GAMES WITH MANY PLAYERS AND ABSTRACT ECONOMIES PERMITTING DIFFERENTIATED COMMODITIES, CLUBS, AND PUBLIC GOODS

by

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Games with many players and abstract economies permitting differentiated commodities, clubs, and public goods

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Abstract

In a seminal paper relating economic and game theoretic structures, Shapley and Shubik (1969) demonstrate that a game is a market game – that is, a game derived from a finite-dimensional private goods exchange economy where all participants have continuous, concave utility functions. In this paper, to accommodate models of economies with public goods, clubs, indivisibilities, and other deviations from the classic model of Shapley and Shubik, we demonstrate an equivalence between homogeneous market games with many players, possibly all with different characteristics, and abstract economies permitting differentiated commodities, clubs, public goods, coalition production, unbounded short sales and other deviations from standard economic models. By a homogeneous market game we mean a game derived from a market where all individuals have the same concave and continuous utility function. We also demonstrate that the condition of small group effectiveness – that small groups of players can realize almost all gains to collective activities – is equivalent to the condition of asymptotic negligibility – that small groups of players cannot have significant impacts on average payoff to large groups of players.

1 Market games

The relationship between games and markets has motivated a vast amount of research. Early papers on this topic data back to Shubik (1959), making the connection between Edgeworth's contract curve and the core of a game. Other influential early papers on this topic include Debreu and Scaf (1963), Aumann (1964,1966), Shapley and Shubik (1967,1975). All these papers, and many since, connect outcomes of economic market equilibrium with solution concepts for cooperative games, such as the core or the Shapley value. Research connecting the *structures* of markets and games rather than solution concepts is relatively sparse. A first paper taking such a step is Shapley and Shubik (1969), relating markets and cooperative games.

Shapley and Shubik (1969) show that a class of market games – games derived from exchange economies with a finite number of commodities and with continuous, concave and quasi-linear utility functions – is equivalent to the class of totally balanced games – games with the property that every subgame of a game in the class (including the game itself) has a nonempty core. Shapley and Shubik also characterize a canonical utility function generated by a game, where the commodities are the players themselves, and demonstrate that for this canonical market, the set of price-taking equilibrium outcomes is equivalent to the core.

Shubik and Wooders (1982), using Wooders's earlier results on nonemptiness of approximate cores of games with many players, demonstrate that a broad class of economies, including ones with indivisibilities, non-convexities, clubs with multiple memberships and possibly ever-increasing returns to club size, and public goods – local and pure – are approximately market games. Shubik and Wooders restrict to replication of the set of players and a finite number of types of players. They use the same assumption of Wooders' earlier research of *per capita boundedness* – that is, finiteness of the supremum of average or per capita payoffs.[1] Wooders (1994) continues this research by relaxing the assumption of replication and characterizing the limiting canonical utility function. This utility function has the property that, when a market (or game) has many players, then the per-capita payoff to any large group[2] of players is approximated by the value of the limiting utility function where the player *types* are the commodities. Although no assumptions of concavity or continuity of utility functions are made on the markets themselves, the limiting utility function is continuous, concave and one-homogeneous.[3] Like Shubik and Wooders, Wooders treats situations with only a finite number of player types. To relax the replication restriction, she requires a slightly stronger assumption than Shubik and Wooders, the assumption of *small group effectiveness*, that almost all gains to collective activities can be realized by cooperation only within elements of some partition of the total player set into groups bounded in size. Wooders (1994, Theorem 4), however, shows that if "scarce types" of players (player types appearing in vanishing small proportions in large games) are ruled out, then her assumption of small group effectiveness is equivalent to per capita boundedness. Small group effectiveness effectively just controls behavior of games with vanishingly small percentages of players of some types.[4] The results of Shubik and Wooders (1982) and Wooders (1994) require only *essential superadditivity* – the property that a group of players can subdivide into smaller groups and each group achieve its own value.[5] The finite-type assumption of these papers is restrictive. Suppose, for example, that there is a finite number of types of players and players have preferences over the set of players with whom they jointly consume or produce commodities – that is, the economy is one with clubs or local public goods or coalition production.

[1]See Wooders (1979,1983) and earlier State University of New York-Stony Brook working papers, available from the SUNY library and the web pages of this author.

[2]We use the term 'group' rather than 'coalition' since the term 'coalition' implicitly suggests some cooperation.

[3]These results are confirmed in a model with a continuum of agents in Azrieli and Lehrer (2007). See also Evstigneev and Flam (2001).

The concavity of the limiting utility function for the TU case of Wooders (1983) was noted by Aumann (1987).

[4]A number of examples making this point appear in Wooders (2008).

[5]See Wooders (2008) for a discussion of essential superadditivity and a statement of the equivalence of the core of a game satisfying essential superadditivity and the core of a 'superadditivie cover game'.

Without some bound on the number of different sorts of clubs or on club sizes, even with a finite number of types of players, the set of commodities becomes infinite as the total player set becomes large.

The current paper extends the market-game equivalence of games and economies with many players to games where the set of player types is a compact metric space and where the worth of a coalition is determined by the numbers of players of each of type in the coalition. Following Shapley and Shubik (1969), we show that games with many players generate markets where all participants in the market have the same concave utility function and the commodity space is equivalent to the space of player types. (Note that this gives us a compact metric space of commodities in the economy generated by a game.) The only conditions required on the games are essential superadditivity, small group effectiveness, and a continuity condition ensuring that players who have similar types make similar contributions to any coalitions they might join. As discussed further below, rather than assuming small group effectiveness, we could equivalently require per capita boundedness and an 'asymptotic negligibility' condition, ensuring that small groups of players cannot significantly affect average payoffs to large groups.

To describe the class of economies that generate games satisfying our conditions, we must allow differentiated commodities. To allows club, public goods, indivisible commodities, coalition production and so on, we specify, for each conceivable group of participants in an economy, an abstract feasibility correspondence. Some examples illustrate the breadth of the model. We require a continuity condition ensuring that economic participants who are similar are close substitutes, that is, they have similar utility fuctions, consumption sets and endowments. Not surprisingly, we also require per capita boundedness and essential superadditivity – our abstract feasibility condition allows groups to partition into smaller groups and carry out collective activities within these smaller groups. We then show that any economy with many participants generates a game satisfying the conditions of our game-theoretic model. In the course of obtaining this result, we show that markets with many participants are approximated by markets where all players have the same concave, one-homogeneous utility function.

Small group effectiveness is a key assumption in our model so we provide some characterization. If each player (or each participant in any economy) has many close substitutes then small group effectiveness is simply per-capita boundedness. Clearly, per capita boundedness is a minimal condition; if it did not hold, then for some sequences of growing games (or economies) the average payoffs to players would approach infinity as the size of the total player set increased. But if there are small groups of players who are of scarce types, then if small group effectiveness does not hold then these players could prevent large games from becoming 'market like' or market games. We demonstrate that small group effectiveness is equivalent to the effectiveness of small groups for improvement – the property that any payoff vector which can be improved upon by the total player set (or some partition of the total player set) can be improved upon (although perhaps by slightly less per capita) by a coalition bounded in size, where the bound depends on the distance between the payoff vector and the total value of the total player set. We also demonstrate the equivalence of small group effectiveness and asymptotic negligibility which is the cornerstone of competitive economies.

1.1 Competitive economies and games

The most interesting characterization of small group effectiveness is its equivalence to asymptotic negligibility, dictating that vanishingly small groups of players have only negligible impacts on aggregate outcomes. We interpret asymptotic negligibility as an asymptotic form of the assumption of a continuum of players. Aumann (1964) remarked that the defining characteristic of perfect competition is that the economy under consideration has a very large number of participants, each of whom has negligible influence on market outcomes. Aumann writes:

> *The influence of an individual participant on the economy cannot be mathematically negligible, as long as there are only finitely many participants. Thus a mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants. We submit that the most natural model for this purpose has a continuum of participants.*

Aumann continues:

> *The reason for this is that one can integrate over a continuum, and changing the integrand at a single point does not affect the value of the integral, that is, the actions of a single individual are negligible.*

More precisely, Aumann is saying that changing the integrand on a *set of measure zero* does not change the value of the integral; no small group of individuals, constituting a negligible fraction of the total economy, can change aggregate economic outcomes. Our research suggests that *the economic reason the notion of perfect competition is built into the continuum model is that negligibility of activities of small groups implies that all gains to collective activities can be achieved by small groups. Thus large groups are inessential and gain no market power from size.* Paradoxically, in large economies large coalitions may have market power if and only if small groups of participants have significant effects on aggregate economic outcomes.

The assumption implicit in a continuum model is that groups of participants constituting negligible fractions of the total economy have negligible effects on aggregate outcomes; that is, sets of measure zero can be ignored. An asymptotic statement of this assumption is that in economies with increasing numbers of participants, relatively small groups of participants can have only vanishingly small effects on aggregate outcomes to large groups– asymptotic negligibility.

Our research shows that the assumption of asymptotic negligibility plays a role even in the existence of equilibrium in large economies. Under certain assumptions Aumann (1966) shows that in an exchange economy with a continuum of participants the convexity assumption used to ensure existence of equilibria in finite economies is not required. We present an example showing that existence of equilibria does not imply asymptotic negligibility. The example is then modified to show that, with unbounded short sales or significant non-monotonicities, when the assumptions on the economy do not ensure asymptotic negligibility an equilibrium may not exist, *no matter how large the economy.* Thus, in general, existence of equilibrium in the continuum depends on the postulate of asymptotic negligibility. We note, however, existence of equilibrium in our setting is an open question.

