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Commitment in symmetric contests

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Abstract

The paper proves that in two-player logit form symmetric contests with concave success function, commitment to a particular strategy does not increase a player's payoff, while in contests with more than two players it does. The paper also provides a contest-like game in which commitment does not increase a player''s payoff for any number of players.

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1. Introduction

It has been noted that in two-player symmetric contests, local commitment to an action does not increase a player's payoff (Dixit 1987), while in contests with more than two players it does. However, a proof of the global result for general contests has not been provided (Kräkel 2002, has a proof of a related result for linear success function).

The local result is based on the slope of the opponent's reaction function being zero at the symmetric equilibrium, thus being equal to the slope of the level curve of a player's payoff function. This is a necessary condition for the result but is not sufficient. A comment on Dixit's paper by Baye and Shin (1999) discusses an example of a contest game where the result does not hold.

This paper proves that when the contest probability of winning has the logit form with concave success function, then the result holds globally, i.e. a player's payoff is not increased from commitment to any strategy, not necessarily close to equilibrium.

Two-player contests may appear special as the result does not extend to contests with more than two players. I present a game in which the result holds for any number of players. The game is a modification of the usual contest game in which each player participates in a contest against the *average* effort of other players.

2. Two-player logit contests

Consider two-player symmetric contests in which Players 1 and 2 simultaneously choose an effort or investment levels $x_i \in [0, \infty)$, $i = 1, 2$. The payoffs are

$$u_1(x_1, x_2) = \frac{f(x_1)}{f(x_1) + f(x_2)} V - x_1, \tag{1}$$

where $V > 0$ is the value of the prize, $f(x) \geq 0$ when $x = 0$, $f'(x) > 0$ for all x, $f''(x) \leq 0$ for all x, and $u_2(x_1, x_2) = u_1(x_2, x_1)$.

The first order conditions for Nash equilibrium are

$$\frac{\partial u_1}{\partial x_1} = \frac{f'(x_1) f(x_2)}{(f(x_1) + f(x_2))^2} V - 1 = 0, \tag{2}$$

$$\frac{\partial u_2}{\partial x_2} = \frac{f'(x_2) f(x_1)}{(f(x_1) + f(x_2))^2} V - 1 = 0. \tag{3}$$

For all interior x_1, x_2, the second order conditions $\partial^2 u_i / \partial x_i^2 = f(x_j)(f''(x_i)(f(x_i) + f(x_j)) - 2(f'(x_i))^2) V / (f(x_i) + f(x_j))^3 < 0$ are satisfied. Therefore the first order conditions define the reaction functions of the players.

The first order conditions are $f'(x^*) V / (4 f(x^*)) - 1 = 0$ at a symmetric equilibrium $x_1 = x_2 = x^*$. If $\lim_{x \to 0} f'(x) V / (4 f(x)) > 1$ and $\lim_{x \to \infty} f'(x) V / (4 f(x)) < 1$, then there is unique interior symmetric equilibrium. The conditions are satisfied for example by function $f(x) = x^r$ for $r \leq 1$.

The second cross-derivative of the payoff function is

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \frac{f'(x_i)f'(x_j)(f(x_i) - f(x_j))}{(f(x_i) + f(x_j))^3}V. \tag{4}$$

From the first order conditions, the slope of the reaction function $\hat{x}_i(x_j)$ of Player i is $d\hat{x}_i/dx_j = -(\partial^2 u_i/\partial x_i \partial x_j)/(\partial^2 u_i/\partial x_i^2)$. Thus

$$\frac{d\hat{x}_i}{dx_j} = -\frac{f'(\hat{x}_i)f'(x_j)(f(\hat{x}_i) - f(x_j))}{f(x_j)(f''(\hat{x}_i)(f(\hat{x}_i) + f(x_j)) - 2(f'(\hat{x}_i))^2)}. \tag{5}$$

