Economics Bulletin

Appendix and Supplemental material not intended for publication-Round 3

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Mathematics Derivations

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Appendices

Solve the UMP with respect to I_{t-1} as,

$$\max_{I_{t-1}} \quad U = U(TH_0 \cdots TH_T; Y_0 \cdots Y_T) \tag{1}$$

$$s.t. \quad \Sigma_{t=0}^{T} \frac{C_t^I + C_t^Y + w_t TUH_t}{(1+r)^t} = R \tag{2}$$

Construct a Lagrangian as,

$$L = U + \lambda(R - \Sigma(\cdot)) \tag{3}$$

Therefore the F.O.C is,

$$\frac{\partial L}{\partial I_{t-1}} = \underbrace{\frac{\partial U}{\partial I_{t-1}}}_{①} + \underbrace{\lambda \frac{\partial(R - \Sigma(\cdot))}{\partial I_{t-1}}}_{②} \tag{4}$$

Then expand the R.H.S separately by ① and ②. For ① $\quad \partial U / \partial I_{t-1}$,

$$\frac{\partial U}{\partial I_{t-1}} = \frac{\partial U}{\partial TH_t} \frac{\partial TH_t}{\partial H_t} \frac{\partial H_t}{\partial I_{t-1}} + \frac{\partial U}{\partial TH_{t+1}} \frac{\partial TH_{t+1}}{\partial H_{t+1}} \frac{\partial H_{t+1}}{\partial I_{t-1}} + \cdots + \frac{\partial U}{\partial TH_T} \frac{\partial TH_T}{\partial H_T} \frac{\partial H_T}{\partial I_{t-1}} \tag{5}$$

Denote

$$\frac{\partial U}{\partial TH_i} = U_{TH_i}; \quad \frac{\partial TH_i}{\partial H_i} = \phi_i', \quad where \;\; i = t \cdots T \tag{6}$$

Then we have,

$$\frac{\partial U}{\partial I_{t-1}} = U_{TH_t} \phi_t' \frac{\partial H_t}{\partial I_{t-1}} + U_{TH_{t+1}} \phi_{t+1}' \frac{\partial H_{t+1}}{\partial I_{t-1}} + \cdots + U_{TH_T} \phi_T' \frac{\partial H_T}{\partial I_{t-1}}. \tag{7}$$

As $H_t = (1 - \delta_{t-1}(I_{t-1}))H_{t-1} + pI_{t-1}$, we have

$$\frac{\partial H_t}{\partial I_{t-1}} = p - \frac{\partial \delta_{t-1}}{\partial I_{t-1}} H_{t-1},$$

and respectively, as $H_{t+1} = (1 - \delta_t(I_t))H_t + pI_t$, we derive $\partial H_{t+1}/\partial I_{t-1}$ as,

$$\frac{\partial H_{t+1}}{\partial I_{t-1}} = (1 - \delta_t)(p - \frac{\partial \delta_{t-1}}{\partial I_{t-1}} H_{t-1}),$$

and for $\partial H_{t+2}/\partial I_{t-1}$, we have,

$$\frac{\partial H_{t+2}}{\partial I_{t-1}} = (1 - \delta_{t+1}) \frac{\partial H_{t+1}}{\partial I_{t-1}} = (1 - \delta_{t+1})(1 - \delta_t)(p - \frac{\partial \delta_{t-1}}{\partial I_{t-1}} H_{t-1}).$$

Therefore, we have

$$\frac{\partial H_T}{\partial I_{t-1}} = (1-\delta_{T-1})\cdots(1-\delta_{t+1})(1-\delta_t)(p-\frac{\partial \delta_{t-1}}{\partial I_{t-1}}H_{t-1}).$$

Denote $\hat{\delta}_{t-1} = \frac{\partial \delta_{t-1}}{\partial I_{t-1}}H_{t-1}$, we get ① as,

$$\begin{aligned}\frac{\partial U}{\partial I_{t-1}} &= U_{TH_t}\phi_t'(p-\hat{\delta}_{t-1}) \\ &+ U_{TH_{t+1}}\phi_{t+1}'(p-\hat{\delta}_{t-1})(1-\delta_t) \\ &+ U_{TH_{t+2}}\phi_{t+2}'(p-\hat{\delta}_{t-1})(1-\delta_t)(1-\delta_{t+1}) \\ &+ \cdots \\ &+ U_{TH_T}\phi_T'(p-\hat{\delta}_{t-1})(1-\delta_t)(1-\delta_{t+1})\cdots(1-\delta_{T-1}).\end{aligned} \tag{8}$$

