') = \int_{H_i} w_d^*(\omega')d\gamma_i(\omega'),$$

and thus,

$$\int_\Omega f_d^*(\omega')d\gamma(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_{H_i} f_d^*(\omega')d\gamma_i(\omega')$$

$$= \sum_{i=1}^{N} \gamma(H_i) \int_{H_i} w_d^*(\omega')d\gamma_i(\omega')$$

$$= \int_\Omega w_d^*(\omega')d\gamma(\omega').$$

□

The results above are essentially Birkhoff's Ergodic Theorems (pointwise and mean) for equilibrium Markov network and coalition formation processes (see for example, Theorems 2.3.4 and 2.3.5 in Hernandez-Lerma and Lasserre 2003).

By part (1) of Theorem 5, each player's time average value $\lim_n p^{*(n)} w_d^*(\omega) = f_d^*(\omega)$ is constant with respect to the starting network-coalition pair on each basin of attraction. In particular,

$$\lim_n p^{*(n)} w_d^*(\omega) = \int_\Omega w_d^*(\omega')d\gamma(\omega') = \int_{H_i} w_d^*(\omega')d\gamma_i(\omega') \text{ for all } \omega \in H_i.$$

By part (2) of Theorem 5, for any given invariant probability measure each player's average of time averages over the entire state space is equal to his state average over the entire state space with respect to the given measure.

7 Strategic Stability and Dynamic Consistency

Under the equilibrium Markov dynamics determined by strategic behavior and the trembles of nature, in order for a set of network-coalition pairs to be stable, not only must the network-coalition pairs contained in the set be favored and therefore chosen by the behavioral strategies of players, but they must also be favored by nature's law of motion (i.e., stated loosely, in order for a set of network-coalition pairs to be stable, the network-coalition pairs contained in the set must not only be chosen but they must also be lucky).

Again let $\sigma_D^{\lambda*}(\cdot)$ be an equilibrium Markov correlated strategy of the dynamic network-coalition formation game with corresponding equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{\lambda*}(\cdot))$, and let

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T,$$

be the finite decomposition of the state space generated by $p^*(\cdot|\cdot)$ with basins of attraction $\{H_1, \ldots, H_N\}$ and transient set T. Finally, let $\mathcal{E}^* = \{\gamma_i(\cdot)\}_{i=1}^{N}$ be the corresponding set of ergodic probability measures with $\gamma_i(H_i) = 1$ for all i.

Player $d's$ parts of the correlated strategy $\sigma_D^{\lambda*}(\cdot)$

$$\omega = (G, S) \longrightarrow \sigma_d^{i*}(\cdot|G, S),\ i = 0, 1, \ldots, m$$

govern the way in which player d tries to influence the process of network and coalition formation across time (as directed by the public randomization device, $\lambda(\cdot)$), and for each given status quo coalition S, the $m+1$ transitions, $\sigma_d^{i*}(\cdot|\cdot, S)$, are the equilibrium Markov transitions on networks governing player d's network proposal process. For each status quo coalition S, we will refer to the equilibrium Markov network transitions, $(\sigma_d^{i*}(\cdot|\cdot, S)_{i=0}^{m})_{d\in D}$, as the *$S$-proposal transitions* and we will refer to the induced equilibrium Markov network-coalition transition, $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{\lambda*}(\cdot))$, as the *state transition.*

To begin, let $\mathcal{L}_{idS}^*$ denote the set of absorbing sets corresponding to player d's S-proposal transition $\sigma_d^{i*}(\cdot|\cdot, S)$, and let $\mathcal{L}_{dS}^{*c} := \cap_{i=0}^{m} \mathcal{L}_{idS}^*$ denote the set of absorbing sets common to all player d's S-proposal transition $\sigma_d^{i*}(\cdot|\cdot, S)$ under correlated strategy $\sigma_D^{\lambda*}(\cdot)$. We will refer to the collection of absorbing sets $\mathcal{L}_{dS}^{*c}$ as player $d's$ correlated absorbing sets. If the set of networks $\mathbb{E}$ is a correlated absorbing set for player d, then for any status quo network $G \in \mathbb{E}$, it is optimal for player $d \in S$ to propose with probability 1 either the status quo network or a new network G' in $\mathbb{E}$ no matter which S-proposal transition $\sigma_d^{i*}(\cdot|\cdot, S)$, $i = 0, 1, \ldots, m$, governs player $d's$ network proposal choice. Moreover, by assumption A-2(2) if $d \notin S$, then player d is constrained to propose only the status quo network. Thus, for *any* player d not in coalition S, $\sigma_d^{i*}(\{G\}|G, S) = 1$ for all status quo networks G under all player $d's$ proposal transitions.[17] If in addition, the set of network proposals $\mathbb{E}$ is a correlated absorbing set for all players in S, that is, if

$$\mathbb{E} \in \cap_{d\in S} \mathcal{L}_{dS}^{*c} := \cap_{d\in S}[\cap_i \mathcal{L}_{idS}^*],$$

then for all status quo networks $G \in \mathbb{E}$, it is optimal for *all* players in S to propose a network contained in $\mathbb{E}$ with probability 1 no matter which S-proposal transition $\sigma_d^{i*}(\cdot|\cdot, S)$ governs player $d's$ network proposal choice. Note, however, that unless $\mathbb{E}$ is a singleton (i.e., $\mathbb{E} = \{G\}$ for some network $G \in \mathbb{G}$), players may not agree on their individual network proposals. However, if $\mathbb{E}$ is a correlated absorbing set for all members of S then at least all members will agree that their proposals should be drawn from $\mathbb{E}$. Thus, we can think of the sets in $\cap_{d\in S} \mathcal{L}_{dS}^{*c}$ as being *strategically stable* for coalition S - as long as coalition S is the status quo coalition. We will denote by $\mathcal{L}_S^{*c}$ the intersection $\cap_{d\in S} \mathcal{L}_{dS}^{*c}$ and we will refer to $\mathcal{L}_S^{*c}$ as an S-strategically stable set.

