Economics Bulletin

Volume 29, Issue 1

The proportional rule for multi-issue bankruptcy problems

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Abstract

We investigate how to extend bankruptcy rules to the more general setting in which claims may refer to different issues. We consider two natural procedures and show that, among all bankruptcy rules, the proportional rule is the only one whose extensions according to the two procedures yield the same outcomes.

I am most grateful to Marek Kaminski for detailed comments and suggestions. Thanks are also due to Carmen Herrero, Dunia López-Pintado, Jordi Massó, Bernardo Moreno and William Thomson. I retain, however, the responsibility for any shortcoming. Financial support from the Spanish Ministry of Science and Innovation (ECO2008-03883/ECON), Junta de Andalucía (P06- SEJ-01645, P08-SEJ-04154) and Fundación Ramón Areces is gratefully acknowledged.

Citation: Juan D. Moreno-Ternero, (2009) "The proportional rule for multi-issue bankruptcy problems", *Economics Bulletin*, Vol. 29 no.1 pp. 474-481.

Submitted: Feb 06 2009. **Published:** March 27, 2009.

1 Introduction

Imagine a country with autonomous regions in which each regional government decides how to spend the resources transferred from the central government. Regional expenditures refer to the different issues lying within the jurisdiction of the regional public system (e.g., health care provision, public education, highways, etc.) Imagine now that the total resources the central government may transfer is below the aggregate expenditure autonomous governments face. How should the central government assign the available amount?

A somewhat related situation occurs in the proposals for financial support to the NSF, or any other foundation promoting scientific research. Each proposal must contain a budget referring to different issues, such as salaries and wages, fringe benefits, equipment, travel, etc. The proposal may request funds under any of the issues, so long as the item and amount are considered necessary to perform the proposed work and are not precluded by specific program guidelines or applicable cost principles. If the whole budget the foundation has to allocate is publicly known and falls below the aggregated budget proposals claim, how should it be allocated?

These are instances of *multi-issue bankruptcy problems*, i.e., problems in which an arbitrator has to allocate a given amount of a perfectly divisible commodity among a group of agents, when the available amount is not enough to satisfy all their claims and these claims refer to different issues.

The problem of dividing when there is not enough is one of the oldest problems in the history of economic thought. Although problems of this sort (and possible solutions for them) are already documented in ancient sources such as the Talmud, Aristotle' books or Maimonides' essays, their formalization was not presented till the early eighties by Barry O'Neill. O'Neill (1982) was the first to provide an extremely simple model in which a variety of situations (such as the bankruptcy of a firm, the division of an insufficient estate to cover all the existing debts, sharing the cost of an indivisible public facility and the collection of a given amount of taxes) fit.[1] Multi-issue bankruptcy problems ("*MIB problems*" hereafter) are the outcome of leading this model one step beyond, so that agents' claims are allowed to refer to different issues. These problems have been studied by Kaminski (2000, 2006), Calleja et al., (2005) and Ju et al., (2007), among others.

As bankruptcy problems, MIB problems are solved by means of *rules*. A multi-issue bankruptcy rule is a mapping that associates with each MIB problem an allocation indicating the amount that each agent obtains for each issue, subject to two conditions: the (non-negative) amount that every agent obtains for each issue is not greater than her claim and the whole amount is distributed.[2] A natural question is how to extend classical bankruptcy rules to the more general setting of MIB problems. We provide in this note two possible answers. The first one considers the amount of each issue that an agent claims, as an independent claim of a *new* agent. This results in a new (unidimensional) bankruptcy problem to which the original rule could be applied. The other alternative is based on a two-stage process: First, the primitive rule is applied to the bankruptcy problem in which claims for the same issue are aggregated into a single claim. The amount allocated to each issue is then distributed among the agents by applying the same rule. We show that there is only one rule whose two extensions according to the two alternatives just described yield the same outcomes. It is the so-called *proportional* rule, probably the rule with the longest tradition of use (whose spirit can be traced back to Aristotle), which allocates the available amount proportionally to agents' claims.

[1] The reader is referred to Moulin (2002) and Thomson (2003, 2006) for reviews of the fast-expanding literature concerning this model.

[2] A different plausible definition of a rule is considered in the concluding section.