2 Games and pregames; The cooperative game-theoretic structures

2.1 Games

A *game* (with side payments) is a pair (N, v) where N is a finite set (the set of *players*) and v is a function, called the *worth function* (of the game), from the set 2^N of subsets of N to the set $\mathbb{R}_+$ of nonnegative real numbers, with the property that $v(\emptyset) = 0$. Nonempty subsets S of N are called *groups* (or *coalitions*) and the number $v(S)$ is the *worth* of the group S. If the player set N is understood, we frequently refer to v itself as the game.

A *payoff vector* for the game (N, v) is a vector x in $\mathbb{R}_+^N$. Given $S \subset N$ define $x(S) := \sum_{i \in S} x_i$. A payoff vector is *feasible* if there is a partition of N into (disjoint) coalitions, say $\{S_1, ..., S_K\}$, such that

$$x(N) \leq \sum_{k=1}^{K} v(S_k). \tag{1}$$

A game (N, v) is *superadditive* if for all disjoint subsets S and $S^{'}$ of N

$$v(S \cup S') \geq v(S) + v(S').$$

If a game satisfies superadditivity (or we treat a superadditive cover game) then (1) is equivalent to the condition that

$$x(N) \leq v(N). \tag{2}$$

Alternatively, one can define the *superadditive cover of a game* (N, v), say (N, v^*), with worth function

$$v^*(S) := \max_{P(S)} \sum_{k=1}^{K} v(S_k)$$

where $\{S^1, ..., S^K\}$ is a partition of S. Our results in this paper require only *essential superadditivity*, which means that the relevant feasibility condition is given by (1). But with that in mind, we can and will, without loss of generality, assume superadditivity – using essential superadditivity just increases notational complexity.[6]

For $\varepsilon \geq 0$, the payoff x is the ε*-core* of (N, v) if x is feasible and

$$x(S) \geq v(S) - \varepsilon|S| \text{ for all coalitions } S,$$

where $|S|$ denotes the number of elements in the set S. When $\varepsilon = 0$ the ε-core is simply the *core*. The ε-core consists of those feasible payoffs with the property that no group of players could be better off by ε per-capita.

[6] For the reader wishing further discussion and clarification, see Wooders (2008).

2.2 Pregames

The framework of a (cooperative) pregame allows us to formalize the notion of a large game for which the worth of a group of players depends only on the attributes of its members, and players with similar attributes are approximately substitutes.

Let Ω be a compact metric space, interpreted as a set of player "types" or attributes. A (player) *profile* on Ω, interpreted as a description of a group of players in terms of numbers of players of each type in the group, is a function f from Ω to the set $\mathbb{Z}_+$ of nonnegative integers for which the *support* $\sigma(f)$ of f, given by

$$\sigma(f) = \{\omega \in \ \Omega : f(\omega) \neq 0\},$$

is finite. A profile is simply a function f from Ω to the nonnegative integers with the property that $f(\omega) \neq 0$ for only a finite number of elements ω in Ω. For each $\omega \epsilon \Omega$, we interpret $f(\omega)$ as the number of players of type ω (or, in other words, with attributes ω), in a group of players described by f. The set of profiles on Ω is denoted by $P(\Omega)$. We write $f \leq g$ if $f(\omega) \leq g(\omega)$ for each ω in Ω; for ω_0 in Ω we write χ_{ω_0} for the profile given by

$$\chi_{\omega_0}(\omega) = \left\{ \begin{array}{c} 0 \text{ if } \omega \neq \omega_0, \\ 1 \text{ if } \omega = \omega_0. \end{array} \right\}$$

By the *norm* of a profile, we mean

$$\|f\| = \sum_{\omega \epsilon \sigma(f)} f(\omega),$$

which is simply the number of players in a group represented by f. This is a finite sum since f has finite support.

A *pregame* is a pair (Ω, Ψ) where Ω is a compact metric space, called the *space of attributes* and $\Psi : P(\Omega) \rightarrow R_+$, called the *worth function (of the pregame)*, is a function with the following properties:

(a) $\Psi(0) = 0$ and;

(b) given any $\varepsilon > 0$ there is a $\delta > 0$ such that
for each pair of player types ω_1 and ω_2 with $dist(\omega_1, \omega_2) < \delta$
it holds that $|\Psi(f + \omega_1) - \Psi(f + \omega_2)| < \varepsilon$ (continuity). (3)

The first condition means that zero players can realize nothing. The second is that players with similar attributes are nearly substitutes. Without loss of generality,[7] we shall restrict attention to superadditive pregames satisfying the condition that $\Psi(f) + \Psi(g) \leq \Psi(f + g)$ for all profiles f and g. This expresses the idea that an option open to a group is to split into several smaller groups.

We frequently refer to the elements of Ω as "types". Players of the same type are substitutes.

2.3 Games induced by pregames

To derive a game from a pregame (Ω, Ψ), we specify a finite set N and a function $\alpha : N \to \Omega$, called an *attribute function.* With any subset S of N we can then associate a player profile, $prof(\alpha|S)$, given by

$$prof(\alpha|S)(\omega) = |\alpha^{-1}(\omega) \cap S|.$$

The profile $prof(\alpha|S)(\omega)$ simply lists the numbers of players of each type in the subset S. We have now determined a game (N, v_α) with worth function given by:

$$v_\alpha(S) = \Psi(prof(\alpha|S))$$

for each $S \subset N$. We have now defined a game (N, v_α).

Let f be a profile. When $\sum f^k = f$ for some collection of profiles $f^1, ..., f^K$, not necessary distinct, we say that the collection is *a partition of* f and each member of the collection is called a *subprofile of* f. Obviously, a partition of a profile is related to a partition of a set of players. If (N, v_α) is a game derived from (Ω, Ψ), and $\{S_1, ..., S_K\}$ is a partition of N, then

$$\{f^k : prof(\alpha|S_k) = f^k, k = 1, ..., K\} \text{ (with repetitions allowed)}$$

is a partition of $prof(\alpha|N)$.

3 Small group effectiveness

A pregame (Ω, Ψ) satisfies *small group effectiveness* if for each positive real number $\varepsilon > 0$ there is an integer $\eta_1(\varepsilon)$ such that for each profile f, for some

[7] See the discussion above regarding essential superadditivity and also Wooders (2008).

partition $\{f^k\}$ of f :

$$\begin{aligned} &\text{(a)} \ \ \|f^k\| \leq \eta_1(\varepsilon) \text{ for each profile } f^k \text{ in the partition,} \\ &\text{(b) } \Psi(f) - \textstyle\sum_k \Psi(f^k) \leq \varepsilon \|f\|. \end{aligned} \tag{4}$$

Small group effectiveness means that given a measure of per capita approximation (an $\varepsilon > 0$) there is an absolute bound on group sizes with the property that almost all gains to collective activities can be realized by groups of players smaller in size than that bound, that is, bounded group sizes nearly (within ε per player) exhaust all gains to scale of collective activities. As noted in the introduction, when player profiles are required to have many substitutes for each player, small group effectiveness is equivalent to per capita boundedness (Wooders 1994, Theorem 4). A pregame (Ω, Ψ) satisfies *per capita boundedness* if there is a constant A such that

$$\frac{\Psi(f)}{\|f\|} \leq A \text{ for all profiles } f \ \in P(\Omega). \tag{5}$$

Thus, small group effectiveness is an extremely mild, yet powerful condition.

The concept of small group effectiveness requires that almost all feasible gains to collective activities can be achieved by groups bounded in absolute size. A related concept requires that almost all improvement be feasible for groups bounded in absolute size. A pregame (Ω, Ψ) satisfies *small group effectiveness for improvement* if for each positive real number $\varepsilon > 0$ there is an integer $\eta_2(\epsilon)$ with the following property:

$$\begin{aligned} &\text{For any profile } f \text{ and any payoff function } x : \sigma(f) \to R_+ \\ &\text{if } x(f) + \epsilon \|f\| < \Psi(f) \text{ then there is a subprofile } g \text{ of } f \text{ such that} \\ &\qquad\qquad \|g\| \leq \eta_2(\epsilon) \text{ and } \ x(g) + \tfrac{\epsilon}{2} \|g\| < \Psi(g). \end{aligned} \tag{6}$$

The following Theorem relates the effectiveness of small groups for the attainment of feasible payoffs to the effectiveness of small groups for improvement on payoffs. (All proofs are contained in the Appendix.)

Theorem 1: **Equivalence of small group effectiveness for feasibility and for improvement.** A pregame (Ω, Ψ) satisfies small group effectiveness for improvement if and only if it satisfies small group effectiveness.