Let $\mathcal{F}$ be a subcollection of the feasible coalitions $\mathcal{C}$. We will say that a set of networks $\mathbb{E}$ is $\mathcal{F}$-strategically stable if it is S-strategically stable for all coalitions $S \in \mathcal{F}$, that is, if

$$\mathbb{E} \in \cap_{S\in\mathcal{F}} \mathcal{L}_S^{*c} := \mathcal{L}_{\mathcal{F}}^{*c},$$

and we will say that $\mathbb{E}$ is strategically stable if $\mathcal{F} = \mathcal{C}$. Thus, if $\mathbb{E}$ is $\mathcal{F}$-strategically stable, then in any status quo state $\omega = (G, S)$ with $G \in \mathbb{E}$ and $S \in \mathcal{F}$, all players in S will find it in their best interest to propose networks in $\mathbb{E}$, while all players not in S will be constrained (under the rules of network formation) to propose the status quo network G - also a network in $\mathbb{E}$. Moreover, the same will be

[17] Thus, for all states $\omega = (G, S)$ and for all players $d \notin S$, the singleton sets $\{G\}$ are absorbing for the $m+1$, S-proposal transitions

$$(\sigma_d^{i*}(\cdot|\cdot, S))_{i=0}^{m}.$$

true in any other state $\omega' = (G', S')$ with $G' \in \mathbb{E}$ and $S' \in \mathcal{F}$, that is, all players in S' will find it in their best interest to propose networks in $\mathbb{E}$, while all players not in S' will be constrained to propose the status quo network G.

Finally, suppose the $\mathcal{F}$-strategically stable set of networks $\mathbb{E}$ is such that nature chooses with probability 1 network-coalition pairs from $\mathbb{E} \times \mathcal{F}$ starting from any status quo network-coalition pair contained in $\mathbb{E} \times \mathcal{F}$; that is, suppose that in addition to $\mathbb{E}$ being $\mathcal{F}$-strategically stable, that $\mathbb{E} \times \mathcal{F}$ is absorbing for the state transition $p^*(\cdot|\cdot) := q(\cdot|\cdot, \sigma_D^{\lambda *}(\cdot))$. We will refer to a $\mathcal{F}$-strategically stable set of networks $\mathbb{E}$ as being *$\mathcal{F}$-dynamically consistent* if $\mathbb{E} \times \mathcal{F}$ is absorbing for $p^*(\cdot|\cdot)$. Thus, a set of networks $\mathbb{E} \in \mathcal{L}_{\mathcal{F}}^{*c}$ is $\mathcal{F}$-dynamically consistent if $\mathbb{E} \times \mathcal{F} \in \mathcal{L}^*$, where as before $\mathcal{L}^*$ is the collection of all absorbing sets corresponding to the state transition $p^*(\cdot|\cdot)$.

We have the following formal definitions.

Definition 7 (*$\mathcal{F}$-Strategic Stability and $\mathcal{F}$-Dynamic Consistency*)
(1) ($\mathcal{F}$-Strategic Stability)
A set of networks $\mathbb{E} \in B(\mathbb{G})$ is $\mathcal{F}$-strategically stable if all players $d \in D$ in all states $(G, S) \in \mathbb{E} \times \mathcal{F}$ propose networks in $\mathbb{E}$ with probability 1, that is, if for all players $d \in D$,

$$\sigma_d^{i*}(\mathbb{E}|G, S) = 1 \text{ for all } (G, S) \in \mathbb{E} \times \mathcal{F} \text{ and } i = 0, 1, \ldots, m.$$

(2) ($\mathcal{F}$-Dynamic Consistency)
A $\mathcal{F}$-strategically stable set of networks $\mathbb{E} \in B(\mathbb{G})$ is $\mathcal{F}$-dynamically consistent if in all states $(G, S) \in \mathbb{E} \times \mathcal{F}$ nature chooses states in $\mathbb{E} \times \mathcal{F}$ with probability 1, that is, if

$$p^*(\mathbb{E} \times \mathcal{F}|G, S) = 1 \text{ for all } (G, S) \in \mathbb{E} \times \mathcal{F}.$$

(3) (Strategic Stability and Dynamic Consistency)
An $\mathcal{F}$-strategically stable set of networks $\mathbb{E} \in B(\mathbb{G})$ is dynamically consistent if it is $\mathcal{F}$-dynamically consistent.

The following result gives necessary conditions for dynamic strategic stability and dynamic consistency. The proof is straightforward.

Theorem 6 *(Dynamic Consistency and Invariance)*
Suppose assumptions [A] hold and let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the equilibrium network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) := q(\cdot|\cdot, \sigma_D^{\lambda *}(\cdot))$.
If $\mathbb{E} \in B(\mathbb{G})$ is dynamically consistent, then starting at any network-coalition pair contained in $E := \mathbb{E} \times \mathcal{F}$, the network-coalition formation process will reach in finite time with probability 1 a nonempty subset of network-coalition pairs $E \cap H_i$, where H_i is a basin of attraction and once there will remain there. Moreover, there exists a p^*-invariant probability measure which assigns positive measure to $E \cap H_i$.

Note that $E \cap H_i$ is absorbing for the state transition $p^*(\cdot|\cdot)$; that is, $E \cap H_i \in \mathcal{L}^*$. Moreover, note that it is possible for E to intersect more than one basin of attraction, but because each basin of attraction is indecomposable, *each* basin of attraction can intersect only one such set $E := \mathbb{E} \times \mathcal{F}$ where $\mathbb{E}$ is dynamically consistent. It is also possible for E to intersect the transient set - but it is not possible for E to be a subset of the transient set. If E intersects basins H_i and $H_{i'}$, and $\gamma(\cdot)$ is a p^*-invariant measure such that $\gamma(E) = 1$, then by part (2) of Theorem 5 above we have,

$$\gamma(E) = \sum_{i''}^{N} \gamma(H_{i''})\gamma_{i''}(E \cap H_{i''}) = \gamma(H_i)\gamma_i(E \cap H_i) + \gamma(H_{i'})\gamma_{i'}(E \cap H_{i'}).$$

Thus, under any p^*-invariant measure $\gamma(\cdot)$ the measure of any absorbing set E is a weighted sum of the probability masses the invariant measures $\gamma(\cdot)$ assigns to each basin H_i.

7.1 Dynamic Path dominance Core and Dynamic Pairwise Stability

One way to extend the definition of the path dominance core introduced in Page and Wooders (2009a) to the dynamic setting considered here is as follows:

Definition 8 (*The Dynamic Path Dominance Core*)
A network $G^* \in \mathbb{G}$ is in the dynamic path dominance core if the set $\{G^*\}$ is dynamically consistent, that is, if $\{G^*\} \in \mathcal{L}^{*c}_{\mathcal{F}}$ and $\{G^*\} \times \mathcal{F} \in \mathcal{L}^*$.

We have the following result giving necessary conditions for a network to be in the path dominance core.