2 The Model

Let $\mathbb{N}$ represent the set of all potential agents (an infinite set) and let $\mathcal{N}$ be the family of all finite (non-empty) subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents $N = \{1, 2, ..., n\}$, where we take $|N| = n \geq 2$. Let $\mathbb{M}$ represent the set of all potential issues (an infinite set) and let $\mathcal{M}$ be the family of all finite (non-empty) subsets of $\mathbb{M}$. An element $M \in \mathcal{M}$ describes a finite set of issues $M = \{1, 2, ..., m\}$, where we take $|M| = m \geq 2$. Let $\mathcal{M}_{m\times n}$ represent the set of matrices with m rows and n columns and $\mathcal{M}^+_{m\times n}$ its subset of matrices whose entries are all non-negative.

A **multi-issue bankruptcy problem** (**MIB problem**) is a 4−tuple (M, N, E, C), where M is the set of issues, N is the set of agents, $E \in \mathbb{R}_+$ is an amount to divide, and $C \in \mathcal{M}^+_{m\times n}$ is a **matrix of claims**. The family of all these problems is denoted by $\mathcal{MIB}$. Every row in C represents an issue. A generic element of C, c_{ij}, denotes the amount of issue i that agent j claims. The very notion of MIB problem requires that $||C|| = \sum_{i\in M} \sum_{j\in N} c_{ij} \geq E$. To simplify notation we write, for any given problem $(M, N, E, C) \in \mathcal{MIB}$, $c_{Mj} = \sum_{i\in M} c_{ij}$ to the aggregate claim of agent $j \in N$, and $c_{iN} = \sum_{j\in N} c_{ij}$ to the aggregate claim according to issue $i \in M$.

*A **multi-issue bankruptcy rule (MIB rule)** is a mapping R that associates with every $(M, N, E, C) \in \mathcal{MIB}$ a unique matrix $R(M, N, E, C) \in \mathcal{M}_{m\times n}$ such that:*
(i) $0 \leq R_{ij}(M, N, E, C) \leq c_{ij}$*, for all* $(i, j) \in M \times N$.
(ii) $\sum_{(i,j)\in M\times N} R_{ij}(M, N, E, C) = E$.

The matrix $R(M, N, E, C)$ represents a desirable way of dividing E among the agents in N, according to the issues in M. Requirement (i) is that each agent receives an award for each issue that is non-negative and bounded above by her claim. Requirement (ii) is that the entire available amount be allocated. These two requirements imply that $R(M, N, E, C) = C$ whenever $E = \sum_{(i,j)\in M\times N} c_{ij}$.

A **bankruptcy problem** is a triple (N, E, c), where N is the set of agents, $E \in \mathbb{R}_+$ is an amount to divide, and $c \in \mathbb{R}^n_+$ is a vector of claims whose ith component is c_i, with $\sum_{i\in N} c_i \geq E > 0$. The family of all those bankruptcy problems is denoted by $\mathcal{B}$. Given a MIB problem, we can associate with it a bankruptcy problem, by disentangling agents and issues, as follows:

Definition 1 *Given $(M, N, E, C) \in \mathcal{MIB}$, its associated bankruptcy problem is $(N^M, E, c^M) \in \mathcal{B}$, where $N^M = \{1, 2, ..., m \cdot n\}$ and $c^M = (c_{11}, ..., c_{1n}, c_{21}, ..., c_{2n}, ..., c_{m1}, ..., c_{mn}) \in \mathbb{R}^{m\cdot n}$.*

*A **bankruptcy rule** is a mapping r that associates with every $(N, E, c) \in \mathcal{B}$ a unique point $r(N, E, c) \in \mathbb{R}^n$ such that:*
(i) $0 \leq r(N, E, c) \leq c$.
(ii) $\sum_{i\in N} r_i(N, E, c) = E$.

Throughout the note we shall only consider anonymous bankruptcy rules meeting the following mild property:

Independence of null claims. *For each $(N, E, c) \in \mathcal{B}$ and each $N' \subset N$, such that $y_i = 0$ for all $i \in N \setminus N'$, we have $R_j(N, E, c) = R_j(N', E, (c_i)_{i\in N'})$* for all $j \in N'$.