4 Equivalence of asymptotic negligibility and small group effectiveness

Roughly, the property of large games we introduce is that relatively small groups of players make only "negligible" contributions to per-capita (or, in other words, average) payoffs of large groups. A pregame (Ω, Ψ) satisfies *asymptotic negligibility* if, for any sequence of profiles $\{f^\nu\}$ where

$$\begin{gathered} \|f^\nu\| \to \infty \text{ as } \nu \to \infty, \\ \sigma(f^\nu) = \sigma(f^{\nu'}) \text{ for all } \nu \text{ and } \nu' \text{ and} \\ \lim_{\nu\to\infty} \tfrac{\Psi(f^\nu)}{\|f^\nu\|} \text{ exists,} \end{gathered} \tag{7}$$

then for any sequence of profiles $\{\ell^\nu\}$ with

$$\lim_{\nu\to\infty} \frac{\|\ell^\nu\|}{\|f^\nu\|} = 0, \tag{8}$$

it holds that

$$\begin{gathered} \lim_{\nu\to\infty} \tfrac{\Psi\|f^\nu+\ell^\nu\|}{\|f^\nu+\ell^\nu\|} \text{ exists, and} \\ \lim_{\nu\to\infty} \tfrac{\Psi\|f^\nu+\ell^\nu\|}{\|f^\nu+\ell^\nu\|} = \lim_{\nu\to\infty} \tfrac{\Psi(f^\nu)}{\|f^\nu\|} \ . \end{gathered} \tag{9}$$

The property of asymptotic negligibility appears quite mild. It simply ensures that in "thick" games – ones with many players of a few types, a small group of possibly distinct player types cannot significantly affect per capita payoffs of members of the large player set. For pregames satisfying per capita boundedness, asymptotic negligibility is equivalent, however, to small group effectiveness. From Theorem 1 and Theorem 2 below, when it is postulated that a small group of participants cannot affect economic aggregates it is also necessarily postulated that all gains to collective activities, both for attainment of the total payoff and for improvement, can be realized by small groups.

Theorem 2: Equivalence of small group effectiveness and asymptotic negligibility: Let (Ω, Ψ) be a pregame satisfying per capita boundedness (5). Then (Ω, Ψ) satisfies small group effectiveness if and only if it satisfies asymptotic negligibility.

We note that the restriction to pregames satisfying per capita boundedness in the above Theorem is required to ensure that asymptotic negligibility implies small group effectiveness. Small group effectiveness itself implies per capita boundedness.

5 Pre-economies with differentiated commodities, clubs, public goods, and unbounded short sales...for example

In this Section we develop the construct of a pre-economy allowing differentiated commodities, as in Mas-Colell (1975) or Jones (1984), for example. Feasibility of allocations, however, will be specified in a manner permitting private goods, public goods, clubs, coalition production, and public goods as special cases. We introduce the concept of social continuity, which ensures that the pregame induced by a pre-economy satisfies asymptotic negligibility and provide examples illustrating the concept. We call this 'social continuity' since it implies that when the economy is large it is 'as if' all participants have the *same* continuous and concave utility function. Our main results in this section, Theorem 5 and Theorem 6, also requires that participants in an economy who are close in the metric on the space of types of participants are close substitutes. Theorem 5 states that the notion underlying the use of a continuum of players in cooperative games and the implicit (but extremely important) assumption that small groups of individuals have negligible effects on market outcomes – that set of measure zero can be ignored– is equivalent to the assumption that the economy can be represented as a market where all participants have the same continuous and 1-homogeneous payoff function. Theorems 4 and 5 also establish that large games with a compact metric space of player types and effective small groups are asymptotically equivalent to market games, thus extending the market-game equivalence results of Shapley and Shubik (1969) and Wooders (1994) to broader classes of economies.

Let $\mathcal{C}$ denote a compact metric space of types of commodities. A *commodity bundle* is a function x with finite support from $\mathcal{C}$ to the nonnegative real numbers; thus each commodity bundle contains only a finite number of distinct commodities. Let $\mathcal{B}$ denote the set of commodity bundles. The space $\mathcal{B}$ is metrizable and an example of a metric for the space is the Prohorov metric (cf., Hildenbrand 1974, p.49). Let ρ denote a metric on the space $\mathcal{B}$.

There is a given compact metric space $\mathcal{T}$ of trader types (or, in other words) attributes. A *type* $t \in \mathcal{T}$ is identified with a triple (e_t, u_t, X_t) consisting of an *endowment* $e_t \in X_t$, a *consumption set* $X_t \subset \mathcal{B}$, and a *utility function* $u_t : X_t \to \mathbb{R}$. An arbitrary element of X_t is called a *consumption.* Note that $\mathcal{T}$ may be a subset of $\mathcal{C}$; this is important for our framework is to encompass clubs or local public goods or other situations where individuals care about the characteristics of other agents. A *popuation profile* is a mapping, with finite support, from $\mathcal{T}$ to the non-negative integers $\mathbb{Z}_+$. In interpretation a population profile f describes a population by the numbers of participants of each type in the population (where, if $\mathcal{T}$ is not a finite set, then there are no players of 'most' types). When no confusion is likely to arise, we shall often call a population profile simply a profile.

To allow our framework to encompass both private and public goods, we postulate an abstract feasibility correspondence. In interpretation this correspondence specifies the consumptions that are feasible and individually rational for a population profile with a given availability of commodities. A *feasibility correspondence* $\widehat{\mathcal{A}}$ specifies, for each profile f, a set of $\|f\|$ consumption bundles,

$$\widehat{\mathcal{A}}(f) \subset \{x_{tq} \in X_t : u_t(x_{tq}) \geq u_t(e_t),\ t \in \sigma(f),\ q = 1, ..., f(t)\}. \tag{10}$$

The commodity bundle x_{tq} is interpreted as the bundle allocated to the q^{th} participant of type in the profile. Unless $\widehat{\mathcal{A}}(f)$ permits an individually rational commodity bundle for each participant, $\widehat{\mathcal{A}}(f)$ may be empty. Since $e_t \in X_t$, however, it holds that, for any profile f with $\|f\| = 1$ (so f describes a population consisting of only one participant) $\widehat{\mathcal{A}}(f) \neq \emptyset$. Note that the commodities could be private goods, public goods, or club goods and come in indivisible units. Also, coalition production may be allowed; it may be that some of the commodities are produced.

We assume that the pre-economy satisfies essential superadditivity – that is, anything that is feasible for a collection of subprofites of a profile f is feasible for f, thus permitting congestion effects that are internalizablie withing groups, local public goods, and/or club goods. More specifically, define a new feasibility correspondence $\mathcal{A}$ by

$$\mathcal{A}(f) = \cup_P \cup_k \widehat{\mathcal{A}}(f^k)$$

where P denotes the set of all partitions of f.

Example 1. An abstract feasibility correspondence allows our model to accommodate private goods and public goods. For example, all members of $\widehat{\mathcal{A}}(f)$ may be required to satisfy the condition that

$$\sum_{t\in\sigma(f)}\sum_{q=1}^{f(t)} x_{tq} = \sum_{t\in\sigma(f)}\sum_{q=1}^{f(t)} f(t)\ e_t; \tag{11}$$

in this case we call the feasibility correspondence an *exchange feasibility correspondence.* An exchange feasibility correspondence permits the types or attributes of players to be part of commoditity bundles. Alternatively, all members of $\widehat{\mathcal{A}}(f)$ may be required to satisfy the condition that $x_{tq} = x_{t'q'}$ for all $tq,\ t'q'$, in which case we call the feasibility correspondence a *public goods feasibility correspondence*. Note that there is no requirement that commodities be indivisible or that utility functions are monotonic or concave. The framework can also accommodate production and local public goods or shared goods.

Let $\{((e_t, u_t, X_t),\ t \in \mathcal{T}), \mathcal{A}\}$ be a pre-economy. Let $n : \mathcal{T} \to \mathbb{Z}_+$ have finite support, so n is a population profile (on $\mathcal{T}$). An *economy* $\mathcal{E}(n)$ with *population* n is given by

$$\mathcal{E}(n) = \{(u_t, e_t, X_t) : t \in\ \sigma(n)\}.$$

In interpretation, for each t in $\mathcal{T}$ there are $n(t)$ participants in the economy with attributes (e_t, u_t, X_t). Since the support of n is finite, $n(t) \neq 0$ for at most a finite number of participant types. Note that since $\mathcal{T}$ is a compact metric space, $\|n\|$may be arbitrarily large and yet no two participants necessarily have the same attributes. An *allocation* for an economy $\mathcal{E}(n)$ is a collection of consumptions $\{x_{tq} \in X_t : t = 1, ..., \|\sigma(n)\|\,,\ q = 1, ., ., ., n(t)\}$.The allocation is *feasible for the population* n (and individually rational) if

$$\{x_{tq} \in X_t : t = 1, ..., \|\sigma(n)\|\,,\ q = 1, ., ., ., n(t)\} \in \mathcal{A}(n). \tag{12}$$

5.1 Social continuity and asymptotic negligibility

Let $\{((e_t, u_t, X_t),\ t \in \mathcal{T}), \mathcal{A}\}$ be a pre-economy. The *pregame induced by the pre-economy* is the pair (T,Λ) where Λ is a function from $\cup_{m=1}^{\infty}\prod_{i=1}^{m}\mathbb{Z}_+$ to

$\mathbb{R}$ defined by

$$\Lambda(n) = \sup_{\{x_{tq}\}} \sum_{t \in \sigma(n)} \sum_{q=1}^{n(t)} u_t(x_{tq}),$$

and the supremum is taken over all feasible allocations $\{x_{tq}\} \in \mathcal{A}(n)$.

