Volume 29, Issue 3

A general strategy-proof fair allocation mechanism revisited

Tommy Andersson
Department of Economics, Lund University

Abstract

This paper revisits the fair and optimal allocation mechanism (Sun and Yang, Economics Letters 81:73-79, 2003) and demonstrates that it is coalitionally strategy-proof. The proof is valid for general preferences, it is simple and it is short.

1. Introduction

A classical problem in the economic literature is how to allocate a number of indivisible objects (e.g. houses, jobs etc.) together with some money among a set of agents under the restriction that each agent must be assigned his most preferred object at the given monetary compensations. This problem is called the fair allocation problem and it has been investigated by e.g. Alkan et al. (1991), Maskin (1987), Svensson (1983) and Tadenuma and Thomson (1991). Sun and Yang (2003) also investigated the fair allocation problem in the case when the number of agents and objects coincide and under the restriction that each object has a maximum compensation limit. The central observation in Sun and Yang (2003) is that agents may behave strategically rather than truthfully. Given this observation, Sun and Yang (2003) defined a mechanism, called the fair and optimal allocation mechanism, and demonstrated that it is (individually) strategy-proof.

This paper studies the model of Sun and Yang (2003) and its contribution is two-fold. First, it generalizes the result in Sun and Yang (2003) by demonstrating that the fair and optimal allocation rule in fact is coalitionally strategy-proof, i.e., it is not possible for any agent or any coalition of agents to successfully manipulate the allocation rule. This result sheds a fresh and deep insight into the model. A second contribution is of technical nature, i.e., the proof of the allocation rule being coalitionally strategy-proof is valid for general preferences, it is simple and it is short. It should also be mentioned that a more general result can be found in Andersson and Svensson (2008). Their strategy-proofness result holds for an arbitrary relation between the number of agents and objects and it also covers the case with and without individual rationality. However, because the result in Andersson and Svensson (2008) covers a variety of different cases, the proof is for obvious reasons not as simple as the proof in this paper. Thus, because allocation models with an equal number of agents and objects are common in the fairness literature, this paper presents a pedagogical and straightforward proof of a surprisingly general and robust strategy-proofness result, and such a result is very helpful in better understanding these basic models.

The paper is outlined as follows. The model and some basic definitions are introduced in Section 2. Section 3 provides a number of fairness definitions and a few results that are related to fair allocation models. The general strategy-proofness result can be found in Section 4.

2. The Model and Basic Definitions

The set of agents and objects are denoted by $N = \{1, ..., n\}$ and $M = \{1, ..., m\}$, respectively, where $n = m$. There is also a divisible good called money. Each agent

$i \in N$ has preferences over consumption bundles $(j, m) \in M \times \mathbb{R}$, represented by a continuous utility function $u_{ij}(m)$. The utility function is supposed to be strictly increasing in money. Moreover, for each agent $i \in N$ and for any two bundles (j, m) and (k, m') there is an amount β of money such that $u_{ij}(m) = u_{ik}(m' + \beta)$. This means that no object is infinitely good or bad for any agent. A list $u = (u_1, ..., u_n)$ of individual utility functions is a (preference) profile. We also adopt the notational convention of writing $u = (u_C, u_{-C})$ for $C \subset N$. The set of profiles with utility functions having the above properties is denoted by $\mathcal{U}$.

An allocation is a list of consumption bundles. It is a pair (a, x), where $a : N \to M$ is an injective mapping assigning object a_i to agent $i \in N$ and where $x \in \mathbb{R}^m$ distributes the quantity x_j of money to object $j \in M$, and, hence, also x_j to agent $i \in N$ if $a_i = j$. We call a the assignment and x the distribution. An allocation is feasible if no two agents are assigned the same object. The set of feasible allocations is denoted by $\mathcal{A}$. A sub-allocation (a', x') for coalition $N' \subset N$ consists of all consumption bundles that are assigned to the agents in N'.

Finally, each object $j \in M$ has an exogenously given maximum compensation limit, $\overline{x}_j$. These compensation limits are gathered in the vector $\overline{x} \in \mathbb{R}^m$.