Theorem 7 *(The Dynamic Path Dominance Core and Invariance)*
Suppose assumptions [A] hold and let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the equilibrium network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) := q(\cdot|\cdot, \sigma_D^{\lambda *}(\cdot))$.
If network $G^* \in \mathbb{G}$ is in the dynamic path dominance core, that is, if $\{G^*\}$ is dynamically consistent, then starting at any network-coalition pair contained in $\{G^*\} \times \mathcal{F}$, the network-coalition formation process will reach in finite time with probability 1 a nonempty subset of network-coalition pairs $(\{G^*\} \times \mathcal{F}) \cap H_i$, where H_i is a basin of attraction and once there will remain there. Moreover, there exists a p^*-invariant probability measure which assigns positive measure to $(\{G^*\} \times \mathcal{F}) \cap H_i$.

Note that if for some network $G^* \in \mathbb{G}$ and some coalition $S^* \in \mathcal{F}$, $\{G^*\} \in \mathcal{L}^{*c}_{S^*}$ and $\{(G^*, S^*)\} \in \mathcal{L}^*$, so that $\{G^*\}$ is $\{S^*\}$-dynamically consistent, this does not necessarily imply that G^* is in the dynamic path dominance core, even if $\{(G^*, S^*)\}$ basin of attraction, because $\{G^*\}$ may not be dynamically consistent. Why? Because while nature will choose with probability 1 the network-coalition pair (G^*, S^*) if the status quo is (G^*, S^*), if the status quo coalition is not S^*, that is, if the status quo state is (G^*, S') for some coalition $S' \in \mathcal{F}$ not equal to S^*, some players in S' may propose a network other than G^* (i.e., it may be the case that $G^* \notin \mathcal{L}^{*c}_{dS'}$ for some player $d \in S'$ or it may be the case that $G^* \notin \mathcal{L}^*_{idS'}$ for some $i = 0, 1, 2, \ldots, m$) and in turn nature may choose a state other than (G^*, S^*). Moreover, if G^* is not strategically stable, but nonetheless $\{G^*\} \times \mathcal{F} \in \mathcal{L}^*$ for some subset of coalitions $\mathcal{F} \subseteq \mathcal{C}$, then if the equilibrium network-coalition formation process reaches any state $(G^*, S) \in \{G^*\} \times \mathcal{F}$, the process will remain in the set $\{G^*\} \times \mathcal{F}$ - despite network proposals to the contrary by players, even players in coalitions in $\mathcal{F}$. In such a case, the state transition overrides the wishes of the players. This leads to the following alternative notion of dynamic path dominance core.

Definition 9 (*The State Transition Core*)
(1) (State Transition Core) A network $G^* \in \mathbb{G}$ is in the state transition core if the set of states $\{G^*\} \times \mathcal{F} \in B(\Omega)$ is an absorbing set for the state transition $p^*(\cdot|\cdot)$.
(2) (Weak State Transition Core) A network $G^* \in \mathbb{G}$ is in the weak state transition core if the set of states $\{G^*\} \times \mathcal{F} \in B(\Omega)$ is an absorbing set for the state transition $p^*(\cdot|\cdot)$ for some subset of coalitions $\mathcal{F} \subseteq \mathcal{C}$.

Under the definition of weak state transition core, for any basin of attraction H_{i*} of the form $H_{i*} = \{(G^*, S^*)\}$, G^* is in the weak state transition core. Moreover, if for some state transition absorbing set E, $E \cap H_{i*}$ is nonempty but E is disjoint from the other basins, then starting at any network-coalition pair in E, the process will reach in finite time with probability 1 the network-coalition pair (G^*, S^*) and will remain there.

Finally, note that if $p^*(\{G^*\} \times \mathcal{F} | G^*, S) = 1$ for all $S \in \mathcal{F} \subseteq \mathcal{C}$, then because the law of motion

$$q(\cdot|(G, S), G_D)$$

is absolutely continuous with respect the probability measure $\mu = \nu \times \gamma$ for all $((G, S), G_D) \in Gr\Phi(\cdot)$, G^* must be an atom of the probability measure ν, that is,

$$G^* \in \{\mathbb{A}_{\alpha 1}, \mathbb{A}_{\alpha 2}, \ldots\} = \{\mathbb{A}_{\alpha k}\}_{k=1}^{\infty} \subset \mathbb{G}.$$

To extend the definition of the pairwise stability introduced in Jackson and Wolinsky (1996) to the dynamic setting considered here, we begin by specializing the feasible set of coalitions to coalitions of size no greater than 2.

Definition 10 (*Dynamic Pairwise Stability*)
Suppose the feasible set of coalitions is given by

$$\mathcal{F}_2 = \{S \in P(D) : |S| \leq 2\}.$$

(i.e., all feasible coalitions consist of at most two players). Then a network $G^* \in \mathbb{G}$ is dynamically pairwise stable if the set $\{G^*\}$ is dynamically consistent, that is, if $\{G^*\} \in \mathcal{L}^{*c}_{\mathcal{F}_2}$ and $\{G^*\} \times \mathcal{F}_2 \in \mathcal{L}^*$.

We have the following characterization

Theorem 8 *(Dynamic Pairwise Stability and Invariance)*
Suppose assumptions [A] hold and let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the equilibrium network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) := q(\cdot|\cdot, \sigma_D^{\lambda *}(\cdot))$.
If network $G^* \in \mathbb{G}$ is dynamically pairwise stable, that is, if $\{G^*\}$ is dynamically consistent, then starting at any network-coalition pair contained in $\{G^*\} \times \mathcal{F}_2$, the network-coalition formation process will reach in finite time with probability 1 a nonempty subset of network-coalition pairs $(\{G^*\} \times \mathcal{F}_2) \cap H_i$, where H_i is a basin of attraction and once there will remain there. Moreover, there exists a p^*-invariant probability measure which assigns positive measure to $(\{G^*\} \times \mathcal{F}_2) \cap H_i$.

Our conclusion that for some basin of attraction H_i, $(\{G^*\} \times \mathcal{F}_2) \cap H_i$ is contained in the support of some p^*-invariant measure is similar to the conclusion reached by Jackson and Watts (2002) for a stochastic process of network formation over a finite set of linking networks governed by Markov chain generated by myopic players. They reach their conclusion by considering a sequence of perturbed irreducible and aperiodic Markov chains (i.e., each with a unique invariant measure) converging to the original Markov chain. This method is similar to a method introduced into games by Young (1993) which in turn is based on some very general perturbation methods found in Freidlin and Wentzell (1984). Here we have reached similar conclusions using very different methods.

