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A characterization of the composite price variable to approximate a price aggregator function in the Quadratic Almost Ideal Demand System

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Abstract

In investigating linear approximations to the Quadratic Almost Ideal Demand System (QUAIDS), Matsuda (2006) proposed a composite variable to approximate a price aggregator function. This paper provides an axiomatic characterization of this composite price variable.

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1. Introduction

In recent times the Quadratic Almost Ideal Demand System (QUAIDS) proposed by Banks et al. (1997) has been the most extensively used model in the analysis of consumer behaviour because of its versatility and flexibility (see Matsuda 2006 and the references therein). As Matsuda (2006) points out, observed price and expenditure data are often non-stationary, and application of nonlinear systems to such data is difficult. In order to be able to properly handle the popular QUAIDS in the context of non-stationary data, Matsuda (2006) proposed a composite variable to approximate one of the price aggregator functions. In this paper we provide an axiomatic characterization of this composite price variable.

2. QUAIDS and Linear Approximation

The cost function underlying QUAIDS is of the form

$$C(u,p)=a(p).\exp(\frac{b(p)}{(1/\ln u)-\lambda(p)}), \qquad (1)$$

where p is the price vector of n commodities, $a(p)$ is a homogeneous function of degree one in prices, given by $\ln a(p)=\alpha_0+\sum_{i=1}^{n}\alpha_i \ln p_i+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}\ln p_i \ln p_j$, with $\sum_i \alpha_i=0$, $\sum_i \delta_{ij}=\sum_j \delta_{ij}=0$; $b(p)$ and $\lambda(p)$ are homogeneous functions of degree zero in prices, given by $b(p)=\prod_{i=1}^{n} p_i^{\beta_i}$, with $\sum_i \beta_i=0$; $\lambda(p)=\sum_{i=1}^{n}\lambda_i \ln p_i$, with $\sum_i \lambda_i=0$ and u denotes the level of utility.

By Shephard’s lemma, the budget share functions corresponding to the cost function “(1)” are of the form

$$w_i=\alpha_i+\sum_j \delta_{ij}\ln p_j+\beta_i \ln\frac{y}{a(p)}+\frac{\lambda_i}{b(p)}(\ln\frac{y}{a(p)})^2, \qquad i=1,2,\ldots,n, \qquad (2)$$

where y denotes nominal income and i denotes item of expenditure.

Linear approximations to the QUAIDS require that both *a(p)* and *b(p)* be replaced by composite variables, which are free of unknown parameters. Various approximations to *a(p)* in the context of the Almost Ideal demand System (AIDS) of Deaton and Muellbauer (1980) have been suggested in the literature (see, for example, Deaton and Muellbauer

1980, Moschini 1995)[1]. In order to approximate *b(p)*, Matsuda (2006) proposed a composite variable, P^z, given by the following equation:

$$ln\,P^z = \sum_{i=1}^{n}(w_i^1 - w_i^0)\,ln\frac{p_i^1}{p_i^0}, \tag{3}$$

where n is the number of commodities in two time periods *t=0 (base period)* and *1(current period)*, $p^t = (p_1^t, p_2^t, ..., p_n^t)$ and $q^t = (q_1^t, q_2^t, ..., q_n^t)$ are the vectors of prices and quantities, respectively, at period *t*, and $w_i^t = \frac{p_i^t q_i^t}{\sum_{i=1}^{n} p_i^t q_i^t}$ is the budget share of commodity *i* at period *t*, *i=1,2,…,n.*[2] Evidently, $\sum_{i=1}^{n} w_i^t = 1$, for *t=0,1*.

In what follows, we provide an axiomatic characterization of $ln\,P^z$.[3]

3. Characterization of P^z

Consider the class of aggregated price relatives

$$ln\,P(p^1, q^1, p^0, q^0) = \sum_{i=1}^{n} m_i(w_i^0, w_i^1)\,ln\frac{p_i^1}{p_i^0},$$

where $m_i : [0,1] \times [0,1] \rightarrow R_+$, $m_i(0,0) = 0$ and m_i is continuous.

Let us look at the following two properties of $P(p^1, q^1, p^0, q^0)$.

<u>Property 1:</u> $P(p^1, q^1, p^0, q^0)$ is homogeneous of degree zero separately in base and current period prices.

