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Matthews--Moore Single- and Double-Crossing

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Abstract

This article provides an introduction to the Matthews—Moore single- and double-crossing properties for screening problems with one-dimensional types. The relationship of Matthews—Moore single--crossing to the Mirrlees single-crossing property is discussed.

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Matthews–Moore Single- and Double-Crossing

Craig Brett and John A. Weymark

The single-crossing property of preferences introduced by Mirrlees (1971) is extensively used in screening models with one-dimensional type spaces. However, its use is limited to applications in which contracts are two dimensional. It is not widely known that a methodology due to Matthews and Moore (1987) provides a fruitful way of analyzing screening problems with a one-dimensional type space that does not require contracts to be two dimensional. The Matthews–Moore methodology employs either a single-crossing property distinct from that of Mirrlees or a related double-crossing property. Here, we provide an introduction to Matthews–Moore single- and double-crossing in the hope that we will thereby encourage the use of these tools in future analyses of screening problems with multi-dimensional contracts.

In a screening problem, individuals have private information about some characteristics of themselves that is of value to, but not known by, the principal. Individuals who share the same private characteristics are said to have the same *type*. Each type chooses its most preferred contract from the set of contracts on offer which, because of the asymmetric information, is the same for every type. This is the *incentive constraint*. An *allocation* consists of a contract for each type. A principal maximizes an objective function subject to the incentive constraint and one or more additional constraints. For example, in the Mussa and Rosen (1978) monopoly nonlinear pricing problem, a firm maximizes profit by choosing a schedule that specifies the payment as a function of the quality of the good chosen subject to the incentive constraint

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and a *participation constraint* that requires each type to obtain some common minimum utility level. In the Mirrlees (1971) optimal nonlinear income tax problem, a government chooses an income tax schedule to maximize a utilitarian social welfare function subject to the incentive constraint and a *materials balance constraint* that requires the total amount consumed of the single good not to exceed the amount produced.

The principal's optimization problem is complex in part because the incentive constraint is itself a set of optimization problems, one for each type. This complexity can be somewhat mitigated by noting that when types choose from a common schedule of options, each type weakly prefers what it chooses to what any other type chooses. A set of contracts with this property is said to satisfy the *self-selection* constraints. As a consequence, instead of having the principal offer a set of contracts from which each type chooses its most preferred contract, the principal can equivalently directly specify its own most preferred allocation subject to the self-selection constraints and any other constraints that might apply without any explicit optimization on the part of the types. Unfortunately, because of the nature of the self-selection constraints, the set of feasible allocations in this problem is non-convex. Hence, knowing that an allocation is locally optimal does not guarantee that it is globally optimal, as would be the case with a convex optimization problem. Moreover, it may be difficult to determine which of the self-selection constraints bind at a solution to the principal's optimization problem. Identifying the pattern of binding incentive constraints is important for characterizing the direction of distortions. This is a further source of complexity.

One of the virtues of the Mirrlees and Matthews–Moore single-crossing properties is that a local approach is nevertheless possible in spite of the non-convexity. When there are a finite number of types, the *local approach* proceeds by analyzing a *relaxed problem* in which only the adjacent self-selection constraints are considered. This approach is valid if all of the self-selection constraints are satisfied whenever the adjacent ones hold. These two single-crossing properties also facilitate the identification of the pattern of binding self-selection constraints. The validity of the local approach can be established using only the self-selection constraints without analyzing the principal's full optimization problem. However, to identify which self-selection constraints bind, it is not sufficient to only consider these constraints. In order to make these ideas precise, we now proceed more formally. Our formal discussion draws extensively on Matthews and Moore (1987).

There are a finite number of types of individuals, $i = 1, \ldots, n$. The ith type's private information is characterized by a scalar θ^i, with the types ordered so that $\theta^1 < \theta^2 < \cdots < \theta^n$. A *contract* is a vector $\mathbf{c} = (a_1, \ldots, a_m, b) \in \mathbb{R}^{m+1}_+$. A *menu of contracts* is a set $\mathcal{C} = \{\mathbf{c}^1, \ldots, \mathbf{c}^n\}$, where the ith of these contracts is the one designed for type i. The *utility function* $U\colon \mathbb{R}^{m+1}_+ \times [\theta^1, \theta^n]$ specifies the utility $U(\mathbf{c}, \theta)$ that an individual of type θ obtains with the contract $\mathbf{c}$. It is assumed that U is continuously differentiable in $\mathbf{c}$ with $U_b < 0$. Whether U is monotone in any of its other arguments depends on the ap-

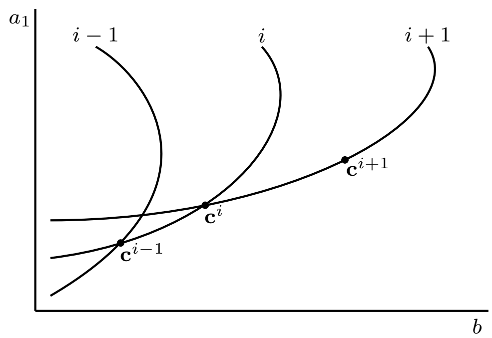

Fig. 1 Mirrlees single-crossing

plication. The first m components of a contract are the *attributes* and the last component is the *outlay.* For example, in the Mussa and Rosen (1978) monopoly pricing problem, the single attribute is the quality of a good and the outlay is the payment. In the Mirrlees (1971) income tax problem, the single attribute is after-tax consumption and the outlay is pre-tax income.