A pre-economy $\{((e_t, u_t, X_t),\ t \in \mathcal{T}), \mathcal{A}\}$ satisfies *social continuity* if the pregame induced by the pre-economy satisfies asymptotic negligibility and per capita boundedness. We call this *social continuity* because, in this case, with many participants the 'aggregate' economy can be described by one where all participants have the same tuility function. In terms of the utility functions and endowments of the participants, the condition is that: There is a constant A such that for any pair of sequences of profiles, say $\{f^v\}$ and $\{\ell^\nu\}$, where

$$\sigma(f^\nu) = \sigma(f^{\nu'}) \text{ for all } \nu, \nu', \text{ and}$$

$$\lim_{\nu \to \infty} \frac{\|\ell^\nu\|}{\|f^\nu\|} = 0$$

it holds that

$$\frac{\sup \sum_{t \in \sigma(f^\nu)} \sum_{q=1}^{f^\nu(t)} u_t(y_{tq})}{\|f^\nu\|} < A \text{ and}$$

$$\lim_{\nu \to \infty} \left(\sup \frac{\sum_{t \in \sigma(f^\nu + \ell^\nu)} \sum_{q=1}^{f^\nu(t)+\ell^\nu(t)} u_t(x_{tq})}{\|f^\nu + \ell^\nu\|} \right) - \left(\sup \frac{\sum_{t \in \sigma(f^\nu)} \sum_{q=1}^{f^\nu(t)} u_t(y_{tq})}{\|f^\nu\|} \right) = 0, \tag{13}$$

where the suprema re taken over all feasible allocations $\{x_{tq}\} \in \mathcal{A}(f^\nu + \ell^\nu)$ for the population $f^\nu + \ell^\nu$ and over all feasible allocations $\{y_{tq}\} \in \mathcal{A}(f^\nu)$ for the population f^ν. Inspection of the condition of social continuity reveals that basically two features of the pre-economy may cause a violation of social continuity; the endowments may be too dissimilar and/or the utility functions may be too dissimilar.

Note that continuity of the utility functions is not sufficient to ensure that social continuity is satisfied, as illustrated by the following example.

Example 2: Suppose that $\mathcal{E}$ is a exchange pre-economy with two types of participants and two types of commodities. Let the utility function, the same for all types of players, be given by

$$u(x,y) = \begin{cases} x+y & \text{if } x,y>0 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that the endowment of a player of type 1 is $e_1 = (1,0)$ and that of a player of type 2 is $e_2 = (0,1)$.

Consider a sequence of economies $\{\mathcal{E}(n^\nu)\}$ where $n^\nu = (1,\nu)$. It is easily seen that small group effectiveness is not satisfied, nor, of course, asymptotic negligibility. The violation of social continuity is a result of the fact that in the sequence of economies, players of type 1 are 'scarce'. Note that if the proportions of each type of agent were bounded away from zero, then asymptotic negligibility (under this constraint) would be satisfied.

Our next example shows that social continuity does not rule out "large" discontinuities of the utility functions. Moreover, individual marginal contributions to populations may be unbounded.

Example 3: For this example, also of an exchange pre-economy, there is only one type of commodity and all participants have the same utility function, given by

$$u(10^{2^k}) = 10^{2^k}(\sum_{i=0}^{k} \tfrac{1}{10^k}),\ k = 1,2,\ldots \text{ and}$$

$$\text{for any } x \in \mathbb{R}_+ \text{ with } 10^{2^k} \leq x < 10^{2^{k+1}},\ u(x) = u(10^{2^k})$$

Since there is only one type of participant in the pre-economy, small group effectiveness is equivalent to per capita boundedness (5). It is a simple matter to verify that the utility function u satisfies per capita boundedness, that is, the limit

$$\lim_{x\to\infty} \frac{u(x)}{x} \text{ exists (and is finite).}$$

5.2 Asymptotic equivalence of socially homogeneous markets and socially continuous pre-economies

Let $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ be a pre-economy. The pre-economy is *socially homogeneous* if there is a utility function u and a consumption set $X \subset$

$\mathcal{B}$ satisfying, for each $t \in \mathcal{T}$, the properties that $u_t = u$ and $X_t = X$. If the utility function u is concave, a socially homogeneous pre-economy is called a *socially homogeneous premarket.* A socially homogeneous premarket is *continuous* if the common utility function is continuous. The pre-economy exhibits *payoff-constant returns* if u is 1-homogeneous. (Note that these are all characteristics of the canonical market introduced in Shapley and Shubik 1969.)

Let $(\mathcal{T}, \Psi)$ and $(\mathcal{T}, \Psi')$ be two pregames with the same sets of attributes Ω. The pregames (Ω, Ψ) and (Ω, Ψ') are *asymptotically equivalent* if, given any real number $\varepsilon > 0$, there is an integer $\eta_1(\varepsilon)$ such that for all profiles f with $\|f\| \geq \eta_1(\varepsilon)$

$$|\Psi'(f) - \Psi(f)| \leq \varepsilon \|f\|. \tag{14}$$

Asymptotic equivalence of two pregames implies that for any large profile the per capita payoffs assignable to that profile by the two worth functions are approximately equal.

Let $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ and $\{((e'_t, u'_t, X'_t), t \in \mathcal{T}), \mathcal{A}'\}$ be two pre-economies whose sets of types $\mathcal{T}$ have the same support. The two pre-economies are *asymptotically equivalent* if their derived pregames, say $(\mathcal{T}, \Lambda)$ and $(\mathcal{T}, \Lambda')$, are well defined and asymptotically equivalent.

The following result extends Theorem 1 of Wooders (1994) to a compact metric space of attributes.

Theorem 3: A continuous, socially homogeneous premarket with payoff-constant returns satisfies social continuity.

In other words, Theorem 3 states that a pregame induced by a pre-economy in which all participants have the same continuous, one-homogeneous utility function (the analogue in this paper of Shapley and Shubik's canonical utility function) satisfies asymptotic negligibility and per capita boundedness.

Our next Theorem demonstrates that with an additional condition ensuring that participants who are "close" to each other in the metric on the space $\mathcal{T}$ are approximate substitutes in the pregame induced by the pre-economy, a pre-economy can be uniformly approximated by a continuous, socially homogeneous premarket with payoff-constant returns. Let $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ be a pre-economy and let $(\mathcal{T}, \Lambda)$ be the pregame induced by the pre-economy. The pre-economy satisfies *substitution* if the pregame ($\mathcal{T}$,Λ) satisfies condition (3(c)).

Theorem 4. Let $\{((e_t, u_t, X_t), t \in \mathcal{T}\ ), \mathcal{A}\}$ be a pre-economy satisfying substitution and with the property that $X_t = X_{t'}$ for all t, t' in $\mathcal{T}$. Then $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ satisfies social continuity if and only if there is a continuous, socially homogeneous premarket $\{((e'_t, u'_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ with payoff-constant returns such that $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ and $\{((e'_t, u'_t, X), t \in \mathcal{T}), \mathcal{A}'\}$ are asymptotically equivalent.

In other words, Theorem 4 states that, when players whose attributes are 'close' are near-substitutes, then a pre-economy with many participants is approximated by a premarket where all participants have the same concave, one-homogeneous and continuous utility function. The properties on the premarket may make it appear quite special but, when large, a diversity of economies are approximated by such a premarket.

The following result, the main result of this section and one of the main results of the paper, is implied by Theorems 3 and 4. The Theorem demonstrates that small groups of individuals have negligible effects on economic aggregates if and only if the economy can be represented as a continuous, socially homogeneous market with payoff-constant returns. This is our analogue of Shapley abd Shubik's market-game equivalence.

Theorem 5: Equivalence of socially homogeneous markets and socially continuous economies. The class of continuous, socially homogeneous premarkets with payoff-constant returns and the class of pre-economies satisfying substitution and social continuity are asymptotically equivalent.

In the approach of this author the set of marketed commodities is viewed as endogenous. The proof of the above Theorem shows that there exists at least one set of commodities for a competitive equilibrium in a large economy satisfying social continuity. The space of types of commodities can be taken as the set of types of participants, since the domain of the utility function u in the proof is "bundles/groups of participants," as in a labor market. It can also be shown that the domain of the utility function of the approximating socially homogeneous market can be taken as the set of commodity bundles $\mathcal{B}$ of the given economy. This requires a "thickness" assumption, ensuring that there are "many" substitutes for each participant in the economy, and the result is not included in this paper.

5.3 Market-game equivalence

Note that in Theorem 4, since our proof proceeds by consideration of the pregame derived from a pre-economy, we have also proven that a pregame satisfies asymptotic negligibility or, equivalently, small group effectiveness, if and only if the pregame is asymptotically equivalent to a continuous, socially homogeneous pre-economy with payoff-constant returns. Thus we have the following Theorem, an extension of the main result of Wooders (1994) to a compact metric space of player attributes.

Theorem 6: Equivalence of games and markets. The class of pregames satisfying small group effectiveness and the class pre-economies satisfying social continuity and substitution are equivalent

6 Discussion

6.1 The pregame framework and asymptotic negligibility

Our next example illustrates that if a sequence of economies does not satisfy social continuity then a continuum model ignoring sets of participants of measure zero may not be appropriate.