That is, $P(p^1, q^1, \lambda p^0, q^0) = P(p^1, q^1, p^0, q^0)$ for any scalar $\lambda > 0$.
and $P(\lambda p^1, q^1, p^0, q^0) = P(p^1, q^1, p^0, q^0)$ for any scalar $\lambda > 0$.

This implies that $\sum_{i=1}^{n} m_i(w_i^0, w_i^1) = 0$.

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[1] Of these, the *Törnqvist* Index has been axiomatically characterized by Balk and Diewert (2001).

[2] Note that the approximation is a function of $w_i^0, w_i^1, p_i^0, p_i^1$. If the base period prices are set to 1, then P^z is a function of current period prices only.

[3] It may, however, be pointed out that no optimization relating P^z to *b(p)* is attempted here. The linear approximation, which is purely for practical convenience (Matsuda 2006), is the starting point.

Property 2: $P(p^1, q^1, p^0, q^0)$ is symmetric with respect to base and current period prices.
That is, $P(p^1, q^1, p^0, q^0) = P(p^0, q^0, p^1, q^1)$.

This implies that $\sum_{i=1}^{n} m_i(w_i^0, w_i^1) ln \frac{p_i^1}{p_i^0} = \sum_{i=1}^{n} m_i(w_i^1, w_i^0) ln \frac{p_i^0}{p_i^1}$ (4)

Or, $\sum_{i=1}^{n} m_i(w_i^0, w_i^1) ln \frac{p_i^1}{p_i^0} = -\sum_{i=1}^{n} m_i(w_i^1, w_i^0) ln \frac{p_i^1}{p_i^0}$. (5)

Theorem: Property 1 and Property 2 hold together if and only if
$m_i(w_i^0, w_i^1) = c.(w_i^1 - w_i^0)$, where c is a positive constant.

Proof:
Since "(5)" is true for all n, it is in particular true for $n=1$. This is turn implies that
$$m_1(w_1^0, w_1^1) = -m_1(w_1^1, w_1^0).$$

Using this successively in "(5)" for $n = 2,3,\ldots$, we can show that
$m_i(w_i^0, w_i^1) = -m_i(w_i^1, w_i^0)$ for all i.
Or, $m_i(w_i^0, w_i^1) + m_i(w_i^1, w_i^0) = 0$ for all i.

Now, this is of the form $F(x,y)+F(y,x)=0$, which is a special case of the general form
$F(x,y)+F(y,z)=F(x,z)$, with $z=x$,
where F is continuous.

The solution of this functional equation is given by
$F(x,y) = g(y)-g(x)$ [see *Acze′l* 1966, page 223, Theorem 1],
where g is continuous.

Therefore, the solution in our case is $m_i(w_i^0, w_i^1) = g(w_i^1) - g(w_i^0)$ for all i.

Observe from Property 1 that $\sum_{i=1}^{n} m_i(w_i^0, w_i^1) = 0$. Hence, $\sum_{i=1}^{n}(g(w_i^1) - g(w_i^0)) = 0$.

Now, assume that $\sum_{i=1}^{n} g(w_i^1) = \sum_{i=1}^{n} g(w_i^0) = \theta$, say. Note that for any $w_i = \frac{1}{k}$, $i=1,2,\ldots,k$,
$g(w_i) = \frac{\theta}{k}$. Since g is continuous, it now follows that the form of g is given by
$g(x) = c.x$, where c is a positive constant.

Hence, $\sum_{i=1}^{n} m_i(w_i^0, w_i^1) ln \frac{p_i^1}{p_i^0} = c.\sum_{i=1}^{n}(w_i^1 - w_i^0) ln \frac{p_i^1}{p_i^0}$.

It is straightforward to verify that $\sum (w_i^1 - w_i^0) ln \frac{p_i^1}{p_i^0}$ satisfies the two properties mentioned above. This completes the proof of the theorem. □

Thus, the price aggregator $ln P(p^1 ,q^1 , p^0 ,q^0)$ is a unique (up to a scalar multiple) member of the class of the price aggregators with $P(p^1 ,q^1 , p^0 ,q^0)$ satisfying the property of homogeneity of degree zero in current period prices and symmetry with respect to base and current period prices.

As already observed earlier, the approximation uses data from a pair of time periods. So, for practical application to data sets covering many time periods, one time period may be chosen as base (time period 0 in the formula).

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