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8 Appendix

8.1 The Hausdorff metric topology for the Space of Directed Networks

Because the set of directed connections, $K := A \times (N \times N)$, is a compact metric space, we can equip the space of networks $P_f(K)$ with the Hausdorff metric h, making it a compact metric space (see Aliprantis and Border (2006), sections 3.16-3.18). Formally, the Hausdorff metric is defined as follows: First, define the distance between a connection $(a,(i_0,i_1)) \in K$ and a network $G \in P_f(K)$ as follows:

$$\rho((a,(i_0,i_1)),G) := \inf_{(a',(i_0',i_1'))\in G} \rho_K\left((a,(i_0,i_1)),(a',(i_0',i_1'))\right),$$

where

$$\rho_K\left((a,(i_0,i_1)),(a',(i_0',i_1'))\right) := \rho_A(a,a') + \rho_N(i_0,i_0') + \rho_N(i_1,i_1')$$

is the product metric on K. The Hausdorff metric h is then defined as

$$h(G,G') := \max\left\{\sup_{(a,(i_0,i_1))\in G}\rho((a,(i_0,i_1)),G'),\sup_{(a',(i_0',i_1'))\in G'}\rho((a',(i_0',i_1')),G)\right\}, \tag{51}$$

for directed networks G and G' in $P_f(K)$.[18]

To better understand how the distance between networks is measured using the Hausdorff metric, consider the notion of a sequence of networks converging to a limit network. Convergence in the space of directed networks $(P_f(K),h)$ can be characterized via the notions of limit inferior and limit superior. Let $\{G^n\}_n$ be a sequence of directed networks. The limit inferior of this sequence, denoted by $Li(G^n)$, is defined as follows:

connection $(a,(i,i')) \in Li(G^n)$ if and only if there is a *sequence* of connections $\{(a^n,(i^n,i'^n))\}_n$ such that $(a^n,(i^n,i'^n)) \in G^n$ for all n and

$$(a^n,(i^n,i'^n)) \underset{\rho_K}{\longrightarrow} (a,(i,i')).$$

The limit superior, denoted by $Ls(G^n)$, is defined as follows:

connection $(a,(i,i')) \in Ls(G^n)$ if and only if there is a *subsequence* of connections $\{(a^{n_k},(i^{n_k},i'^{n_k}))\}_k$ such that $(a^{n_k},(i^{n_k},i'^{n_k})) \in G^{n_k}$ for all k and

$$(a^{n_k},(i^{n_k},i'^{n_k})) \underset{\rho_K}{\longrightarrow} (a,(i,i')).$$

A directed network $G \in P_f(K)$ is said to be the limit of networks $\{G^n\}_n$ if

$$Ls(G^n) = G = Li(G^n).$$

Moreover, because the set of connections $A \times (N \times N)$ is a compact metric space,

$$Ls(G^n) = G = Li(G^n) \text{ if and only if } h(G^n,G) \longrightarrow 0$$

(i.e., the sequence of networks $\{G^n\}_n$ converges to network $G \in P_f(K)$ under the Hausdorff metric h - see Theorem 3.93 in Aliprantis and Border (1999)).[19]

8.2 The Existence of Correlated Stationary Markov Equilibrium

8.2.1 The Continuity Lemma

A key ingredient in proving the existence of a correlated stationary Markov equilibrium is the one-shot, state-contingent game given by

$$\mathcal{G}_v(\omega) := (\mathcal{P}(\Phi_d(\omega)), u_d(\omega,\cdot)(v_d))_{d\in D} \tag{52}$$

where for each state $\omega \in \Omega$, player $d's$ strategy set is $\mathcal{P}(\Phi_d(\omega))$ and player $d's$ payoff function is

$$\sigma_D \longrightarrow u_d(\omega,\sigma_D)(v_d) := (1-\beta_d)r_d(\omega,\sigma_D) + \beta_d \int_\Omega v_d(\omega')q(\omega'|\omega,\sigma_D). \tag{53}$$

[18] It is important to note that because the space of connections K is compact, all metrics compatible with the product topology on $K := A \times (N \times N)$ generate the same Hausdorff metric topology on $P_f(K)$ (see Theorem 3.87 in Aliprantis and Border, 2006).

[19] Both $Li(G^n)$ and $Ls(G^n)$ are networks, that is, both $Li(G^n)$ and $Ls(G^n)$ are contained in $P_f(K)$. Moreover, in general,

$$Li(G^n) \subseteq Ls(G^n).$$

As in the literature on discounted stochastic games (e.g., see Nowak and Raghavan, 1992), the space of players' value function profiles, $v := (v_d)_{d \in D} := (v_1, \ldots, v_m)$, is given by

$$\mathcal{L}^{\infty}_{X} := \mathcal{L}^{\infty}_{X_1} \times \cdots \times \mathcal{L}^{\infty}_{X_m},$$

where for each player $d = 1, 2, \ldots, m$, $\mathcal{L}^{\infty}_{X_d}$ is space of μ-equivalence classes of functions, $v_d : \Omega \longrightarrow R$, such that $v_d(\omega) \in X_d$ a.e. $[\mu]$. For each player d, X_d is the closed bounded interval, $[-M, M]$, the same for each player. Players' payoffs (both immediate and discounted) reside in closed, bounded, convex subset, $X := X_1 \times \cdots \times X_m = [-M, M]^m$, and thus, players' value function profiles reside in the space, $\mathcal{L}^{\infty}_{X}$, a metrizable, weak star compact, convex subset of $\mathcal{L}^{\infty}_{R^m}$.

Formally, let $\mathcal{L}^1_R(\mu, \Omega) := \mathcal{L}^1_R$ denote the separable Banach space of μ-equivalence classes of μ-integrable functions, $u : \Omega \longrightarrow R$ with norm

$$\|u\|_1 := \int_{\Omega} |u| \, d\mu.$$

Also, denote by L^1_R the prequotient of $\mathcal{L}^1_R$ (i.e., the space of all real-valued, integrable functions), and let

$$\mathcal{L}^1_{R^m} := \underbrace{\mathcal{L}^1_R \times \cdots \times \mathcal{L}^1_R}_{m \text{ times}}$$

denote the separable Banach space of μ-equivalence classes of μ-integrable functions, $U : \Omega \longrightarrow R^m$, $U := (U_1, \ldots U_d, \ldots, U_m)$, with norm

$$\|u\|_1 := \int_{\Omega} |u| \, d\mu.$$

Also, denote by L^1_R the prequotient of $\mathcal{L}^1_R$ (i.e., the space of all real-valued, integrable functions), and let

$$\mathcal{L}^1_{R^m} := \underbrace{\mathcal{L}^1_R \times \cdots \times \mathcal{L}^1_R}_{m \text{ times}}$$

denote the separable Banach space of μ-equivalence classes of μ-integrable functions, $U : \Omega \longrightarrow R^m$, $U := (U_1, \ldots U_d, \ldots, U_m)$, with norm

$$\|U\|_1 = \sum_{d=1}^{m} \|U_d\|_1 .$$

Next, let $\mathcal{L}^{\infty}_R$ denote the Banach space of μ-equivalence classes of μ-essentially bounded functions, $v : \Omega \longrightarrow R$ with norm

$$\|v\|_{\infty} := ess\sup v := \inf \{x \in R : \mu\{\omega : |v(\omega)| > x\} = 0\}.$$

$\mathcal{L}^{\infty}_R$ is the norm dual of $\mathcal{L}^1_R$. Equip $\mathcal{L}^{\infty}_R$ with the weak star topology, denoted by w^* or $\sigma(\mathcal{L}^{\infty}_R, \mathcal{L}^1_R)$. We will denote by L^{∞}_R the prequotient of $\mathcal{L}^{\infty}_R$ (i.e., the space of all real-valued, μ-essentially bounded functions).