For ②, since $\frac{\partial R}{\partial I_{t-1}} = 0$,

$$\begin{aligned}\lambda\frac{\partial(R-\Sigma(\cdot))}{\partial I_{t-1}} &= -\lambda\frac{\partial \Sigma_0^T \frac{C_t^I + C_t^Y + w_t TUH_t}{(1+r)^t}}{\partial I_{t-1}} \\ &= -\lambda[\frac{\partial \frac{C_{t-1}^I}{(1+r)^{t-1}}}{\partial I_{t-1}} + \underbrace{\frac{\partial \Sigma_0^T \frac{w_t TUH_t}{(1+r)^t}}{\partial I_{t-1}}}_{③}] \\ &= -\lambda[\frac{\pi_{t-1}^I}{(1+r)^{t-1}} + ③],\end{aligned} \tag{9}$$

where denote $\pi_{t-1}^I = \partial C_{t-1}^I/\partial I_{t-1}$ the marginal cost of gross investment in period $t-1$,

$$\begin{aligned}③ \quad = \quad & \frac{w_t}{(1+r)^t}\frac{\partial TUH_t}{\partial TH_t}\frac{\partial TH_t}{\partial H_t}\frac{\partial H_t}{\partial I_{t-1}} + \frac{w_{t+1}}{(1+r)^{t+1}}\frac{\partial TUH_{t+1}}{\partial TH_{t+1}}\frac{\partial TH_{t+1}}{\partial H_{t+1}}\frac{\partial H_{t+1}}{\partial I_{t-1}} + \\ \cdots \quad & + \frac{w_T}{(1+r)^T}\frac{\partial TUH_T}{\partial TH_T}\frac{\partial TH_T}{\partial H_T}\frac{\partial H_T}{\partial I_{t-1}}.\end{aligned} \tag{10}$$

By notation (6) and $\partial TUH_i/\partial TH_i = -1$ (since $TUH_t = \Omega - TH_t$), ② is simplified to be,

$$\begin{aligned}\lambda\frac{\partial(R-\Sigma(\cdot))}{\partial I_{t-1}} = \quad & -\lambda\{\frac{\pi_{t-1}^I}{(1+r)^{t-1}} \\ & -\frac{w_t}{(1+r)^t}\phi_t'(p-\hat{\delta}_{t-1}) \\ & -\frac{w_{t+1}}{(1+r)^{t+1}}\phi_{t+1}'(p-\hat{\delta}_{t-1})(1-\delta_t) \\ & -\frac{w_{t+2}}{(1+r)^{t+2}}\phi_{t+2}'(p-\hat{\delta}_{t-1})(1-\delta_t)(1-\delta_{t+1}) - \cdots \\ & -\frac{w_T}{(1+r)^T}\phi_T'(p-\hat{\delta}_{t-1})(1-\delta_t)(1-\delta_{t+1})\cdots(1-\delta_{T-1})\}\end{aligned} \tag{11}$$

Hence, the F.O.C that ① + ② = 0 is,

$$\begin{aligned}\frac{\pi^I_{t-1}}{(1+r)^{t-1}} &= \phi'_t[\frac{U_{TH_t}}{\lambda} + \frac{w_t}{(1+r)^t}](p - \hat{\delta}_{t-1}) \\ &+ \phi'_{t+1}[\frac{U_{TH_{t+1}}}{\lambda} + \frac{w_{t+1}}{(1+r)^{t+1}}](p - \hat{\delta}_{t-1})(1-\delta_t) \\ &+ \cdots \\ &+ \phi'_T[\frac{U_{TH_T}}{\lambda} + \frac{w_T}{(1+r)^T}](p - \hat{\delta}_{t-1})(1-\delta_t)(1-\delta_{t+1})\cdots(1-\delta_{T-1}).\end{aligned} \tag{12}$$

To derive discounted change of equilibrium net from δ from period $t-1$ to period t,

$$\frac{\pi^I_{t-1}}{(1+r)^{t-1}(1-\delta_t)} - \frac{\pi^I_t}{(1+r)^t} = \frac{\phi'_t(p - \hat{\delta}_{t-1})}{(1-\delta_t)}[\frac{U_{TH_t}}{\lambda} + \frac{w_t}{(1+r)^t}], \tag{13}$$

by rearranging,

$$\begin{aligned}\phi'_t(p - \hat{\delta}_{t-1})[w_t + \frac{U_{TH_t}}{\lambda}(1+r)^t] &= \pi^I_{t-1}(1+r) - (1-\delta_t)\pi^I_t \\ &= \pi^I_{t-1}[r + (\delta_t - 1)\tilde{\pi}_{t-1} + \delta_t],\end{aligned} \tag{14}$$

where $\tilde{\pi}_{t-1} = (\pi_t - \pi_{t-1})/\pi_{t-1}$. Assuming $\delta_t \tilde{\pi}^I_{t-1} \approx 0$,

$$\frac{\phi'_t}{\pi^I_{t-1}}[w_t + \frac{U_{TH_t}}{\lambda}(1+r)^t] = \frac{r - \tilde{\pi}_{t-1} + \delta_t}{p - \hat{\delta}_{t-1}} = \frac{r - \tilde{\pi}_{t-1} + \delta_t}{p - \frac{\partial \delta_{t-1}}{\partial I_{t-1}} H_{t-1}}.$$

Hence we have equation (7) in the manuscript.