3. Fairness

In the classical definition of fairness (due to Foley, 1967) an allocation is fair if it is envy-free.

Definition 1 *For a given profile $u \in \mathcal{U}$, a distribution x is said to be fair if there is an assignment a such that $u_{ia_i}(x_{a_i}) \geq u_{ij}(x_j)$ for all $i \in N$ and all $j \in M$.*

The concept of fair and optimal allocations was introduced by Sun and Yang (2003).

Definition 2 *Let $(a, x) \in \mathcal{A}$ be an allocation and $\overline{x} \in \mathbb{R}^m$ an exogenously given vector of maximum compensation limits. For a given profile $u \in \mathcal{U}$, the distribution x is said to be fair and optimal with respect to $\overline{x}$ (w.r.t. $\overline{x}$, henceforth) if:*

(i) *x is a fair distribution,*

(ii) *$x_j \leq \overline{x}_j$ for all $j \in M$,*

(iii) *$\sum_{j \in M} x_j$ is maximal, subject to (i) and (ii).*

If the distribution x is fair and optimal w.r.t. $\overline{x}$, allocation (a, x) is said to be fair and optimal w.r.t. $\overline{x}$.

The following result can be found in Sun and Yang (2003, Theorem 2.5)

Lemma 1 *For each profile $u \in \mathcal{U}$, there exists a fair and optimal allocation (a, x^*). Moreover, the distribution x^* is unique.*

Consider now sub-allocation (a', x') for coalition $N' \subset N$. The sub-allocation (a', x') is said to be fair for coalition N' if $u_{ia_i}(x_{a_i}) \geq u_{ia_j}(x_{a_j})$ for all $i \in N'$ and all $j \in N'$, i.e., if no agent in N' envies any other agent in N' at sub-allocation (a', x'). The next result is due to Alkan et al. (1991, Perturbation Lemma) and it will be helpful in the proof of the main theorem.

Lemma 2 *If (a', x') is a fair sub-allocation for coalition N', then for each $\varepsilon > 0$ there exists another fair sub-allocation (c', z') for coalition N' where $x'_{c'_i} < z'_{c'_i} < x'_{c'_i} + \varepsilon$ for all $i \in N'$, $c'_i \in a'$ for all $i \in N'$ and no to agents in N' are assigned the same object.*

The meaning of Lemma 2 is that if (a', x') is a fair sub-allocation, then it is always possible to fairly reallocate the objects in a' among the agents in N' and at the same time increase the monetary compensation for each object in a'.

4. The Main Result

Before stating a formal definition of coalitional strategy-proofness (CSP), we define a fair and optimal allocation rule φ as follows. For a given vector $\overline{x} \in \mathbb{R}^m$ and for all $u \in \mathcal{U}$, $\varphi(u)$ is a non-empty set such that:

$$\varphi(u) \subset \{(a, x) \in \mathcal{A} \mid (a, x) \text{ is fair and optimal w.r.t. } \overline{x}\}.$$

The correspondence φ defined in this way is essentially single-valued, i.e., each agent is indifferent among the various outcomes that the allocation rule may have. That is, if $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(u)$, then $y = x$, by Lemma 1, so $u_{ib_i}(y_{b_i}) = u_{ib_i}(x_{b_i}) = u_{ia_i}(x_{a_i})$ for all $i \in N$, by fairness and symmetry.

Definition 3 *An allocation rule φ is manipulable at a profile $u \in \mathcal{U}$ by a coalition $C \subset N$ if there is a profile $v \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(v_C, u_{-C})$, such that $u_{ib_i}(y_{b_i}) > u_{ia_i}(x_{a_i})$ for all $i \in C$. If the allocation rule is not manipulable by any coalition, at any profile, it is said to be coalitionally strategy-proof (CSP).*

We are now ready to state the main result.