For $d = 1, 2, \ldots, m$, let X_d be the closed bounded interval $[-M, M] \subset R$, and let

$$\mathcal{L}^{\infty}_{X_d} := \{v \in \mathcal{L}^{\infty}_R : v(\omega) \in X_d \text{ a.e. } [\mu]\}.$$

Equip $\mathcal{L}^{\infty}_{X_d}$ with the compact and metrizable relative weak star topology, denoted by w^*_d or $\sigma(\mathcal{L}^{\infty}_{X_d}, \mathcal{L}^1_{X_d})$.[20] To fix the metric and hence the notation, let $\rho_{w^*_d}$ be the metric on $\mathcal{L}^{\infty}_{X_d}$ compatible with the weak star topology. Also, let ρ_{X_d} denote the metric on X_d where for x and x' in X_d, $\rho_{X_d}(x, x') := |x - x'|$.

[20] Because the Borel σ-field B_{Ω} is countably generated, the space of μ-equivalence classes of μ-integrable functions, $\mathcal{L}^1_R$, is separable. As a consequence, the set of value function μ-equivalence classes $\mathcal{L}^{\infty}_{X_d}$ is a compact, convex, and metrizable subset of $\mathcal{L}^{\infty}_R$ for the weak star topology (e.g., see Nowak and Raghavan, 1992).

Finally, let $X := X_1 \times \cdots \times X_m$ and consider the Cartesian product,

$$\mathcal{L}^\infty_X := \mathcal{L}^\infty_{X_1} \times \cdots \times \mathcal{L}^\infty_{X_m},$$

equipped with the the sum metric,

$$\rho_{w^*} := \textstyle\sum_{d=1}^m \rho_{w_d^*},$$

a metric compatible with the relative weak star product topology, w^*, on $\mathcal{L}^\infty_X$, and equip X with the sum metric

$$\rho_X := \textstyle\sum_{d=1}^m \rho_{X_d}.$$

In order to establish existence, we must show that in each state $\omega \in \Omega$ and for each m-tuple of player value functions, $v = (v_d) \in \mathcal{L}^\infty_X$, the one-shot game $\mathcal{G}_v(\omega)$ has a nonempty, compact set of Nash equilibria, $\mathcal{N}_v(\omega)$. But more importantly, we must show that $\mathcal{N}_v(\cdot)$ is measurable in ω for each v and that $\mathcal{N}_{(\cdot)}(\omega)$ is upper semicontinuous in v for each ω. In order to accomplish the latter, we will first show that

$$(v, \sigma_D) \longrightarrow (u_d(\omega, \cdot)(\cdot))_d$$

is continuous for each $\omega \in \Omega$.

Lemma (*The Continuity Lemma***)**
Suppose assumptions [A] hold and let $\{(v^n, \sigma^n_D)\}_n$ *be any sequence in* $\mathcal{L}^\infty_X \times \prod_d \mathcal{P}(\Phi_d(\omega))$. *If* $v^n \underset{w^*}{\longrightarrow} v^*$ *and* $\sigma^n_D \longrightarrow \sigma^*_D$ *narrowly,then for each player* d

$$u_d(\omega, \sigma^n_D)(v^n) \longrightarrow u_d(\omega, \sigma^*_D)(v^*) \text{ for all } \omega \in \Omega.$$

Proof. Let $\{(v^n, \sigma^n_D)\}_n$ be a sequence such that $v^n \underset{w^*}{\longrightarrow} v^*$ and $\sigma^n_D \longrightarrow \sigma^*_D$ narrowly. Let ω be given and fixed, and observe that for all players d:

$$|u_d(\omega, \sigma^n_D)(v^n_d) - u_d(\omega, \sigma^*_D)(v^*_d)|$$

$$\leq \underbrace{|u_d(\omega, \sigma^n_D)(v^n_d) - u_d(\omega, \sigma^*_D)(v^n_d)|}_{A^n} + \underbrace{|u_d(\omega, \sigma^*_D)(v^n_d) - u_d(\omega, \sigma^*_D)(v^*_d)|}_{B^n}.$$

We will carry out our proof for one player d, keeping in mind that the argument can easily be made to hold for all players simultaneously. Consider B^n first. We have

$$B^n = \beta_d \left| \int_\Omega v^n_d(\omega')q(\omega'|\omega, \sigma^*_D) - \int_\Omega v^*_d(\omega')q(\omega'|\omega, \sigma^*_D) \right|.$$

Let $z(\cdot|\omega, \sigma^*_D)$ be a density of $q(\cdot|\omega, \sigma^*_D)$ with respect to μ. Given that $v^n_d \underset{w^*}{\longrightarrow} v^*_d$, we have (by the very notion of weak star convergence),

$$\textstyle\int_\Omega v^n_d(\omega')q(\omega'|\omega, \sigma^*_D) = \int_\Omega v^n_d(\omega')z(\omega'|\omega, \sigma^*_D)d\mu(\omega')$$

$$\longrightarrow \textstyle\int_\Omega v^*_d(\omega')z(\omega'|\omega, \sigma^*_D)d\mu(\omega') = \int_\Omega v^*_d(\omega')q(\omega'|\omega, \sigma^*_D).$$

Thus, $B^n \overset{n}{\longrightarrow} 0$.