Suppose that Player 1 can commit to an action x_1, observable by Player 2 who then chooses x_2. Player 1 then maximizes $u_1(x_1, \hat{x}_2(x_1))$. The first order condition for maximization is

$$\frac{du_1}{dx_1} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{d\hat{x}_2}{dx_1} = 0.$$

At the symmetric equilibrium x^*, $\partial u_1/\partial x_1 = 0$. From (4), at this equilibrium $d\hat{x}_2/dx_1 = 0$. Therefore $du_1/dx_1 = 0$ at $x_1 = x^*$. The necessary condition for maximization is satisfied at the simultaneous move game equilibrium x^*. This is the result noted by Dixit (1987). However, whether $x_1 = x^*$ is indeed a global maximum is left open, although Dixit notes that this depends on the curvatures of the best response function and of level contours.

The second order condition for maximization is

$$\frac{d^2u_1}{dx_1^2} = \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2 \partial x_1}\frac{d\hat{x}_2}{dx_1} + \left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2}\frac{d\hat{x}_2}{dx_1}\right)\frac{d\hat{x}_2}{dx_1} + \frac{\partial u_1}{\partial x_2}\frac{d^2\hat{x}_2}{dx_1^2} < 0.$$

At symmetric equilibrium $d\hat{x}_2/dx_1 = 0$, thus $\partial^2 u_1/\partial x_1^2 + \partial u_1/\partial x_2 \cdot d^2\hat{x}_2/dx_1^2 < 0$. From (5), $d^2\hat{x}_2/dx_1^2 = (f'(x^*))^3/(2f(x^*)(f''(x^*)f(x^*) - (f'(x^*))^2))$ because $d(d\hat{x}_2/dx_1)/dx_1 = \partial(d\hat{x}_2/dx_1)/\partial x_1$ at equilibrium. Since $\partial u_1/\partial x_2 = -f(x_1)f'(x_2)/(f(x_1) + f(x_2))^2$, at x^*

$$\frac{d^2u_1}{dx_1^2} = \frac{f''(x^*)f(x^*) - (f'(x^*))^2}{4(f(x^*))^2}V - \frac{f'(x^*)}{4f(x^*)}V\frac{(f'(x^*))^3}{2f(x^*)(f''(x^*)f(x^*) - (f'(x^*))^2)}.$$

From the first order conditions, at equilibrium $f'(x^*)V = 4f(x^*)$. Then

$$\frac{d^2u_1}{dx_1^2} = \frac{2(f''(x^*)V - 16f(x^*))^2 - 16^2 f(x^*)^2}{8(f''(x^*)f(x^*) - (f'(x^*))^2)}.$$

The minimum of the numerator is achieved when $f'' = 0$. Then the expression is positive, thus the numerator is positive for all x^*. Thus $d^2u_1/dx_1^2 < 0$ at equilibrium. Therefore locally the second order condition for a maximum is satisfied. This can also be checked by using the condition in Baye and Shin (1999) on the derivatives of the contest winning probability function.

To prove that $x_1 = x^*$ is a global maximum, consider the following. The level curve of Player 1 passing through the equilibrium x^* is

$$\frac{f(x_1)}{f(x_1)+f(x_2)}V - x_1 = \frac{1}{2}V - x^*. \tag{6}$$

The reaction function of Player 2 is given by (3). If one can show that the level curve and the reaction function have only x^* in common, then the local second order condition is sufficient for a global maximum.

From the level curve, $f(x_1)+f(x_2) = f(x_1)V/(V/2-(x^*-x_1))$, or

$$f(x_2) = f(x_1)\frac{V/2+(x^*-x_1)}{V/2-(x^*-x_1)}. \tag{7}$$

Substituted into the reaction function, $f(x_1)f'(x_2)V = (f(x_1)V/(V/2-(x^*-x_1)))^2$, or

$$f'(x_2) = f(x_1)\frac{V}{(V/2-(x^*-x_1))^2}. \tag{8}$$

The derivative of the right-hand side of (8) w.r.t. x_1 is $V(f'(x_1)(V/2-(x^*-x_1)) - 2f(x_1))/(V/2-(x^*-x_1))^3$. Since $V/2-(x^*-x_1) > 0$, the denominator is positive. The derivative of the numerator w.r.t. x_1 is $f''(x_1)(V/2-(x^*-x_1)) - f'(x_1) < 0$. Since the numerator is zero when $x_1 = x^*$ and is decreasing, $f(x_1)V(1/(V/2-(x^*-x_1)))^2$ is decreasing when $x_1 > x^*$ and increasing when $x_1 < x^*$.