Example 4: Consider a sequence of economies $\{\mathcal{E}(n^{\nu})\}$ with two "types" of participants and two commodities. In the economy $\mathcal{E}(n^{\nu})$ one participant is endowed with ν units of the first commodity and each of the other participants is endowed with one unit of the second commodity. Thus, the endowments are given by $(\nu, 0)$ for the "first" participant and (0,1) for each of the other participants. All participants have the same utility function

$$u(x, y) = xy.$$

Here, there is a changing sequence of players. There is no limit economy since "in the limit," one participant must be endowed with an infinite amount of one good. The sequence of economies satisfies neither small group effectiveness nor asymptotic negligibility. To assume a continuum of players where one player can be ignored is meaningless as a limit of such a sequence of finite economies.A continuum limit

6.2 The number of commodities and the number of goods

There has long been the intuition that for large economies to be competitive it must be the case that markets are, in some sense, "thick." Rustichini and Yannelis (1991) develop a precise statement of the equivalence of nonatomicity and the property that there are many more participants than commodities. The work of Rustichini and Yannelis is in the framework of a model with a continuum of participants and is thus not directly comparable to ours. With a finite dimensional commodity space, the asymptotic analogue of Rustichini and Yannelis may already appear in the result that when markets are "thick," so the percentage of players of each type is bounded away from zero then "many agents relative to numbers of commodities" implies that small groups are effective (Wooders 1994, Theorem 4). Our research suggests that substitutability of commodities and the scale of activities required to realize all gains to collective activities are also important determinants of the "competitiveness" of an economy.

7 Conclusions

Classical economists argued that large numbers of participants justified the hypothesis of perfect competition, specifically price-taking behavior. Von Neumann and Morgenstern perceived that even though individuals are unable to influence market prices and cannot benefit from strategic behavior in large markets, large "coalitions" might form. Von Neumann and Morgenstern write

> It is neither certain nor probable that a mere increase in the number of participants might lead *in fine* to the conditions of free competition. The classical definitions of free competition all involve further postulates besides this number. E.g., it is clear that if certain great groups of individuals will – for any reason whatsoever– act together, then the great number of participants may not become effective; the decisive exchanges may take place directly between large "coalitions," few in number and not between individuals, many in number acting independently. ... Any satisfactory theory ...will have to explain when such big coalitions will or will not be formed –i.e., when the large num-

bers of participants will become effective and lead to more or less free competition.

The assumption that small groups of individuals cannot affect market aggregates, virtually taken for granted by von Neumann and Morgenstern, lies behind the answer to the question they pose. Our results suggest that the great number of participants will become effective and lead to more or less free competition only when small groups of participants cannot significantly affect market outcomes. With small group effectiveness, since all or almost all gains to collective activities can be captured by small groups, large groups gain no market power from size; in other words, large groups are inessential.

Other writers, seeking justification for the price-taking hypothesis, have shown equivalence of outcomes of cooperation and of price-taking behavior. We refer to the distinguished literature arising from Edgeworth (1881), and including Shubik (1959), Debreu and Scarf (1963), and Aumann (1964). While our concern in this paper has been with equivalence of economic structures, some aspects of the literature on the equivalence of the core and the competitive outcomes are relevant. In the generality of the economic model of this paper (except for the limiting model where all participants have the same concave continuous utility function) without additional assumptions competitive equilibrium may welll not exist. For example, for economies with clubs or local public goods, the assumptions of Shafer and Sonnenshein (1975) will typically not hold. With, however, some frictions in the model, and a complete price system, and in view of recent results demonstrating equivalence of the core and competitive equilibrium with unbounded club sizes, we conjecture that similar results would hold for the class of economic models considered in this paper.

8 Appendix

Theorem 1: **Equivalence of small group effectiveness for feasibility and for improvement.** A pregame (Ω, Ψ) satisfies small group effectiveness for improvement if and only if it satisfies small group effectiveness.

Proof: Suppose a pregame (Ω, Ψ) satisfies small group effectiveness but does not satisfy small group effectiveness for improvement. Then there is a positive real number ε_0 and a sequence of profiles $\{f^\nu\}$ such that for each ν, there is a payoff function x^ν with the properties that

(a) $x^\nu(f^\nu) + \varepsilon_0 \|g^\nu\| < \Psi(f^\nu)$, and

(b) for each subprofile g of f^ν with $\|g\| \leq \nu$ it holds that $x^\nu(g) + \frac{\varepsilon_0}{2}\|g\| \geq \Psi(g)$.

Let $\eta_1(\frac{\varepsilon_0}{2})$ be the bound on group sizes given by the definition of small group effectiveness. From small group effectiveness and the definition of $\eta_1(\frac{\varepsilon_0}{2})$ it follows that for each ν there is a partition $\{f^{\nu k}\}$ of f^ν satisfying

$$\Psi(f^\nu) - \sum_k \Psi(f^{\nu k}) < \frac{\varepsilon_0}{2}\|f^\nu\|. \tag{15}$$

It now follows from (b) above that

$$\sum_k (x^\nu(f^{\nu k}) + \frac{\varepsilon_0}{2}\left\|f^{\nu k}\right\|) \geq \sum_k \Psi(f^{\nu k}). \tag{16}$$

From (15) it follows that

$$\sum_k (\Psi(f^{\nu k}) > \Psi(f^\nu) - \frac{\varepsilon_0}{2}\left\|f^{\nu k}\right\|). \tag{17}$$

But from (16) and (17) it follows that

$$\sum_k x^\nu(f^{\nu k}) + \tfrac{\varepsilon_0}{2}\left\|f^{\nu k}\right\|) \geq \sum_k \Psi(f^{\nu k}) > \Psi(f^\nu) - \tfrac{\varepsilon_0}{2}\|f^\nu\| \text{ and}$$
$$\sum_k x^\nu(f^{\nu k}) = x^\nu(f^\nu) \geq \Psi(f^\nu) - \tfrac{\varepsilon_0}{2}\|f^\nu\|,$$

and, from (a) it follows that $\Psi(f^\nu) - \frac{\varepsilon_0}{2}\|f^\nu\| > x^\nu(f^\nu) + \frac{\varepsilon_0}{2}\|f^\nu\|$. This, however, implies that

$$x^\nu(f^\nu) \geq \Psi(f^\nu) - \frac{\varepsilon_0}{2}\|f^\nu\| > x^\nu(f^\nu) + \frac{\varepsilon_0}{2}\|f^\nu\|$$

a contradiction. Thus, small group effectiveness implies small group effectiveness for improvement.

For ease in exposition, given any positive integer ν, a partition $\{g^k\}$ of a profile g is called ν-*bounded* if $\left\|g^k\right\| < \nu$ for each k.

Now suppose that (Ω, Ψ) satisfies small group effectiveness for improvement but not small group effectiveness (for feasibility). Then there is a positive real number ε_0 and a sequence of profiles $\{f^\nu\}$ such that for each ν, for every ν-bounded partition $\{f^{\nu k}\}$ of f^ν satisfying

$$\Psi(f^\nu) - \sum_k \Psi(f^{\nu k}) \geq 3\varepsilon_0\|f^\nu\|. \tag{18}$$

Given ε_0, let $\eta_2(\varepsilon_0)$ be the integer given by small group effectiveness for improvement for the game (Ω,Ψ). Define another pregame (Ω,Λ) by setting

$$\Lambda(f)=\max_{\mathcal{P}}\sum_{g\in\mathcal{P}}\Psi(g),$$

where the maximum is taken over all partitions $\mathcal{P}$ of f with $\|g\|\leq\eta_2(\varepsilon_0)$ for each $g\in\mathcal{P}$. Note that the pregame (Ω,Λ) satisfies small group effectiveness. Let (Ω,Λ^b) denote the balanced cover of (Ω,Λ). (The balanced cover function Λ^b is the smallest-valued function such that $\Lambda^b(f)\geq\Lambda(f)$ for every profile f and such that every game induced by (Ω,Λ^b) has a non-empty core; see Bondareva (1962) and Shapley 1967. Wooders 1992a, for example, provides a definition.) It can be shown that for all sufficiently profiles g it holds that

$$\Lambda^b(g)-\Lambda(g)\leq\varepsilon_0\,\|g\|\,; \tag{19}$$

this is an immediate consequence of the main Theorem of Wooders (1992a).[8] For the case where Ω is a finite set, the result is proven in Wooders (1994).

For the purpose of obtaining a contradiction suppose that for all sufficiently large terms ν in the sequence it holds that $\Psi(f^\nu)-\Lambda(f^\nu)<\varepsilon_0\,\|f^\nu\|$ .Since the pregame (Ω,Λ) satisfies small group effectiveness, from the fact that $\Lambda(h)=\Psi(h)$ for all profiles h with $\|h\|\leq\eta_2(\varepsilon_0)$ and from equation (19), it follows that for each ν sufficiently large there is a partition $\{f^{\nu k}\}$ of f^ν such that

$$\begin{gathered}\left|\Psi(f^\nu)-\textstyle\sum_k\Psi(f^{\nu k})\right|\\ <\left|\Psi(f^\nu)-\Lambda(f^\nu)\right|+\left|\Lambda(f^\nu)-\textstyle\sum_k\Lambda(f^{\nu k})\right|+\left|\textstyle\sum_k\Lambda(f^{\nu k})-\sum_k\Psi(f^{\nu k})\right|\\ <3\,\varepsilon_0\,\|f^\nu\|\,,\end{gathered} \tag{20}$$

which is a contradiction to (18). Therefore $\Psi(f^\nu)-\Lambda(f^\nu)>\varepsilon_0\,\|f^\nu\|$ for ν sufficiently large .