Next, consider A^n. We have

$$A^n \leq (1-\beta_d)\underbrace{|r_d(\omega, \sigma^n_D) - r_d(\omega, \sigma^*_D)|}_{A^n_1}$$

$$+\beta_d \underbrace{\left| \int_\Omega v^n_d(\omega')q(\omega'|\omega, \sigma^n_D) - \int_\Omega v^n_d(\omega')q(\omega'|\omega, \sigma^*_D) \right|}_{A^n_2}.$$

Continuity of $r_d(\omega,\cdot)$ and $\sigma_D^n \longrightarrow \sigma_D^*$ imply that $A_1^n \xrightarrow{n} 0$. To see that $A_2^n \xrightarrow{n} 0$, observe that by Scheffee's Theorem we have

$$\underbrace{\left|\int_\Omega v_d^n(\omega')q(\omega'|\omega,\sigma_D^n) - \int_\Omega v_d^n(\omega')q(\omega'|\omega,\sigma_D^*)\right|}_{A_2^n}$$

$$\leq M \left\| q(\cdot|\omega,\sigma_D^n) - q(\cdot|\omega,\sigma_D^*)\right\|_\infty \xrightarrow{n} 0.$$

□

8.2.2 Proof of Existence of a Correlated Stationary Markov Equilibrium

Again consider the one-shot game $\mathcal{G}_v(\omega)$ and let $\mathcal{N}_v(\omega)$ denote the set of Nash equilibria of $\mathcal{G}_v(\omega)$.

The proof will proceed in 6 steps:

Step 1: (*$\omega \longrightarrow \mathcal{N}_v(\omega)$ is measurable*)

Following Nowak and Raghavan (1992) let

$$V(\omega,\sigma_D)(v) := \textstyle\sum_d \left(u_d(\omega,(\sigma_d,\sigma_{-d}))(v_d) - \max_{\sigma\in\mathcal{P}(\Phi_d(\omega))} u_d(\omega,(\sigma,\sigma_{-d}))(v_d)\right),$$

and consider the correspondence

$$\omega \longrightarrow \mathcal{N}_v(\omega) := \{\sigma_D \in \mathcal{P}(\Phi_d(\omega)) : V(\omega,\sigma_D)(v) = 0\}. \tag{54}$$

Note that $\sigma_D = (\sigma_d)_d \in \mathcal{N}_v(\omega)$ if and only if for each player $d \in D$,

$$u_d(\omega,(\sigma_d,\sigma_{-d}))(v_d) \geq u_d(\omega,(\sigma,\sigma_{-d}))(v_d) \text{ for all } \sigma \in \mathcal{P}(\Phi_d(\omega)).$$

Given that $q(F|\omega,\cdot)$ is continuous on $\Phi(\omega)$ for closed $F \in B(\Omega)$, it follows from Delbaen's Lemma (1974) that the function

$$G_d \longrightarrow \int_\Omega v_d(\omega')q(\omega'|\omega,(G_d,G_{-d}))$$

is also continuous on $\Phi_d(\omega)$ for all players d, states $\omega \in \Omega$, and value functions $v(\cdot) \in \mathcal{L}_X^\infty$. Therefore, by weak continuity, the function

$$\sigma_d \longrightarrow \int_\Omega v_d(\omega')q(\omega'|\omega,\sigma_d,\sigma_{-d})$$

is continuous on $\mathcal{P}(\Phi_d(\omega))$ for all players d, states $\omega \in \Omega$, and value functions $v(\cdot) \in \mathcal{L}_X^\infty$. Moreover, because each player's payoff function,

$$\sigma_d \longrightarrow u_d(\omega,(\sigma_d,\sigma_{-d}))(v_d),$$

is continuous and affine on $\mathcal{P}(\Phi_d(\omega))$, and because the feasible sets, $\mathcal{P}(\Phi_d(\omega))$, are compact and convex, the game $\mathcal{G}_v(\omega)$ has a Nash equilibrium $\sigma_D^* \in \prod_d \mathcal{P}(\Phi_d(\omega))$. Thus, $\mathcal{N}_v(\omega)$ is nonempty and compact. Finally, because $\sigma_D \longrightarrow V(\omega,\sigma_D)(v)$ is continuous, it follows from Theorem 6.4 in Himmelberg (1975) that $\omega \longrightarrow \mathcal{N}_v(\omega)$ is measurable.

Step 2: (*Properties of the Nash Correspondence $v \longrightarrow \mathcal{N}_v(\omega)$*)

The correspondence $v \longrightarrow \mathcal{N}_v(\omega)$ has a closed graph for all $\omega \in \Omega$. To see this, let $\{(v^n,\sigma_D^n)\}$ be a sequence such that $\sigma_D^n \in \mathcal{N}_{v^n}(\omega)$ for all n and let $v^n \underset{w^*}{\longrightarrow} v^*$ and $\sigma_D^n \longrightarrow \sigma_D^*$ narrowly. We must show that $\sigma_D^* \in \mathcal{N}_{v^*}(\omega)$. Suppose that $\sigma_D^* \notin \mathcal{N}_{v^*}(\omega)$. Thus, σ_D^* is not Nash equilibrium for the game $\mathcal{G}_{v^*}(\omega)$. Therefore for some player d and some action $\sigma_d \in \mathcal{P}(\Phi_d(\omega))$,

$$u_d(\omega,(\sigma_d,\sigma_{-d}^*))(v_d^*) > u_d(\omega,(\sigma_d^*,\sigma_{-d}^*))(v_d^*).$$

By the Continuity Lemma, we have for sequences $\{(v^n, \sigma_D^n)\}_n$ and $\left\{(v^n, (\sigma_d, \sigma_{-d}^n))\right\}_n$

$$u_d(\omega, (\sigma_d, \sigma_{-d}^n))(v_d^n) \xrightarrow{n} u_d(\omega, (\sigma_d, \sigma_{-d}^*))(v_d^*)$$
$$\text{and}$$
$$u_d(\omega, (\sigma_d^n, \sigma_{-d}^n))(v_d^n) \xrightarrow{n} u_d(\omega, (\sigma_d^*, \sigma_{-d}^*))(v_d^*).$$

Thus, for n sufficiently large,

$$u_d(\omega, (\sigma_d, \sigma_{-d}^n))(v_d^n) > u_d(\omega, (\sigma_d^n, \sigma_{-d}^n))(v_d^n)$$

contradicting the fact that $\sigma_D^n \in \mathcal{N}_{v^n}(\omega)$ for all n.

Step 3: ($v \longrightarrow \Sigma(coP_v(\cdot))$ *has a closed graph*)

Consider the Nash payoff correspondence given by

$$P_v(\omega) := \{(U_d) \in R^m : (U_d) = (u_d(\omega, \sigma_D)(v_d)) \text{ for some } \sigma_D \in \mathcal{N}_v(\omega)\},$$

where, recall

$$u_d(\omega, \sigma_D)(v_d) := (1 - \beta_d)r_d(\omega, \sigma_D) + \beta_d \int_\Omega v_d(\omega')q(\omega'|\omega, \sigma_D).$$

By Theorem 6.5 in Himmelberg (1975) the payoff correspondence $\omega \longrightarrow P_v(\omega)$ is measurable with nonempty, compact values, and by Theorem 9.1 in Himmelberg (1975) the correspondence

$$\omega \longrightarrow coP_v(\omega)$$

is measurable with nonempty, compact convex values.