Some of the classical bankruptcy rules are the following. The *proportional* rule makes awards proportional to claims, i.e., $p(N, E, c) = \lambda c$, where $\lambda = \frac{E}{\sum c_i}$. The *constrained equal-awards* rule distributes the amount equally among all agents, subject to no agent receiving more than she claims, i.e., $a(N, E, c) = (\min\{c_i, \lambda\})_{i\in N}$, where $\lambda > 0$ is chosen so that $\sum \min\{c_i, \lambda\} = E$. The *constrained equal-losses* rule imposes that losses are as equal as possible subject to no agent receiving a negative amount, i.e., $l(N,E,c) = (\max\{0, c_i - \lambda\})_{i\in N}$, where $\lambda > 0$ is chosen so that $\sum \max\{0, c_i - \lambda\} = E$. Finally, the *Talmud* rule behaves like the constrained equal-awards rule or the constrained equal losses rule, depending on whether the amount to divide exceeds or falls short of half of the aggregate claim, and using half-claims instead of the claims themselves. Formally, $t(N,E,c) = (\min\{\frac{1}{2}c_i, \lambda\})_{i\in N}$ *if* $E \leq \frac{1}{2}\sum c_i$ and $t(N,E,c) = (\max\{\frac{1}{2}c_i, c_i - \mu\})_{i\in N}$ if $E \geq \frac{1}{2}\sum c_i$, where λ and μ are chosen so that $\sum t_i(N,E,c) = E$.

3 The problem

In this note we address the problem of how to extend bankruptcy rules to the multidimensional framework of MIB problems. We present two alternatives. The first one is the composition of the bankruptcy rule and the application that associates with each MIB problem a bankruptcy problem, as in Definition 1. Formally:

Definition 2 *Given a bankruptcy rule r, its* ***one-stage extension*** *to MIB problems, R^1, is the MIB rule that, for each $(M,N,E,C) \in \mathcal{MIB}$, and $(i,j) \in M \times N$, yields*

$$R^1_{ij}(M,N,E,C) = r_k(N^M, E, c^M),$$

where $k = n \cdot (i-1) + j$.

This one-stage extension, however, is not sensible to the different issues that appear in a MIB problem. We propose another possible way of extending rules, that captures such aspect: first, the primitive rule is applied to the bankruptcy problem in which claims for the same issue are aggregated into a single claim. The amount allocated to each issue is distributed then among agents, according to the same rule.[3] Formally:

Definition 3 *Given a bankruptcy rule r, its* ***two-stage extension*** *to MIB problems, R^2, is the MIB rule that, for each $(M,N,E,C) \in \mathcal{MIB}$, and $(i,j) \in M \times N$, yields*

$$R^2_{ij}(M,N,E,C) = r_j(N, r_i(M,E,(c_{kN})_{k\in M}), (c_{ik})_{k\in N}).$$

The following example shows that most of the classical bankruptcy rules fail to agree on these two extensions to the multi-issue setting.

Example 1 *Let $M = N = \{1,2\}$, $E = 5$ and $C = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}$. Then,*

- $A^1(M,N,E,C) = \begin{pmatrix} 1 & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} \end{pmatrix} \neq \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{5}{4} & \frac{5}{4} \end{pmatrix} = A^2(M,N,E,C)$
- $L^1(M,N,E,C) = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & \frac{3}{2} \\ \frac{13}{4} & \frac{1}{4} \end{pmatrix} = L^2(M,N,E,C)$
- $T^1(M,N,E,C) = \begin{pmatrix} \frac{1}{2} & \frac{7}{4} \\ \frac{7}{4} & 1 \end{pmatrix} \neq \begin{pmatrix} \frac{1}{2} & 2 \\ \frac{3}{2} & 1 \end{pmatrix} = T^2(M,N,E,C).$

Indeed, as Theorem 1 in the next section shows, the proportional rule is the only exception to these negative results.

[3] The obvious dual two-stage extension, i.e., first allocate among claimants and then among their specific claims, will not be considered here as it seems less intuitive to impose agents a precise way of distributing among claims, once they have been awarded with their overall amount.

4 The result

Theorem 1 *The proportional rule is the only (anonymous) bankruptcy rule, satisfying independence of null claims, whose two extensions to the multi-issue framework coincide.*

Proof.

Let P^1 and P^2 denote the one-stage and two-stage extensions, respectively, of the proportional rule to MIB problems. Let $(M, N, E, C) \in \mathcal{MIB}$ and $(i, j) \in M \times N$ be given. Then, it is straightforward to show that

$$P^1_{ij}(M, N, E, C) = \frac{E}{||C||} \cdot c_{ij}.$$

On the other hand,

$$P^2_{ij}(M, N, E, C) = p_j \left(N, \frac{E}{||C||} \cdot c_{iN}, (c_{ik})_{k \in N} \right) = \left(\frac{E}{||C||} \cdot c_{iN} \right) \cdot \frac{c_{ij}}{c_{iN}} = \frac{E}{||C||} \cdot c_{ij},$$

Hence, $P^1_{ij}(M, N, E, C) = P^2_{ij}(M, N, E, C)$, as claimed.