The menu $\mathcal{C}$ satisfies the *self-selection constraints* if

$$U(\mathbf{c}^i, \theta^i) \geq U(\mathbf{c}^j, \theta^i) \text{ for all } j \neq i$$

and it satisfies the *adjacent self-selection constraints* if

$$U(\mathbf{c}^i, \theta^i) \geq U(\mathbf{c}^j, \theta^i) \text{ for all } i \text{ and for } j \in \{i-1, i+1\}.$$

In the first case, a type weakly prefers the contract designed for it to the contract designed for any other type, whereas in the second, a type is only required to weakly prefer its own contract to those of the adjacent types.

The utility function U satisfies the MRS-*ordering property* if for all $\mathbf{c}$,

$$-\frac{U_k(\mathbf{c}, \theta)}{U_b(\mathbf{c}, \theta)} \text{ is (i) increasing in } \theta \text{ for all } k \text{ or (ii) decreasing in } \theta \text{ for all } k.$$

In other words, the marginal rates of substitution between an attribute and the outlay are ordered by type in the same way for each attribute. When $m = 1$ (in which case contracts are two dimensional), the utility function U satisfies the *Mirrlees single-crossing property* if

> an indifference curve of any type intersects an indifference curve of any other type at most once.

This is simply the MRS-ordering property for $m = 1$. Mirrlees single-crossing is illustrated in Figure 1 for the case in which the MRS is decreasing in type.

The *contract utility curve* for the contract $\mathbf{c}$ is the graph of the function $U(\mathbf{c}, \cdot)$ on the domain $[\theta^1, \theta^n]$. It shows what the utility is of each possible

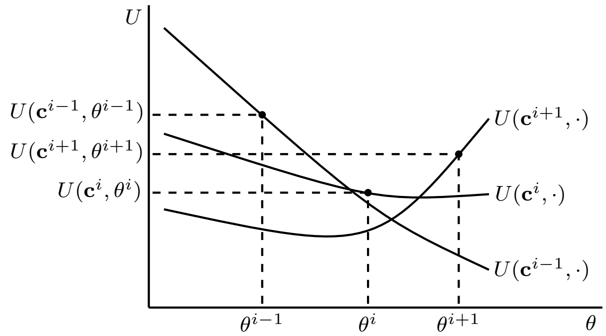

Fig. 2 Matthews–Moore single-crossing

type in the interval $[\theta^1, \theta^n]$ with the contract $\mathbf{c}$. Associated with the menu of contracts $\mathcal{C} = \{\mathbf{c}^1, \ldots, \mathbf{c}^n\}$ is the *menu of contract utility curves* $\mathcal{U}^{\mathcal{C}} = \{U(\mathbf{c}^1, \cdot), \ldots, U(\mathbf{c}^n\}$. A menu of contracts $\mathcal{C}$ satisfies the *Matthews–Moore single-crossing property* if

> no two distinct contract utility curves in $\mathcal{U}^{\mathcal{C}}$ (i) intersect more than once or (ii) are tangent to each other at any point in (θ^1, θ^n).

Matthews–Moore single-crossing is illustrated in Figure 2.

Because utility levels and types are both scalars, it is possible to satisfy the Matthews–Moore single-crossing property no matter how many attributes there are in a contract. In contrast, it is not possible for indifference contours to cross only once when there is more than one attribute, so it is not possible to extend the Mirrlees single-crossing property to higher dimensions.

The menu $\mathcal{C}$ is *weakly attribute ordered* if

$$\text{for each pair } \{i, j\}, \text{(i) } c_k^i \leq c_k^j \text{ for all } k \text{ or (ii) } c_k^i \geq c_k^j \text{ for all } k$$

and it is *attribute ordered* if

$$\text{(i) for all } i < j,\ c_k^i \leq c_k^j \text{ for all } k \text{ or (ii) for all } i < j,\ c_k^i \geq c_k^j \text{ for all } k.$$

In the first case, for any pair of types, one of them has weakly more of every attribute than the other. In the second, the attributes are either all nondecreasing or all nonincreasing in type. Clearly, a menu is weakly attribute ordered if it is attribute ordered.

Matthews and Moore (1987) show that weak attribute ordering provides a link between the MRS-ordering property and their single-crossing property.

Theorem 1. *If the utility function U satisfies the* MRS-*ordering property and the menu of contracts $\mathcal{C}$ is weakly attribute ordered, then $\mathcal{C}$ satisfies the Matthews–Moore single-crossing property.*

When $m = 1$, a menu of contracts is necessarily weakly attribute ordered.[1] Thus, the Mirrlees single-crossing property implies the Matthews–Moore single-crossing property and, hence, the latter is less restrictive.[2]

The following theorem due to Matthews and Moore (1987) shows that the local approach is justified if their single-crossing property holds.