Step 4: (*The Nowak-Raghavan Lemma*)

Let $\Sigma(coP_v(\cdot))$ be the set of all μ-equivalence classes of measurable selectors of $\omega \longrightarrow coP_v(\omega)$, $v \in \mathcal{L}_X^\infty$ (i.e., $U(\cdot) \in \Sigma(coP_v(\cdot))$ if and only if $U(\omega) \in coP_v(\omega)$ for all $\omega \in \Omega \backslash N_U$, where N_U is a μ-null set, $\mu(N_U) = 0$). The Nowak-Raghavan (NR) Lemma states that the payoff selection correspondence $v \longrightarrow \Sigma(coP_v(\cdot))$ is upper semicontinuous with nonempty convex, weakly compact values. Convexity, weak compactness, and nonemptiness are straightforward. We need only prove upper semicontinuity. Thus, we must show that if $U^n(\cdot) \in \Sigma(coP_{v^n}(\cdot))$ for all n and $U^n(\cdot) \underset{w^*}{\longrightarrow} U^*(\cdot)$ and $v^n(\cdot) \underset{w^*}{\longrightarrow} v^*(\cdot)$, then $U^*(\cdot) \in \Sigma(coP_{v^*}(\cdot))$ (i.e., $U^*(\omega) \in coP_{v^*}(\omega)$ a.e. $[\mu]$).

The proof of the NR Lemma proceeds in three steps:

First, we have $U^n(\cdot) \underset{w^*}{\longrightarrow} U^*(\cdot)$ and $v^n(\cdot) \underset{w^*}{\longrightarrow} v^*(\cdot)$, where for all n, $U^n(\cdot) \in \Sigma(coP_{v^n}(\cdot))$ and $v^n(\cdot) \in \mathcal{L}_X^\infty$. Let $N^\infty = \cup N_{U^n}$ be the μ-null set where for each n, N_{U^n} is such that for all $\omega \in \Omega \backslash N_{U^n}$, $U^n(\omega) \in coP_{v^n}(\omega)$. By Komlos' Theorem (1967), we can assume without loss of generality that for some μ-null set $\widehat{N}$ (i.e., $\mu(\widehat{N}) = 0$)

$$\frac{1}{n}\sum_{k=1}^{n} U^k(\omega) \xrightarrow{n} \widehat{U}(\omega) \in R^m \text{ for all } \omega \in \Omega \backslash \widehat{N}.$$

Therefore,

$$\frac{1}{n}\sum_{k=1}^{n} U^k(\omega) \xrightarrow{n} \widehat{U}(\omega) \text{ for all } \omega \in \Omega \backslash N \text{ where } N = \widehat{N} \cup N^\infty.$$

By Proposition 1 in Page (1991),

$$\widehat{U}(\omega) \in coLs\{U^n(\omega)\} \text{ and we know already that } \widehat{U}(\omega) = U^*(\omega) \text{ for all } \omega \in \Omega \backslash N.$$

Here "*co*" denotes convex hull and $Ls\{U^n(\omega)\}$ is the set of cluster points of the sequence $\{U^n(\omega)\}_n$.

Second, applying the Kuratowski-Ryll-Nardzewski Theorem (1965), let $\widetilde{U}(\cdot)$ be a measurable selector of $coLs\{U^n(\cdot)\}$. Thus, we have $\widetilde{U}(\omega) \in coLs\{U^n(\omega)\}$ *for all* $\omega \in \Omega$, and therefore,

$$\widetilde{U}(\omega) = \widehat{U}(\omega) = U^*(\omega) \text{ for all } \omega \in \Omega \backslash N.$$

By Theorem 8.2 in Wagner (1977), $\widetilde{U}(\cdot)$ has a Caratheodory representation $\widetilde{U}(\omega) = \sum_{i=0}^{m} \widetilde{\alpha}^i(\omega)\widetilde{U}^i(\omega)$, where the R^m-valued functions $\widetilde{U}^0(\cdot), \widetilde{U}^1(\cdot), \ldots, \widetilde{U}^m(\cdot)$ are measurable selectors of $Ls\{U^n(\cdot)\}$ and the nonnegative functions $\widetilde{\alpha}^0(\cdot), \widetilde{\alpha}^1(\cdot), \ldots, \widetilde{\alpha}^m(\cdot)$ are measurable with $\sum_{i=0}^{m} \widetilde{\alpha}^i(\omega) = 1$ for all ω. Thus, for each i and each ω, $U^{in_k}(\omega) \xrightarrow{k} \widetilde{U}^i(\omega)$ in R^m for some subsequence $\{U^{in_k}(\omega)\}_k \subset R^m$ where $U^{in_k}(\omega) \in coP_{v^{n_k}}(\omega)$ for all k.

Third, Given that $\widetilde{U}(\omega) = \sum_{i=0}^{m} \widetilde{\alpha}^i(\omega)\widetilde{U}^i(\omega)$, the proof (that the payoff selection correspondence $v \longrightarrow \Sigma(coP_v(\cdot))$ is upper semicontinuous) will be complete if we can show that for each $\omega \in \Omega \backslash N$, $\widetilde{U}^i(\omega) \in coP_{v^*}(\omega)$ for $i = 0, 1, \ldots m$. To accomplish this, we need the following

Lemma (*): If $U^n(\omega) \xrightarrow{n} \widetilde{U}^i(\omega)$ in R^m, where $U^n(\omega) \in coP_{v^n}(\omega)$ for all n and if $v^n(\cdot) \underset{w^*}{\longrightarrow} v^*(\cdot)$, then $\widetilde{U}^i(\omega) \in coP_{v^*}(\omega)$.