Conversely, let r be any bankruptcy rule different from the proportional rule. Let R^1 and R^2 denote its one-stage and two-stage extensions, respectively, to MIB problems. We show that $R^1 \neq R^2$.

Since $r \neq p$, it can be shown (e.g., Ju et al., 2007; Corollary 11) that there exist $(N, E, c) \in \mathcal{B}$, and $i, j \in N$ such that

$$r_i(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) \neq r_i(N, E, c) + r_j(N, E, c).$$

Assume, without loss of generality, that $N = \{1, ..., n\}$ and $i < j$. Then, since

$$\sum_l r_l(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) = E = \sum_l r_l(N, E, c),$$

it follows that there exists $k \in N \setminus \{i, j\}$ such that

$$r_k(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) \neq r_k(N, E, c). \tag{1}$$

For ease of notation, we assume that $k < j$.

Let $C \in \mathcal{M}^+_{(n-1) \times 2}$ be the following matrix:

$$C = \begin{pmatrix} c_1 & 0 \\ c_2 & 0 \\ \vdots & \vdots \\ c_{i-1} & 0 \\ c_i & c_j \\ c_{i+1} & 0 \\ \vdots & \vdots \\ c_{j-1} & 0 \\ c_{j+1} & 0 \\ \vdots & \vdots \\ c_n & 0 \end{pmatrix}$$

Then, $(N \setminus \{j\}, \{1,2\}, E, C) \in \mathcal{MIB}$. By Definition 2, anonymity, and independence of null claims, it is straightforward to show that

$$R^1(N \setminus \{j\}, \{1,2\}, E, C) = \begin{pmatrix} r_1(N,E,c) & 0 \\ r_2(N,E,c) & 0 \\ \vdots & \vdots \\ r_{i-1}(N,E,c) & 0 \\ r_i(N,E,c) & r_j(N,E,c) \\ r_{i+1}(N,E,c) & 0 \\ \vdots & \vdots \\ r_{j-1}(N,E,c) & 0 \\ r_{j+1}(N,E,c) & 0 \\ \vdots & \vdots \\ r_n(N,E,c) & 0 \end{pmatrix},$$

In particular,

$$R^1_{k,1}(N \setminus \{j\}, \{1,2\}, E, C) = r_k(N,E,c). \tag{2}$$

By Definition 3, anonymity, and independence of null claims, it is straightforward to show that

$$R^2(N \setminus \{j\}, \{1,2\}, E, C) = \begin{pmatrix} r_1(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) & 0 \\ r_2(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) & 0 \\ \vdots & \vdots \\ r_{i-1}(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) & 0 \\ x & y \\ r_{i+1}(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) & 0 \\ \vdots & \vdots \\ r_{j-1}(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) & 0 \\ r_{j+1}(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) & 0 \\ \vdots & \vdots \\ r_n(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})) & 0 \end{pmatrix},$$

where $(x,y) = r(\{i,j\}, r_i(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})), (c_i, c_j))$, i.e., x and y are such that $0 \leq x \leq c_i$, $0 \leq y \leq c_j$ and $x + y = r_i(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j}))$.

Therefore, in particular,

$$R^2_{k,1}(N \setminus \{j\}, \{1,2\}, E, C) = r_k(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})). \tag{3}$$

For the sake of contradiction, assume now that $R^1 = R^2$. Then,

$$R^1_{k,1}(N \setminus \{j\}, \{1,2\}, E, C) = R^2_{k,1}(N \setminus \{j\}, \{1,2\}, E, C).$$

Equivalently, from (2) and (3),

$$r_k(N,E,c) = r_k(N \setminus \{j\}, E, (c_i + c_j, (c_k)_{k \neq i,j})),$$

which contradicts (1). ■

5 Further insights

We have basically shown in this note that the proportional rule is the only rule in the classical bankruptcy model that can be extended indifferently to the multi-issue bankruptcy model, according to two natural procedures. This result provides additional support for the idea of proportional division in a claims problem which, despite being deeply rooted in the classical unidimensional context, has not been sufficiently explored in the multidimensional context. Practical (real-life) examples fitting the multi-issue bankruptcy model might call for the two extensions we propose in this note.[4] The fact that the two options only agree for the proportional rule is a valuable information that should ease the debate about the resolution of these practical examples.