We claim that for all ν sufficiently large, $\Lambda^b(f^\nu)\leq\Psi(f^\nu)+\varepsilon_0\,\|f^\nu\|$. This follows from the observations that $\Psi(f^\nu)+\varepsilon_0\,\|f^\nu\|\ \geq\ \Lambda(f^\nu)+\varepsilon_0\,\|f^\nu\|\ \geq\ \Lambda^b(f^\nu)$.

For each ν let x^ν be a payoff in the core of the game with profile of players equal to f^ν and with worth function given by Λ^b (restricted to subprofiles of

[8] The result is also immediate from Wooders (1983) and earlier working papers by the same author for transferable utility games.

f^{ν}). From the Bondareva-Shapley Theorem there is such a payoff x^{ν}. From the preceding paragraph, for all ν sufficiently large, $\Lambda^b(f^{\nu}) = x^{\nu}(f^{\nu}) \leq \Psi(f^{\nu})$. We claim that $x^{\nu}(g) \geq \Psi(g) - \varepsilon_0 \|g\|$ for all profiles $g \leq f^{\nu}$. If not, from the fact that (Ω, Ψ) satisfies small group effectiveness for improvement there is a profile g with $\|g\| \leq \eta_2(\varepsilon_0)$ satisfying $x^{\nu}(g) + \frac{\varepsilon_0}{2}\|g\| \geq \Psi(g)$ and thus we have a contradiction to the fact that x^{ν} is in the core of a game with profile $f^{\nu} \geq g$ and with worth function given by Λ^b. It now follows that x^{ν} is in the ε_0-core of the game with player profile f^{ν} and with worth function given by Ψ. This, however, implies that $\left|\Psi(f^{\nu}) - \Lambda^b(f^{\nu})\right| < \varepsilon_0 \|f^{\nu}\|$ and since $\Lambda^b(f^{\nu}) - \Lambda(f^{\nu}) \leq \varepsilon_0 \|f^{\nu}\|$, it also implies that $|\Psi(f^{\nu}) - \Lambda(f^{\nu})| < 2\varepsilon_0 \|f^{\nu}\|$. But, from the definition of Λ this yields a contradiction to (18). ■

Theorem 2: Equivalence of small group effectiveness and asymptotic negligibility: Let (Ω, Ψ) be a pregame satisfying per capita boundedness (5). Then (Ω, Ψ) satisfies small group effectiveness if and only if it satisfies asymptotic negligibility.

Proof. *Part 1: Small group effectiveness implies asymptotic negligibility.* We proceed by supposing the assertion is false. Then there are sequences of profiles $\{f^{\nu}\}$ and $\{\ell^{\nu}\}$, and, for some integer T, a vector $f \in \mathbb{R}^T$ and a subset $\{\omega_1, ..., \omega_T\} \subset \Omega$ such that

$$\sigma(f^{\nu}) = \{\omega_1, ..., \omega_T\} \text{ for each } \nu,$$

$$\tfrac{f^{\nu}(\omega_t)}{\|f\|} \text{ converges to } f(\omega_t) \text{ for each } \omega_t \text{ in } \{\omega_1, ..., \omega_T\},$$

$$\lim_{\nu\to\infty} \tfrac{\|\ell^{\nu}\|}{\|f^{\nu}\|} \to 0 \text{ as } \nu \text{ becomes large, and}$$

$$\lim_{\nu\to\infty} \tfrac{\Psi(f^{\nu})}{\|f^{\nu}\|} \text{ exists,}$$

but either

$$\frac{\Psi(f^{\nu} + \ell^{\nu})}{\|f^{\nu} + \ell^{\nu}\|} \text{ does not exist,}$$

or for some $\varepsilon_0 > 0$ it holds that

$$\left| \lim_{\nu\to\infty} \frac{\Psi(f^{\nu} + \ell^{\nu})}{\|f^{\nu} + \ell^{\nu}\|} - \lim_{\nu\to\infty} \frac{\Psi(f^{\nu})}{\|f^{\nu}\|} \right| > 2\,\varepsilon_0. \tag{21}$$

From per capita boundedness (5), there is a constant A such that $\lim_{\nu\to\infty} \frac{\Psi(f^{\nu})}{\|f^{\nu}\|} \leq A$ and for all terms ν, $\frac{\Psi(f^{\nu}+\ell^{\nu})}{\|f^{\nu}+\ell^{\nu}\|} \leq A$. (This is easy to show,) We can suppose, without any loss of generality, that both $\{\frac{\Psi(f^\nu)}{\|f^\nu\|}\}$ and $\{\frac{\Psi(f^\nu+\ell^\nu)}{\|f^\nu+\ell^\nu\|}\}$ converge. For ease in notation for each ν define

$$g^\nu = f^\nu + \ell^\nu.$$

Since (Ω, Ψ) satisfies small groups effectiveness (4) there is an integer $\eta_1(\varepsilon_0)$ such that for each ν there is a partition $\{g^{\nu k} : k = 1, ..., K\}$ of g^ν satisfying

$$\|g^{\nu k}\| \le \eta_1(\varepsilon_0) \text{ for each } \ k \text{ and}$$
$$\Psi(g^\nu) - \sum_k \Psi(g^{\nu k}) \le \varepsilon_0 \|g^\nu\|.$$

From the definition of g^ν there are at most $\|\ell^\nu\|$ members of the partition $\{g^{\nu k}\}$ with the property that $\sigma(g^{\nu k}) \cap \sigma(\ell^\nu) \neq \emptyset$. Thus, by renumbering profiles if necessary we can suppose that for some integer K',

$$K' \ge K - \|\ell^\nu\|, \text{ and}$$
$$\sum_{k=1}^{K'} g^{\nu k} \le f^\nu.$$

From (5) it follows that

$$\Psi(g^\nu) - \sum_{k=1}^{K'} \Psi(g^{\nu k}) - A\|\ell^\nu\|\eta_1(\epsilon_0) \le \varepsilon_0 \|g^\nu\|.$$

Since $\frac{\|\ell^\nu\|}{\|f^\nu\|} \to 0$ as $\nu \to \infty$ it follows that $A\frac{\|\ell^\nu\|}{\|f^\nu\|}\eta_1(\varepsilon_0) \ \to 0$ as $\nu \to \infty$. Thus, for all ν sufficiently large,

$$\Psi(g^\nu) - \sum_{k=1}^{K'} \Psi(g^{\nu k}) \le 2\ \varepsilon_0 \|g^\nu\|.$$

Since $\sum_{k=1}^{K'} g^{\nu k} \le f^\nu$ from (3(c)) it holds that $\sum_{k=1}^{K'} \Psi(g^{\nu k}) \le \Psi(f^\nu)$. It now follows that for all ν sufficiently large,

$$\Psi(g^\nu) \ge \Psi(f^\nu) \ge \sum_{k=1}^{K'} \Psi(g^{\nu k}).$$

Since $0 \leq \Psi(g^\nu) - \sum_{k=1}^{K'} \Psi(g^{\nu k}) \leq 2\varepsilon_0 \|g^\nu\|$ it follows that

$$0 \leq \Psi(g^\nu) - \Psi(f^\nu) \leq 2\,\varepsilon_0 \|g^\nu\|.$$

a contradiction to (21).

Part 2: Asymptotic negligibility implies small group effectiveness. Recall that given any positive integer ν , a partition $\{g^k\}$ of a profile g is called *ν-bounded* if $\|g^k\| < \nu$ for each k. Suppose (Ω, Ψ) satisfies asymptotic negligibility but does not satisfy small group effectiveness. Then there is a positive real number $\varepsilon_0 > 0$ and a sequence of profiles $\{f^\nu\}$ such that for each ν, for every ν-bounded partition $\{f^{\nu k}\}$ of f^ν it holds that

$$\Psi(f) - \sum \Psi(f^{\nu k}) > 3\,\varepsilon_0 \|f^\nu\|. \tag{22}$$

Let δ be a positive real number satisfying the property that whenever $dist(\omega, \omega') < \delta$, then for all profiles f, it holds that $|\Psi(f + \chi_\omega) - \Psi(f + \chi_{\omega'})| < \varepsilon_0$. Let $\Omega_1, ..., \Omega_Q$ be a partition of Ω into nonempty subsets, each contained in a ball of radius less that δ, and for each $q \in \{1, ..., Q\}$ let ω_q be a point in Ω_q . For each ν define the profile g^ν by

$$g^\nu(\omega_q) = \sum_{\omega \in \sigma(f^\nu) \cap \Omega_q} f^\nu(\omega) \text{ and}$$

$$g^\nu(\omega) = 0 \text{ for } \omega \notin \{\omega_1, ..., \omega_Q\}.$$

From continuity (3(c)) it follows that

$$|\Psi(f^\nu) - \Psi(g^\nu)| \leq \varepsilon_0 \|f^\nu\|.$$

To see this we construct a string of profiles where each profile differs from the next only in that one player is of a different type, and where the string goes from f^ν to g^ν. Specifically, let $a_1, ..., a_J$ denote the elements in the support of f^ν with each appearing as many times in the list as its multiplicity (so $J = \|f^\nu\|$). Similarly, let $b_1, ..., b_J$ be a list of the elements in the support of g^ν. For each $j = 1, ..., J$ define f_j^ν as the profile with the list of elements in its support given by $b_1, b_2, ..., b_j, a_{j+1}, ..., a_J$. Observe that

$$f^\nu - g^\nu = f^\nu - f_1^\nu + f_1^\nu - f_2^\nu + f_2^\nu - ... + f_{J-1}^\nu - f_{J-1}^\nu - g^\nu,$$

and from continuity (3(c)),

$$|\Psi(f^\nu) - \Psi(g^\nu)|$$
$$\leq |\Psi(f^\nu) - \Psi(f_1^\nu)| + |\Psi(f_1^\nu) - \Psi(f_2^\nu)| + ... + \left|\Psi(f_{J-1}^\nu) - \Psi(g^\nu)\right| \leq \varepsilon_0 \|f^\nu\|.$$