Proof of Lemma (*): Again by Theorem 8.2 in Wagner (1977) each

$$U^n(\cdot) \in \Sigma(coP_{v^n}(\cdot))$$

has a Caratheodory representation

$$U^n(\omega) = \sum_{i=0}^{m} \alpha^{ni}(\omega)U^{ni}(\omega) \text{ for all } \omega \in \Omega,$$

where for all n, $U^{ni}(\omega) \in P_{v^n}(\omega)$ and $\sum_{i=0}^{m} \alpha^{ni}(\omega)(\omega) = 1$, $\alpha^{ni}(\omega)(\omega) \geq 0$ for $i = 0, 1, \ldots, m$. For each n, let $\sigma_D^{ni} \in \mathcal{N}_{v^n}(\omega)$ be such that for each player d, $U_d^{ni}(\omega) = u_d(\omega, \sigma_D^{ni})(v_d^n)$ and without loss of generality, assume that $\sigma_D^{ni} \longrightarrow \sigma_D^{*i}$, and

$$(\alpha^{n0}(\omega), \alpha^{n1}(\omega), \ldots, \alpha^{nm}(\omega)) \longrightarrow (\alpha^{*0}(\omega), \alpha^{*1}(\omega), \ldots, \alpha^{*m}(\omega)).$$

By the Continuity Lemma, we have for all players d,

$$U_d^n(\omega) = \textstyle\sum_{i=0}^{m} \alpha^{ni}(\omega)(\omega)U_d^{ni}(\omega) = \sum_{i=0}^{m} \alpha^{ni}(\omega)(u_d(\omega, \sigma_D^{ni})(v_d^n))$$

$$\underset{n}{\longrightarrow} \textstyle\sum_{i=0}^{m} \alpha^{*i}(\omega)(u_d(\omega, \sigma_D^{*i})(v_d^*)) = \sum_{i=0}^{m} \alpha^{*i}(\omega)U^{*i}(\omega) = \widetilde{U}_d^i(\omega).$$

Because $\nu \longrightarrow \mathcal{N}_v(\omega)$ has a closed graph, we know that $\sigma_D^{*i} \in \mathcal{N}_{v^*}(\omega)$. Thus, we conclude that each $U^{*i}(\omega) \in P_{v^*}(\omega)$, and thus we have for all $\omega \in \Omega$,

$$\sum_{i=0}^{m} \alpha^{*i}(\omega)U^{*i}(\omega) = \widetilde{U}_d^i(\omega) \in coP_{v^*}(\omega),$$

completing the proof of the Nowak-Raghavan Lemma.

Step 5: (*The Fixed Point Argument*)

Applying the Kakutani-Glicksberg Fixed Point Theorem (1952) to $v \longrightarrow \Sigma(coP_v(\cdot))$ we obtain an m-tuple of value functions

$$v(\cdot) = (v_d(\cdot)) \in \mathcal{L}_X^\infty$$

such that

$$v(\omega) \in coP_v(\omega) \text{ for all } \omega \in \Omega \backslash N \text{ where } \mu(N) = 0.$$

Let $v^*(\cdot) = (v_d^*(\cdot)) \in \mathcal{L}_X^\infty$ be a measurable selection of $coP_v(\cdot)$ such that $v^*(\omega) = v(\omega)$ for all $\omega \in \Omega \backslash N$. Thus, $v^*(\omega) \in coP_v(\omega)$ for all $\omega \in \Omega$ and because $coP_v(\omega) = coP_{v^*}(\omega)$ for all $\omega \in \Omega$, we have $v^*(\omega) \in coP_{v^*}(\omega)$ for all $\omega \in \Omega$.

Step 6: (*Construction of a Correlated Stationary Markov Equilibrium*)

By Theorem 8.2 in Wagner (1977) $v^*(\cdot)$ has a Caratheodory representation

$$v^*(\omega) = \sum_{i=0}^{m} \lambda^{i*}(\omega)v^{i*}(\omega) \text{ for all } \omega$$

where for all $i = 0, 1, \ldots, m$, $v^{i*}(\cdot) \in \mathcal{L}_X^\infty$ and $v^{i*}(\cdot) \in P_{v^*}(\omega)$ for all $\omega \in \Omega$. By the Measurable Implicit Function Theorem (Theorem 7.1 in Himmelberg 1975), there exists for each $i = 0, 1, \ldots, m$, a measurable selection of $\mathcal{N}_{v^*}(\cdot)$, that is, a measurable function

$$\omega \longrightarrow \sigma_D^{i*}(\omega) \in \prod_d \mathcal{P}(\Phi_d(\omega))$$

with $\sigma_D^{i*}(\omega) \in \mathcal{N}_{v^*}(\omega)$ for all ω, such that for each player $d \in D$, $i = 0, 1, \ldots, m$, and $\omega \in \Omega$

$$\begin{aligned} &v_d^{i*}(\omega) \\ &= u_d(\omega, \sigma_D^{i*}(\omega))(v_d^*) \\ &:= (1-\beta_d) r_d(\omega, \sigma_D^{i*}(\omega)) + \beta_d \textstyle\int_\Omega v_d^*(\omega') q(\omega'|\omega, \sigma_D^{i*}(\omega)). \end{aligned}$$

Thus, for each player $d \in D$, and $\omega \in \Omega$

$$v_d^*(\omega) = \textstyle\sum_{i=0}^m \lambda^{i*}(\omega) v_d^{i*}(\omega)$$

$$= \textstyle\sum_{i=0}^m \lambda^{i*}(\omega) \left[(1-\beta_d) r_d(\omega, \sigma_D^{i*}(\omega)) + \beta_d \int_\Omega v_d^*(\omega') q(\omega'|\omega, \sigma_D^{i*}(\omega))\right]$$

$$= (1-\beta_d) r_d(\omega, \underbrace{\sum_{i=0}^m \lambda^{i*}(\omega) \sigma_D^{i*}(\omega)}_{\sigma_D^{\lambda*}(\omega)}) + \beta_d \textstyle\int_\Omega v_d^*(\omega') q(\omega'|\omega, \underbrace{\displaystyle\sum_{i=0}^m \lambda^{i*}(\omega) \sigma_D^{i*}(\omega)}_{\sigma_D^{\lambda*}(\omega)})$$

For $d \in D$, let $w_d^*(\cdot) := \frac{v_d^*(\cdot)}{1-\beta_d}$. Substituting, we have for all $\omega \in \Omega$

$$w_d^*(\omega) = r_d(\omega, \sigma_D^{\lambda*}(\omega)) + \beta_d \int_\Omega w_d^*(\omega') q(\omega'|\omega, \sigma_D^{\lambda*}(\omega)). \; (**)$$

where $\sigma_D^{\lambda*}(\omega) = \sum_{i=0}^m \lambda^{i*}(\omega) \sigma_D^{i*}(\omega)$ and $\sigma_D^{i*}(\omega) \in \mathcal{N}_{w^*}(\omega)$ for all ω and $i = 0, 1, 2, \ldots, m$.

By classical results on discounted dynamic programming (e.g., Blackwell 1965), we conclude from (**) that for all players $d \in D$ and all starting states $\omega \in \Omega$

$$w_d^*(\omega) = E_d r_d(\sigma_D^{\lambda*})(\omega) := \sum_{n=1}^\infty \beta_d^{n-1} E_d r_d^n(\sigma_D^{\lambda*})(\omega).$$