The derivative of the right-hand side of (7) w.r.t. x_1 is $(f'(x_1)(V^2/4-(x^*-x_1)^2) - Vf(x_1))/(V/2-(x^*-x_1))^2$. The derivative of the numerator is $f''(x_1)(V^2/4-(x^*-x_1)^2) + f'(x_1)(2(x^*-x_1)-V)$. From (6), $|x^*-x_1| < V/2$ for positive x_1, x_2. Therefore the derivative of the numerator is negative. Since the numerator is zero when $x_1 = x^*$, the same conclusion as in the previous paragraph follows: $f(x_1)(V/2+(x^*-x_1))/(V/2-(x^*-x_1))$ is decreasing when $x_1 > x^*$ and increasing when $x_1 < x^*$.

Consider $x_1 < x^*$. Since the right-hand side of equation (7) is increasing, $f(x_1)(V/2+(x^*-x_1))/(V/2-(x^*-x_1)) < f(x^*)$. Then the x_2 that satisfies (7) is less than x^*. Then $f'(x_2) \geq f'(x^*) = 4f(x^*)/V$. Since the right-hand side of equation (8) is also increasing, $f(x_1)V(1/(V/2-(x^*-x_1)))^2 < 4f(x^*)/V$. Thus (8) is not satisfied. A similar reasoning shows that the two equations cannot be satisfied for $x_1 > x^*$. Therefore $x_1 = x_2 = x^*$ is the unique point that satisfies the two equations.

Since the local second order conditions are satisfied for $x_1 = x_2 = x^*$ and this is the only point that the level curve passing through it has in common with the reaction function of Player 2, other points on the reaction function give lower payoff to Player 1. Thus

Theorem 1 *Suppose that in two-player symmetric contests with payoff function (1), where the contest success function satisfies* $f(0) \geq 0$, $f'(x) > 0, f''(x) \leq 0$ *for all* $x > 0$, *there exists a simultaneous move interior symmetric equilibrium* x^*. *Then the subgame perfect equilibrium outcome when Player 1 can commit to an observable action before Player 2 is* $x_1 = x_2 = x^*$.

The equality of the slopes of the reaction function and of the level curve at the simultaneous move equilibrium is necessary but not sufficient. Baye and Shin (1999) discuss contest games where a player gains because the local second order condition does not hold. They show that among such games are logit form contests with $f(x) = x^r$ for $r \in (\sqrt{2}, 2]$.

3. Contests with more than two players

A symmetric n-player contest with the logit form probability of winning the prize has payoff functions

$$u_i(x_1, \ldots, x_n) = \frac{f(x_i)}{\sum_{j=1}^{n} f(x_j)} V - x_i, \tag{9}$$

with the same assumptions on $f(x)$ as in the previous section, $f(0) \geq 0$, $f'(x) > 0$, $f''(x) \leq 0$ for all $x > 0$.

The derivatives of the payoff function are:

$$\frac{\partial u_i}{\partial x_i} = \frac{f'(x_i) \sum_{k \neq i} f(x_k)}{\left(\sum_{k=1}^{n} f(x_k)\right)^2} V - 1, \quad \frac{\partial u_i}{\partial x_j} = \frac{-f(x_i) f'(x_j)}{\left(\sum_{k=1}^{n} f(x_k)\right)^2} V,$$

$$\frac{\partial^2 u_i}{\partial x_i^2} = \frac{\left(f''(x_i) \sum_{k=1}^{n} f(x_k) - 2(f'(x_i))^2\right) \sum_{k \neq i} f(x_k)}{\left(\sum_{k=1}^{n} f(x_k)\right)^3} V,$$