As mentioned in the introduction, the classical bankruptcy model is able of accommodating many different situations with, perhaps, the bankruptcy of a firm as the usual running example. In actual bankruptcy laws, however, the creditors' legal characteristics are typically divided among categories (e.g., secured claims, unsecured claims, taxes, and trustee expenses) which calls for a more complex model in which, rather than single claims, vectors of claims, each indicating the claim for a given characteristic, are considered (e.g., Kaminski, 2006). Hence another instance of the usefulness of a multi-issue bankruptcy model.[5] Our choice for solving the multi-issue bankruptcy model has been to consider rules that assign for each agent a vector of quantities, each one associated with each issue. In doing so, we are implicitly precluding reallocations or transfers among claims of a same agent that might give rise to manipulations of the outcome.[6] Another choice that has been considered in the literature refers to rules that only specify the overall amount each agent gets, rather than the vector (e.g., Kaminski, 2000, 2006; Calleja et al., 2005; Ju et al., 2007).[7] Here the assumption of precluding manipulations via reallocation of claims can be made explicitly and if so, it leads (essentially) to a suitable extension of the proportional rule to this context (e.g., Ju et al., 2007).[8] Other justifications for our *suballocation* approach are, for instance, that a donor may require that the claimant makes specific cuts in the proposed expenses, or the desirability of the division of different streams of revenue within, say, a bank.[9] To summarize, what lies behind this approach is the idea that the main allocation in a multi-issue bankruptcy problem can impose certain constraints on the secondary allocation. Other options different to ours would have arisen upon imposing weaker constraints, such as making specific cuts in a given expense but allowing freedom with the remaining expenses.

[4] Think of, for instance, the case of university departments asking for funds to the university to be spent on different issues, such as maintenance, teaching material, seminars, etc., for which it is plausible to consider both the one-stage and two-stage approaches described above.

[5] A somewhat related, but more general problem, also described by Kaminski (2006), is the so-called shareholders problem (or the problem of the liquidation of a company) in which a shareholder can hold stock in a company, its bonds, and/or can be a lender. Such a setup would go beyond the framework of the multi-issue bankruptcy model, as an agent's type would not necessarily be a vector of claims (for a shareholder, the stock part is typically a proportion of stock or similar).

[6] In real bankruptcy cases, for example, the claims of different types exist in some legal form that actually precludes such a reallocation.

[7] Nevertheless, these multi-dimensional models could be in some cases unambiguously translated into our framework. For instance, if we consider a hydraulic method (e.g., Kaminski, 2000) with different vessels for different types of claims of the same claimant (as is the case with American bankruptcy law), then the content of every vessel would correspond to the *suballocations* we consider in this note.

[8] The American bankruptcy rule, which is indeed a mixed lexicographic-proportional rule can be singled out by adding a requirement of priority among asset types (e.g., Kaminski, 2000; Ju et al., 2007).

[9] The allocation of profits/losses among credit card and mortgage business may have tax, accounting and other consequences.

This work has several potential avenues for further research. To begin with, one might think of plausible extensions of axioms in the classical bankruptcy model to the multi-issue setting. It is plausible to conjecture that some combinations of these extensions would lead to parallel characterizations (to the existing ones in the classical bankruptcy model) of rules extended according to the two natural procedures we analyze in this note. Now, whereas some of the classical axioms in the unidimensional context could apply here directly, others would not. A more interesting (and challenging) route would be to formalize new axioms with inherent appeal for the multidimensional setting and to explore their implications, ideally obtaining characterization results for different rules. Somewhat related, it would be interesting to explore non-anonymous (multi-issue) rules, especially if we deal with claims coming in various categories, or if we simply have reasons a priori (e.g., heterogeneity in preferences of the agents in society over the different issues) to treat different issues differently. Finally, the classical bankruptcy model implicitly assumes that issues of production, preferences, and other characteristics are unobservable or irrelevant. This does not need to be the case in the multi-issue setting we have presented here and therefore it would be worth exploring the existence of axioms (and, ultimately, rules) reflecting some of these issues.

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