We can suppose without loss of generality that the sequence $\{\frac{1}{\|g^\nu\|}g^\nu\}$ converges, say to g. Since the pregame (Ω, Ψ) satisfies per capita boundedness, from Wooders (1994, Theorem 4) there is an integer η such that for each ν, there is a partition of g^ν into subprofiles $\{g^{\nu k}\}$ with $\left\|g^{\nu k}\right\| < \eta$ for each subprofile and

$$\left|\Psi(g^\nu) - \sum_k \Psi(g^{\nu k})\right| < \varepsilon_0 \left\|g^\nu\right\|.$$

Construct a collection of subprofiles $\{f^{\nu k}\}$ of f^ν from the collection of subprofiles $\{g^{\nu k}\}$ so that for each k and each q,

$$\sum_{\omega \in \Omega_q} g^{\nu k}(\omega) = \sum_{\omega \in \Omega_q} f^{\nu k}(\omega).$$

It follows from continuity (3(c)) that for each ν and for each k

$$\left|\Psi(g^{\nu k}) - \Psi(f^{\nu k})\right| < \varepsilon_0 \left\|f^{\nu k}\right\|.$$

Therefore

$$\left|\Psi(f^\nu) - \sum_k \Psi(f^{\nu k})\right| <$$

$$|\Psi(f^\nu) - \Psi(g^\nu)| + \left|\Psi(g^\nu) - \sum_k \Psi(g^{\nu k})\right| + \left|\sum_k \Psi(g^{\nu k}) - \sum_k \Psi(f^{\nu k})\right|$$

$$< 3\varepsilon_0 \left\|f^\nu\right\|,$$

which is a contradiction to (22). ■

Theorem 3: A continuous, socially homogeneous premarket with payoff-constant returns satisfies social continuity.

Proof: First, note that from concavity and the fact that all participants have the same utility function it follows that the pre-market satisfies per capita boundedness (5). Suppose the conclusion of the statement of the Theorem is false. Then there is a socially homogeneous premarket $\{((e_t, u, X), t \in \mathcal{T}), \mathcal{A}\}$, a positive real number $\varepsilon_0 > 0$, and two sequences of profiles, say $\{f^v\}$ and $\{\ell^\nu\}$, satisfying

$$\sigma(f^\nu) = \sigma(f^{\nu'}) \text{ for all } \nu, \nu', \text{ and}$$

$$\lim_{\nu\to\infty}\frac{\|\ell^\nu\|}{\|f^\nu\|}=0,$$

such that

$$\lim_{\nu\to\infty}\left(\sup_{\{_{tq}\}x}\frac{\sum\limits_{t\in\sigma(f^\nu+\ell^\nu)}\sum\limits_{q=1}^{f^\nu(t)+\ell^\nu(t)}u(x_{tq})}{\|f^\nu+\ell^\nu\|}\right)-\left(\sup_{\{y_{tq}\}}\frac{\sum\limits_{t\in\sigma(f^\nu)}\sum\limits_{q=1}^{f^\nu(t)}u(y_{tq})}{\|f^\nu\|}\right)>\varepsilon_0, \tag{23}$$

where the suprema are taken over all feasible allocations $\{x_{tq}\}\in\mathcal{A}(f^\nu+\ell^\nu)$ and $\{y_{tq}\}\in\mathcal{A}(f^\nu)$. From per capita boundedness the pregame induced by the market is well-defined.

From concavity and 1-homogeneity of the function u we can assume that for each ν, it holds that

$$\sup_{\{x_{tq}\}}\frac{\sum\limits_{t\in\sigma(f^\nu+\ell^\nu)}\sum\limits_{q=1}^{f^\nu(t)+\ell^\nu(t)}u(x_{tq})}{\|f^\nu+\ell^\nu\|}=u\left(\frac{\sum\limits_{t\in\sigma(f^\nu)}(f_t^\nu+\ell_t^\nu)\;e_t}{\|f^\nu+\ell^\nu\|}\right)\text{ and}$$

$$\sup_{\{y_{tq}\}}\frac{\sum\limits_{t\in\sigma(f^\nu)}\sum\limits_{q=1}^{f^\nu(t)}u(y_{tq})}{\|f^\nu\|}=u\left(\frac{\sum\limits_{t\in\sigma(f^\nu)}f^\nu(t)\;e_t}{\|f^\nu\|}\right).$$

As ν grows large, however,

$$\lim_{\nu\to\infty}\rho\left(\frac{\sum\limits_{t\in\sigma(f^\nu+\ell^\nu)}(f^\nu(t)+\ell^\nu(t))\;e_t}{\|f^\nu+\ell^\nu\|}-\frac{\sum\limits_{t\in\sigma(f^\nu)}f^\nu(t)e_t}{\|f^\nu\|}\right)=0.$$

From continuity of the function u,

$$\lim_{\nu\to\infty}\left|u\left(\frac{\sum\limits_{t\in\sigma(f^\nu)}((f^\nu(t)+\ell^\nu(t))\;e_t}{\|f^\nu+\ell^\nu\|}\right)-u\left(\frac{\sum\limits_{t\in\sigma(f^\nu)}f^\nu(t)\;e_t}{\|f^\nu\|}\right)\right|=0,$$

and therefore

$$\lim_{\nu\to\infty}\frac{\sup_{\{x_{tq}\}}\sum\limits_{t\in\sigma(f^\nu+\ell^\nu)}\sum\limits_{q=1}^{f^\nu(t)+\ell^\nu(t)}u(x_{tq})}{\|f^\nu+\ell^\nu\|}-\frac{\sup_{\{y_{tq}\}}\sum\limits_{t\in\sigma(f^\nu)}\sum\limits_{q=1}^{f^\nu(t)}u(y_{tq})}{\|f^\nu\|}=0.$$

This gives the desired contradiction. ■

Theorem 4. Let $\{((e_t, u_t, X_t), t \in \mathcal{T} \ ), \mathcal{A}\}$ be a pre-economy satisfying substitution and with the property that $X_t = X_{t'}$ for all t, t' in $\mathcal{T}$. Then $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ satisfies social continuity if and only if there is a continuous, socially homogeneous premarket $\{((e'_t, u', X_t), t \in \mathcal{T}), \mathcal{A}\}$ with payoff-constant returns such that $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ and $\{((e'_t, u', X), t \in \mathcal{T}), \mathcal{A}'\}$ are asymptotically equivalent.

Proof: First, we suppose that $\{((e_t, u, X_t), t \in \mathcal{T}), \mathcal{A}\}$ satisfies social continuity. Since $\{((e_t, u, X_t), t \in \mathcal{T}), \mathcal{A}\}$ satisfies social continuity the derived pregame satisfies asymptotic negligibility, as is immediately clear from the definitions. Thus to obtain the result it suffices to show that there is a socially homogeneous premarket whose induced pregame is asymptotically close to the pregame derived from the pre-economy $\{((e_t, u, X_t), t \in \mathcal{T}), \mathcal{A}\}$. For the case where $\mathcal{T}$ is a finite set, the result follows from Wooders (1993a, Theorem 2). We will extend that result to the case where $\mathcal{T}$ is a compact metric space. Let $(\Lambda, \mathcal{T})$ denote the pregame induced by the pre-economy $\{((e_t, u, X_t), t \in \mathcal{T}), \mathcal{A}\}$. That is, the value of the worth function Λ at a profile f is given by

$$\Lambda(f) = \sup_{\{x_{tq}\}} \sum_{t \in \sigma(f)} \sum_{q=1}^{f^\nu(t)} u(x_{tq}),$$

where the supremum is taken over all feasible allocations $\{x_{tq}\} \in \mathcal{A}(f^\nu)$.

Let $\mathcal{X}$ denote the set of commodity bundles on $\mathcal{T}$, that is, $\mathcal{X}$ is the set of all functions from $\mathcal{X} \to \mathbb{R}_+$ with finite support. For each commodity bundle $x \in \mathcal{X}$, define $u(x)$ by

$$u(x) = \|x\| \lim_{\nu \to \infty} \frac{\Lambda(f^\nu)}{\|f^\nu\|},$$

where $\{f^\nu\}$ is any sequence of profiles with the properties that $\|x\| \lim_{\nu\to\infty} \frac{1}{\|f^\nu\|} f^\nu$ $= x$ and $\|f^\nu\| \to \infty$. From social continuity the function u is well-defined and continuous. Observe that this data determines a socially homogeneous premarket. Also observe that from concavity of the function u the pregame induced by the premarket is the pair $(u, \mathcal{T})$ where u is restricted to profiles on $\mathcal{T}$.