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \frac{f'(x_i) f'(x_j)(f(x_i) - \sum_{k \neq i} f(x_k))}{\left(\sum_{k=1}^{n} f(x_k)\right)^3} V.$$

Since $\partial^2 u_i / \partial x_i^2 < 0$ for all interior points, at an interior symmetric equilibrium $(x^*, \ldots, x^*)$ $\partial u_i / \partial x_i = (n-1) f'(x^*)/n^2 f(x^*) \cdot V - 1 = 0$. Such an equilibrium exists when $\lim_{x \to 0} (n-1) f'(x) V/(n^2 f(x)) > 1$ and $\lim_{x \to \infty} (n-1) f'(x) V/(n^2 f(x)) < 1$. These conditions are satisfied e.g. by $f(x) = x^r$ for $r \leq 1$.

Suppose that Player 1 can commit to an action x_1, observable by all other players who then choose their actions simultaneously. The first order condition for maximization of $u_1(x_1, \hat{x}_2(x_1), \ldots, \hat{x}_n(x_1))$ is

$$\frac{du_1}{dx_1} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \frac{d\hat{x}_2}{dx_1} + \ldots + \frac{\partial u_1}{\partial x_n} \frac{d\hat{x}_n}{dx_1}.$$

Equilibrium reaction functions $\hat{x}_i(x_1)$ are given implicitly by the first order conditions $\partial u_i/\partial x_i(x_1,\ldots,x_n)=0$, $i=2,\ldots,n$. Differentiating each of them w.r.t. x_1 gives

$$\frac{\partial^2 u_i}{\partial x_1 \partial x_i}+\frac{\partial^2 u_i}{\partial x_2 \partial x_i}\frac{d\hat{x}_2}{dx_1}+\ldots+\frac{\partial^2 u_i}{\partial x_n \partial x_i}\frac{d\hat{x}_n}{dx_1}=0,\ i=2,\ldots,n.$$

At symmetric equilibrium $\partial^2 u_i/\partial x_i^2=(n-1)(nf''(x^*)f(x^*)-2(f'(x^*))^2)V/(n^3 f(x^*)^2)$ for all i and $\partial^2 u_i/\partial x_j \partial x_i=(2-n)(f'(x^*))^2 V/(n^3 f(x^*)^2)$ for all $i,j\neq i$. Summing up the $n-1$ equations in the above display gives

$$(n-1)\frac{\partial^2 u_i}{\partial x_j \partial x_i}+\left(\frac{\partial^2 u_i}{\partial x_i^2}+(n-2)\frac{\partial^2 u_i}{\partial x_j \partial x_i}\right)\left(\frac{d\hat{x}_2}{dx_1}+\ldots+\frac{d\hat{x}_n}{dx_1}\right)=0. \tag{10}$$

Since at equilibrium $\partial u_1/\partial x_i=-f'(x^*)V/(n^2 f(x^*))\neq 0$ for all $i\neq 1$, and $\partial u_1/\partial x_1=0$, the first order condition for maximization becomes $du_1/dx_1=\partial u_1/\partial x_i\cdot(d\hat{x}_2/dx_1+\ldots+d\hat{x}_n/dx_1)=0$. However, from (10) $d\hat{x}_2/dx_1+\ldots+d\hat{x}_n/dx_1\neq 0$ for $n>2$, since $\partial^2 u_i/\partial x_j \partial x_i\neq 0$ then. Therefore

Proposition 1 *In symmetric n-player contests with payoff function (9), the subgame perfect equilibrium outcome when one player can commit to an observable action before other players is different from the simultaneous move equilibrium x^* when $n>2$.*

The result is a particular case of a result in Dixit (1987) where asymmetric contests are also allowed. The result depends on whether $\partial^2 u_i/\partial x_j \partial x_i=0$ as this determines whether $d\hat{x}_i/dx_1=0$.