Theorem 1 *An allocation rule φ that is fair and optimal w.r.t. $\overline{x} \in \mathbb{R}^m$ is coalitionally strategy-proof.*

Proof. Suppose that φ is fair and optimal w.r.t. $\overline{x}$, but that φ is not CSP, i.e., that there is a coalition $C \subset N$ and two allocations, say $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(v_C, u_{-C})$, that are fair and optimal w.r.t. $\overline{x}$, where $u_{ib_i}(y_{b_i}) > u_{ia_i}(x_{a_i})$ for all $i \in C$.

3

Let $T = \{j \in M \mid y_j > x_j\}$. Note first that if $a_i \in T$ then $b_i \in T$ because if $b_i \in M - T$ then by fairness and monotonicity:

$$u_{ia_i}(y_{a_i}) > u_{ia_i}(x_{a_i}) \geq u_{ib_i}(x_{b_i}) \geq u_{ib_i}(y_{b_i}).$$

Hence, if $i \in C$ then $u_{ia_i}(x_{a_i}) \geq u_{ib_i}(y_{b_i})$ which contradicts the assumption that $u_{ib_i}(y_{b_i}) > u_{ia_i}(x_{a_i})$, and if $i \notin C$ then $u_{ia_i}(y_{a_i}) > u_{ib_i}(y_{b_i})$ which contradicts that (b, y) is fair. Thus, $T \neq \varnothing$ because $C \neq \varnothing$ and if $a_i \in T$ then $b_i \in T$. Now it follows that if $a_i \in M - T$ then $b_i \in M - T$ since $n = m$. Let now $N' = \{i \in N \mid a_i \in T\} \neq \varnothing$. By fairness and monotonicity, the following condition holds for all $i \in N - N'$ and all $j \in T$:

$$u_{ia_i}(x_{a_i}) \geq u_{ib_i}(x_{b_i}) \geq u_{ib_i}(y_{b_i}) \geq u_{ij}(y_j) > u_{ij}(x_j). \tag{1}$$

Consider next the sub-allocation (a', x') for coalition N' at allocation (a, x). For each $\varepsilon > 0$ there exists another (feasible) sub-allocation (c', z') for coalition N' where $x'_{c'_i} < z'_{c'_i} < x'_{c'_i} + \varepsilon$ for all $i \in N'$ by Lemma 2. Finally, consider allocation (c, z) where $c_i = c'_i$ and $z_{c_i} = z'_{c_i}$ for all $i \in N'$ and $c_i = a_i$ and $z_{c_i} = x_{c_i}$ for all $i \in N - N'$. Because (a, x) is fair and (c', z') is a fair sub-allocation for coalition N', allocation (c, z) is fair by monotonicity and condition (1) for $\varepsilon > 0$ sufficiently small. Moreover, allocation (c, z) also respects the maximum compensation limits $\overline{x}$ for $\varepsilon > 0$ sufficiently small. But this contradicts the assumption that (a, x) is fair and optimal w.r.t. $\overline{x}$ because $z_{c_i} \geq x_{c_i}$ for all $i \in N$ with strict inequality for all $i \in N'$. Hence, φ is CSP. ∎

References

Andersson, T., and L-G. Svensson (2008) "Non-manipulable assignment of individuals to positions revisited" *Mathematical Social Sciences* **56**, 350-354.

Alkan, A., Demange, G., and D. Gale (1991) "Fair allocation of indivisible goods and criteria of justice" *Econometrica* **59**, 1023-1039.

Foley, D. (1967) "Resource allocation and the public sector" *Yale Economic Essays* **7**, 43-98.

Maskin, E. (1987) "On the fair allocation of indivisible goods" in Arrow and the foundations of the theory of public policy by G. Feiwell, Ed., MacMillan Press: London, 342-349.

Sun, N., and Z. Yang, (2003) "A general strategy proof fair allocation mechanism" *Economics Letters* **81**, 73-79.

Svensson, L-G. (1983) Large indivisibles: "An analysis with respect to price equilibrium and fairness" *Econometrica* **51**, 939-954.

Tadenuma, K., and W. Thomson (1991) "No-envy and consistency in economies with indivisible goods" *Econometrica* **59**, 1755-1767.