We claim that the pregame $(u, \mathcal{T})$ is asymptotically equivalent to the pregame $(\Lambda, \mathcal{T})$. For the case where $\mathcal{T}$ is a finite set this is immediate from

Wooders (1994, Theorem 2). Thus, suppose $\mathcal{T}$ is not finite and that the claim is false. Then there is a positive real number $\varepsilon_0 > 0$ and sequence of profiles $\{f^\nu\}$ such that $\|f^\nu\|$ goes to infinity as ν goes to infinity and for each for each ν it holds that

$$\left| \frac{\Lambda(f^\nu)}{\|f^\nu\|} - \frac{u(f^\nu)}{\|f^\nu\|} \right| > 3\varepsilon_0. \tag{24}$$

Let $dist$ denote the metric on the space $\mathcal{T}$ and let $\delta > 0$ be a positive real number with the property that if $dist(t_1, t_2) < \delta$, then for all profiles f it holds that

$$\left| \Lambda(f + \chi_{t_1}) - \Lambda(f + \chi_{t_2}) \right| < \varepsilon_0;$$

this is possible from the substitution property of the pre-economy. Let $\mathcal{T}_1, ..., \mathcal{T}_J$ be a partition of $\mathcal{T}$ such that for each $j = 1, ..., J$ the set $\mathcal{T}_j$ is contained in a ball of radius less than δ. For each j select $t_j \in \mathcal{T}_j$. Define a sequence of profiles $\{g^\nu\}$ as follows:

$$g^\nu(t_j) = \textstyle\sum_{t \in \sigma(f^\nu) \cap \mathcal{T}j} f^\nu(t) \text{ and}$$

$$g^\nu(t) = 0 \text{ for } t \notin \{t_1, ..., t_J\}.$$

From Wooders (1994) for all ν sufficiently large it holds that

$$\left| \frac{\Lambda(g^\nu)}{\|g^\nu\|} - \frac{u(g^\nu)}{\|g^\nu\|} \right| < \varepsilon_0.$$

From the substitution property of the pre-economy it follows that for all ν sufficiently large,

$$\left| \frac{\Lambda(g^\nu)}{\|g^\nu\|} - \frac{\Lambda(f^\nu)}{\|f^\nu\|} \right| < \varepsilon_0.$$

From continuity and 1-homogeneity of u it also follows that for all ν sufficiently large

$$\left| \frac{u(g^\nu)}{\|g^\nu\|} - \frac{u(f^\nu)}{\|f^\nu\|} \right| = \left| u\left(\frac{1}{\|g^\nu\|} g^\nu \right) - u\left(\frac{1}{\|f^\nu\|} f^\nu \right) \right| < \varepsilon_0. \tag{25}$$

Thus we have

$$\left| \tfrac{\Lambda(f^\nu)}{\|f^\nu\|} - \tfrac{u(f^\nu)}{\|f^\nu\|} \right| \leq$$
$$\left| \tfrac{\Lambda(f^\nu)}{\|f^\nu\|} - \tfrac{\Lambda(g^\nu)}{\|g^\nu\|} \right| + \left| \tfrac{\Lambda(g^\nu)}{\|g^\nu\|} - \tfrac{u(g^\nu)}{\|g^\nu\|} \right| + \left| \tfrac{u(g^\nu)}{\|g^\nu\|} - \tfrac{u(f^\nu)}{\|f^\nu\|} \right|$$
$$\leq 3\varepsilon_0,$$

a contradiction to (24).

To show the other direction of the Theorem, let $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ be a pre-economy and let $\{((e'_t, u', X), t \in \mathcal{T}), \mathcal{A}'\}$ be a continuous, socially homogeneous premarket with payoff-constant returns. Suppose that $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ and $\{((e'_t, u', X), t \in \mathcal{T}), \mathcal{A}'\}$ are asymptotically equivalent. Let $(\Lambda, \mathcal{T})$ be the pregame derived from the pre-economy $\{((e_t, u_t, X_t), t \in \mathcal{T}), \mathcal{A}\}$ and let $(\Lambda', \mathcal{T})$ be the pregame derived from the pre-economy $\{((e'_t, u', X), t \in \mathcal{T}), \mathcal{A}'\}$. The result can now be proven by contradiction, first for the case where $\mathcal{T}$ is a finite set, as in Wooders (1994, Theorem 2), and then for the general case by using continuity and approximation of the general case by the finite-type case. ■

References

[1] Aumann, R.J. (1964) "Markets with a Continuum of Traders," *Econometrica* 32, 39-50.

[2] Aumann, R.J. (1966) "Existence of Competitive Equilibria in Markets with a Continuum of Traders," *Econometrica* 32, 39-50.

[3] Aumann, R.J. (1987) "Game Theory," *The New Palgrave: A Dictionary of Economics*, J. Eatwell, M. Milgate, and P. Newman, eds. The Macmillan Press.

[4] Azrieli, Y. and E. Lehrer (2007) "Market Games in Large Economies with a Finite Number of Types," *Economic Theory* 31:327-342.

[5] Bondareva, O. (1962) 'Theory of the Core in an n-Person Game', Vestnik, LGU13, 141-142 (in Russian), (Leningrad State University, Leningrad).

[6] Debreu G. and Scarf, H. (1963) "A Limit Theorem on the Core of an Economy," *International Economic Review* 4, 235-246.

[7] Edgeworth, F.Y. (1881) *Mathematical Psychics*, Kegan Paul: London.

[8] Evstigneev, I. and S. Flam (2001) "Sharing Nonconvex Costs," *Journal of Global Optimization*, Special Issue on Applications to Economics, v. 20, 257-271.

[9] Jones, L. (1984) "A Competitive Model of Commodity Differentiation," *Econometrica* 2, 507-530.

[10] Hildenbrand, W. (1974) *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, New Jersey.

[11] Kannai, Y. (1992) "The Core and Balancedness," in the *Handbook of Game Theory*, R.J. Aumann and S. Hart, eds. North Holland, Amsterdam.

[12] Khan, M. Ali (1974) "Some Remarks on the Core of a "Large" Economy," *Econometrica* 42, 633-642.

[13] Manelli, A. (1991) "Monotone Preferences and Core Equivalence," *Econometrica* 59, 123-138.

[14] Mas-Colell, A. (1975) "A Model of Equilibrium with Differentiated Commodities," *Journal of Mathematical Economics* 2, 263-295.

[15] Mas-Colell, A. (1979), "A Refinement of the Core Equilibrium Equivalence Theorem," *Economic Letters.*

[16] von Neumann. J. and Morgenstern, O. (1953) *Theory of Games and Economic Behavior*, Princeton University Press, Princeton New Jersey.

[17] Rustichini, A. and N. Yannelis (1991) "What is Perfect Competition?" in *Equilibrium Theory in Infinite Dimensional Spaces,* Springer-Verlag: Berlin.

[18] Shafer, W. and H. Sonnenschein (1975) "Some Theorems on the Existence of Competitive Equilibrium," *Journal of Economic Theory* 11, 83-93.

[19] Shapley, L.S. (1967) "On Balanced Sets and Cores," *Navel Research Logistics Quarterly* 9, 45-48.

[20] Shapley, L.S. and Shubik, M. (1969) "On Market Games," *Journal of Economic Theory* 1, 9-25.

[21] Shapley, L.S. and Shubik, M. (1975) 'Competitive Outcomes in the Cores of Market Games', *International Journal of Game Theory* 4, 229-237.

[22] Shubik, M. (1959) 'Edgeworth Market Games', in Luce, F.R. and Tucker, A.W. (eds.), *Contributions to the Theory of Games* IV, *Annals of Mathematical Studies* 40, Princeton: Princeton University Press, 277-278.

[23] Shubik, M., Wooders, M.H., 1982. Clubs, markets, and near-market games. In: Wooders, M.H. (Ed.), Topics in Game Theory and Mathematical Economics: Essays in Honor of Robert J. Aumann. Field Institute Communication Volume, American Mathematical Society, originally Near Markets and Market Games, Cowles Foundation, Discussion Paper No. 657.

[24] Wooders, M.H. (1979) "Asymptotic Cores and Asymptotic Balancedness of Large Replica Games," S.U.N.Y.-Stony Brook Working Paper No. 215 (Revised 1980).

[25] Wooders, M.H. (1983) "The Epsilon Core of a Large Replica Game, " *Journal of Mathematical Economics* 11, 277-300.

[26] Wooders, M.H. (1992a) "Inessentiality of Large Groups and the Approximate Core Property; an Equivalence Theorem," *Economic Theory* 2, 129-147.

[27] Wooders, M.H. (1992b) "Large Games and Economies with Effective Small Groups," *Game Theoretic Approaches to General Equilibrium Theory,* eds. J.F. Mertens and S. Sorin).

[28] Wooders, M.H. (1994) "Equivalence of Games and Markets," *Econometrica* 62, 1141-1160.