Contests appear to have a difference between cases $n=2$ and $n>2$. However, the payoff functions can be modified to construct games in which commitment does not give an advantage for other values of n or indeed for any value of n. Consider the payoff function

$$u_i(x_1,\ldots,x_n)=\frac{kf(x_i)}{kf(x_i)+m\sum_{j\neq i}f(x_j)}V-x_i.$$

The second cross-derivative of this payoff function is

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j}=\frac{kmf'(x_i)f'(x_j)(kf(x_i)-m\sum_{j\neq i}f(x_j))}{\left(kf(x_i)+m\sum_{j\neq i}f(x_j)\right)^3}V$$

and at symmetric equilibrium x^*

$$\frac{\partial^2 u_1}{\partial x_i \partial x_j}=\frac{km(f'(x^*))^2(k-m(n-1))}{(k+m(n-1))^3 f(x^*)^2}V.$$

This expression is zero when $k=m(n-1)$. For various values of n, one can construct games so that the necessary condition for maximization of $u_1(x_1,\hat{x}_2(x_1),\ldots,\hat{x}_n(x_1))$ is satisfied. For example, when $n = 3$, then $m = 1/2$ and $k = 1$ gives a game for which it is satisfied.

Taking $m = 1$ and $k = n - 1$ gives a game in which the slope of the reaction function is zero at symmetric equilibrium for all n. Rewriting this game's payoff function as

$$u_i(x_1, \ldots, x_n) = \frac{f(x_i)}{f(x_i) + \frac{1}{n-1}\sum_{j \neq i} f(x_j)} V - x_i \tag{11}$$

gives an interpretation that each player engages in a contest with a potentially variable total prize against the *average* effort of other players. For this game, independently of n, commitment may have no advantage.

To show that there is a game in which the possibility of commitment by Player 1 gives no advantage for any n, consider the case $f(x) = x$. The symmetric interior equilibrium for this game is found from the first order conditions

$$\frac{\frac{1}{n-1}\sum_{j \neq i} x_j}{(x_i + \frac{1}{n-1}\sum_{j \neq i} x_j)^2} V - 1 = 0.$$

When $x_i = x_j = x^*$ for all i, j, then $x^* = V/4$.

From the first order condition of Player i, $(x_1 + \sum_{j \neq i} \hat{x}_j)V/(n-1) = (\hat{x}_i + (x_1 + \sum_{j \neq i} \hat{x}_j)/(n-1))^2$. Subtracting the first order condition for Player j from the one of Player i gives $(\hat{x}_j - \hat{x}_i)V/n = (\hat{x}_j - \hat{x}_i)(1/(n-1) - 1)(1 + 1/(n-1)(\hat{x}_i + \hat{x}_j) + 2/(n-1)(x_1 + \sum_{k \neq i,j} \hat{x}_k))$. If $\hat{x}_j \neq \hat{x}_i$, then $\hat{x}_j - \hat{x}_i$ can be cancelled from the two sides. But then the left-hand side is positive while the right-hand side is negative. Therefore $\hat{x}_j = \hat{x}_i$ at an interior equilibrium.

When $\hat{x}_i = \hat{x}$ for all $i \neq 1$, the level curve of Player 1 through equilibrium $x_i = x^*$ is

$$\frac{x_1}{x_1 + \hat{x}} - x_1 = \frac{V}{4} \tag{12}$$

and $\hat{x}$ satisfies the first order condition for Players i, $i \neq 1$

$$\frac{\frac{1}{n-1}(x_1 + (n-2)\hat{x})}{(\hat{x} + \frac{1}{n-1}(x_1 + (n-2)\hat{x}))^2} V - 1 = 0. \tag{13}$$

The first order condition can be rewritten as $F(x_1, \hat{x}) = (n-1)(x_1 + (n-2)\hat{x})V - ((2n-3)\hat{x} + x_1)^2 = 0$. Substituting $\hat{x}$ from the level curve, simplifying and factorizing the expression gives $(4x_1 - V)^2(x_1(4n^2 - 4n + 4) + V(8n - 3n^2 - 5)) = 0$. The last parenthesis is zero when $x_1 = V(3n^2 - 8n + 5)/(4n^2 - 4n + 4)$. The right-hand side is larger than $V/4$ for $n > 1$ and then $\hat{x} < 0$ from the level curve. Thus the only points satisfying both equations (12) and (13) with both $x_1, \hat{x}$ positive is $x_1 = \hat{x} = V/4$.