Theorem 2. *If the menu of contracts $\mathcal{C}$ satisfies the adjacent self-selection constraints and the Matthews–Moore single-crossing property, then $\mathcal{C}$ satisfies the self-selection constraints.*

By Theorem 1, the analogous result holds for the Mirrlees single-crossing property when $m = 1$. Figure 2 can be used to illustrate Theorem 2. For any type θ, satisfaction of the self-selection constraints requires that the contract utility curve for this type's contract must lie weakly above the curves for the other contracts at θ. This is the case for the three types in the figure. If the adjacent self-constraints are satisfied but, say, type θ^{i-1} prefers $\mathbf{c}^{i+1}$ to its own contract, then the contract utility curve for $\mathbf{c}^{i+1}$ would have to intersect the contract utility curve for $\mathbf{c}^{i}$ at least twice for this to be possible, violating the Matthews–Moore single-crossing property.

Even when the local approach cannot be employed, it may nevertheless be possible to characterize the solution to a screening problem by solving a relaxed problem in which some of the self-selection constraints are not considered, particularly if it can be determined which of them are binding. For example, when $m = 1$, if the Mirrlees single-crossing property is satisfied and the menu of contracts is attribute ordered, all of the self-selection constraints are satisfied if all of the adjacent downward (or adjacent upward) self-section constraints are binding (see Figure 1).

Matthews and Moore (1987) consider an extension of the Mussa and Rosen (1978) monopoly pricing problem in which a warranty level is an additional attribute. In their analysis, they introduce the following property. A menu of contracts $\mathcal{C}$ satisfies the *Matthews–Moore double-crossing property* if

no two distinct contract utility curves in $\mathcal{U}^{\mathcal{C}}$ intersect more than twice.

In their model, this double-crossing property of contract utility curves enables Matthews and Moore to characterize the optimal solution without considering the upward self-selection constraints. In order to show that the solution to

[1] A menu of contracts is attribute ordered if the Mirrlees single-crossing property and the adjacent self-selection constraints are satisfied. However, if, as in Figure 1, the utility function is not monotonic in this attribute, the outlays need not be monotone in type.

[2] When a set of alternatives and preference types are both one dimensional, a profile of preferences satisfies the *preference single-crossing property* if the direction of preference between any two alternatives reverses only once as the type increases. In a screening problem with $m = 1$, if the two contract components are monotone in type, the menu of contracts is effectively one dimensional, in which case there are single-crossing preferences if and only if the Mirrlees single-crossing property holds. See Gans and Smart (1996).

this relaxed problem solves the unrelaxed problem, they first show that all of the adjacent downward self-section constraints bind.

Since they were introduced, the Matthews–Moore single- and double-crossing properties have rarely been employed. Matthews–Moore single-crossing is used by Bohn and Stuart (2013) and by van Egteren (1996) to study majority voting over income tax schedules and the regulation of a public utility, respectively.[3] Brett (1998) uses Matthews–Moore double-crossing to investigate when workfare should supplement income taxation.

The MRS-ordering property is a natural assumption in many screening problems with multi-dimensional contracts and a one-dimensional type space. When it holds, it is only necessary to determine if the menu of contracts is weakly attribute ordered in order to conclude that the Matthews–Moore single-crossing property is satisfied and, hence, that the local approach is valid. It is more difficult to identify situations in which the Matthews–Moore double-crossing property is satisfied, but as in the models of Matthews and Moore (1987) and Brett (1998), reasonable restrictions may imply that it is.

An obvious limitation of the single- and double-crossing properties considered here is that they assume that the type space is one dimensional. With a multi-dimensional type space, it is possible for the projections of the indifference curves or the utility contract curves for each dimension of the type space to satisfy one of these properties. For the case of the Mirrlees single-crossing property, Armstrong and Rochet (1999) and Brett (2007) illustrate the usefulness of this observation for two-dimensional type spaces.

References

Armstrong M, Rochet JC (1999) Multi-dimensional screening: A user's guide. European Economic Review 43:959–979

Bohn H, Stuart C (2013) Revenue extraction by median voters, unpublished manuscript, Department of Economics, University of California, Santa Barbara

Brett C (1998) Who should be on workfare? The use of work requirements as part of an optimal tax mix. Oxford Economic Papers 50:607–622

Brett C (2007) Optimal nonlinear taxes for families. International Tax and Public Finance 14:225–261

Gans JS, Smart M (1996) Majority voting with single-crossing preferences. Journal of Public Economics 59:219–237

Matthews S, Moore J (1987) Monopoly provision of quality and warranties: An exploration in the theory of multi-dimensional screening. Econometrica 55:441–467

Mirrlees JA (1971) An exploration in the theory of optimum income taxation. Review of Economic Studies 38:175–208

Mussa M, Rosen S (1978) Monopoly and product quality. Journal of Economic Theory 18:301–317

van Egteren H (1996) Regulating an externality-generating public utility: A multi-dimensional screening approach. European Economic Review 40:1773–1797

[3] Bohn and Stuart do not cite Matthews and Moore (1987), which suggests that they independently recognized the significance of having single-crossing contract utility curves.