Player 1 maximizes $u_1(x_1, \hat{x}(x_1), \ldots, \hat{x}(x_1))$. The local second order condition for maximum is $d^2u_1/dx_1^2 < 0$. At equilibrium $d^2u_1/dx_1^2 = \partial^2 u_1/\partial x_1^2 + (n-1)\partial u_1/\partial \hat{x} \cdot d^2\hat{x}/dx_1^2$.

From the reaction function of the other players, $d\hat{x}/dx_1 = -(\partial F/\partial x_1)/(\partial F/\partial \hat{x})$ and at equilibrium $\partial F/\partial x_1 = 0$. Then $d^2\hat{x}/dx_1^2 = -(\partial^2 F/\partial x_1^2)/(\partial F/\partial \hat{x})$. Evaluating the derivatives at $x_1 = \hat{x} = V/4$ gives $d^2 u_1/dx_1^2 = (-4(n-1)+2)/(V(n-1)) < 0$ for $n > 1$.

Since the local second order condition is satisfied and the reaction function of Players i, $i \neq 1$ has only the symmetric equilibrium point $x_1 = x^* = V/4$ in common with the level curve of Player 1, choosing x_1 different from $V/4$ cannot give Player 1 higher payoff.

Proposition 2 *In the game with payoff function (11) with $f(x) = x$, the outcome of the subgame perfect equilibrium when Player 1 can commit to an observable action is the same as the outcome in the equilibrium $x^* = V/4$ of simultaneous move game for any n.*

4. Conclusion

This paper proves that in two-player logit form symmetric contests with concave success functions, the possibility of commitment does not benefit the player who can commit. In strategic situations the necessary condition for this is that at equilibrium the reaction function of the other player has the same slope as the player's payoff level curve. This condition is not sufficient but in the contests analyzed, commitment indeed does not lead to a higher payoff.

In n-player symmetric contests with $n > 2$ commitment is always advantageous. Modifying the payoff function to represent the game as a contest against the average effort of other players leads to a game where commitment does not work for any n.

Commitment may have several interpretations. For example, delegation (e.g. Vickers 1985) or indirect evolution of preferences (Güth and Yaari 1992) use commitment. In those cases the committed player has a different reaction function and therefore the outcome shifts along the reaction function of the opponent. The results of this paper and of Possajennikov (2008) imply that in games where commitment to an action does not increase a player's payoff, preferences coinciding with material payoffs are stable, or delegates are provided with incentives to maximize principal's payoff. Two-player contests and contests against the average effort of other players are examples of such games.

References

Baye, M. R., and O. Shin (1999) "Strategic Behavior in Contests: Comment" *American Economic Review* **89**(3), 691–693.

Dixit, A. (1987) "Strategic Behavior in Contests" *American Economic Review* **77**(5), 891–898.

Güth, W., and M. Yaari (1992) "Explaining Reciprocal Behavior in Simple Strategic Games: An Evolutionary Approach" in *Explaining Forces and Changes: Approaches to Evolutionary Economics* by U. Witt, Ed., University of Michigan Press.

Kräkel, M. (2002) "Delegation and Strategic Incentives for Managers in Contests" *Managerial and Decision Economics* **23**, 461–470.

Possajennikov, A. (2008) "On the Survival of Payoff Maximizing Behavior and Delegation in Contests" University of Nottingham, working paper, available on http://www.nottingham.ac.uk/~lezap/.

Vickers, J. (1985) "Delegation and the Theory of the Firm" *Economic Journal* **95**, 138–147.