Economics Bulletin

Volume 31, Issue 2

Specialized advertising and price competition with endogenous advertising fees

Jose M. Hernandez
University of Zaragoza

Lola Esteban
University of Zaragoza

Abstract

This paper studies how the strategic interaction between a specialized communication platform and two firms, competing in prices and advertising efforts, determines the equilibrium advertising fee and the pattern of competition between sellers. We show that the link between information and price competition in the product market leads the platform to ask for high prices for its advertising services, in the sense that the resulting product prices rise to the level where the marginal potential consumer achieves zero utility, so firms can exercise a high degree of market power and consumers achieve a low surplus.

Financial support from both FEDER and the Spanish Government, through Grant ECO2008-05024/ECON, is gratefully acknowledged.
Citation: Jose M. Hernandez and Lola Esteban, (2011) "Specialized advertising and price competition with endogenous advertising fees", *Economics Bulletin*, Vol. 31 no.2 pp. 1686-1695.
Submitted: Apr 14 2011. **Published:** June 07, 2011.

1. Introduction

An important question regarding the functioning of differentiated product markets is how *promotional (advertising) competition* relates to *price competition.* The usual approach to this issue (see e.g., Grossman and Shapiro, 1984) considers that the product market and the media market are isolated, so firms compete in the product market with exogenous advertising fees. This paper studies oligopolistic interaction in a new framework, where advertising fees are *endogenously* determined, so we can analyze how the connection between product sellers and the advertising media determines the market outcome. To that end, we begin by noting that the current enormous proliferation of highly specialized communication media has brought about the progressive fragmentation of audiences so, nowadays, many communication channels are highly orientated to one particular segment of consumers. In particular, we observe that, to reach a set of potential clients in a local market with a high degree of precision, firms frequently have either one local free-to-air radio (or TV) station or a small number of media, but one of them *dominates* the advertising market. Within this framework of progressive advertising specialization, it is interesting to study how the *strategic* interaction between a subscription-free specialized (*dominant*) communication platform and a horizontally-differentiated product market, with firms supplying goods and demanding advertising space, can affect the pricing strategy of the platform and the intensity of competition between sellers.[1]

2. The Model

Let us consider two firms launching a new product and competing in prices, (p_1, p_2), for a unit mass of potential consumers distributed uniformly on the segment $[0, 1]$ who demand, at most, one unit of a good. The two firms are located at the extremes of the linear potential market, so a consumer located at $1 \geq x \geq 0$ achieves a utility $U = v - T(x) - p_1$ from buying firm 1's product, where v denotes the consumers' common valuation of the goods and $T(x)$ represents the transportation cost, with $T'(x) > 0$. Similarly, this consumer will achieve a utility $U = v-T(1-x)-p_2$ from buying firm's 2 product, and the location of the fully informed consumer who is indifferent between the two firms is defined by the following condition $v-T(x)-p_1 = v-T(1-x)-p_2$. Solving out this equation for x, we obtain firm 1's full-information demand function, $x(p_2 - p_1, t)$.

Consumers are unaware of the existence of the goods, so a potential customer cannot be an actual buyer unless sellers invest in advertising. We assume that there is a single subscription-

[1] The relationship between platform pricing and competition in the product market has also been addressed by Dukes (2004), who emphasizes *competition* between radio or TV stations by modeling both the product market and the media market with the circle framework of Grossman and Shapiro (1984).

free local TV (or radio) station that provides advertising space for firms and denote by $n_i(\phi_i)$ the number of ads that firm i has to insert in the TV in order to randomly reach a fraction ϕ_i of the potential demand $[0,1]$. Following the standard assumption in the advertising literature, we consider $n'(\phi) \geq 0$, $n''(\phi) > 0$. Further, we can characterize different advertising technologies based on the following property of the advertising elasticity, $E_{n,\phi} = \frac{n'(\phi)\phi}{n(\phi)}$:

Definition 1 *We say that $n(\phi)$ is increasingly elastic (IE) if $\frac{dE_{n,\phi}}{d\phi} \geq 0$.*

An advertising technology is IE if a decrease in the advertising reach reduces (or does not change) the sensitivity of the number of ads (n) to variations in the advertising reach (ϕ). We note that (i) the set of advertising technologies more used in the literature $n(\phi) \in \{-Log(1-\phi); \phi^{\alpha}\}$ satisfy this definition, so this property seems plausible in real-life advertising technologies, and (ii) if $n(\phi)$ is IE, then $n''(\phi) > 0$, but the reverse implication does not hold, which means that the IE property is more restrictive that the convexity of $n(\phi)$.

To investigate the interaction between the platform and the firms, we first need to determine the relationship between advertising efforts and the firms' demands. To that end, we note that, for example, firm 1's demand is comprised of two parts: a fragment of consumers $\phi_1(1-\phi_2)$ only know firm 1's product, so the firm can monopolize this segment of the market. The demand corresponding to these consumers can be obtained from the condition $U = v - T(x) - p_1 = 0$, which yields a *monopolistic* demand function $x_m(p_1,t)$. Differentiating this expression we obtain $x_m'(p_1,t) = -\frac{1}{T'(x)} < 0$, $x_m''(p_1,t) = \frac{T''(x)\, x_m'(p_1,t)}{[T'(x)]^2} \gtrless 0$, so the curvature of $x_m(p_1,t)$ depends on the properties (concavity or convexity) of the transportation cost function. We denote by p^m the *monopoly price* over the segment $[0,1]$, that is, if $p^* = \arg\max\{p\; x_m(p,t)\}$, then $p^m = Max\,[v - T(1); p^*]$, $x^m = Min\,[1;\; x_m(p^*,t)]$, and $x^m = 1$ if $v \geq T(1) + T'(1)$.[2] The second fragment of firm 1's demand, $\phi_1\phi_2$, is comprised of consumers who knows both firms' products, so firm 1 will compete for these potential buyers according to the demand function $x(p_2 - p_1, t)$. Accordingly, firm 1 faces a demand $D_1 = \phi_1\{(1-\phi_2)\; Min\,[1, x_m(p_1,t)] + \phi_2\; x(p_2 - p_1, t)\}$.

The timing of the game is as follows. In the first period, the platform sets the advertising fee, a. Later, the firms simultaneously decide their pricing strategy, p_i, and their advertising effort, ϕ_i. Finally, the informed consumers make a decision about which product to purchase. To find the equilibrium of the game, we first analyze the competition between the firms in the second period, for a given advertising fee. Let us assume that, at the equilibrium prices, all the informed consumers buy a product, i.e. $p_l \leq v - T(1)$, $l = 1,2$, which implies $x_m(p_l,t) \geq 1$. Under this condition, firm

[2] We normalize production costs to zero and assume that $x^m > \frac{1}{2}$, so firms compete for the fully informed potential demand.

1 faces the following problem:

$$Max_{p_1,\phi_1}\ \Pi_1 = p_1\phi_1\left[(1-\phi_2)+\phi_2\ x(p_2-p_1,t)\right]-a\ n_1(\phi_1) \tag{1}$$

Let us denote the symmetric solution of (1) as $p^c(a)$, $\phi^c(a)$, with $\frac{dp^c(a)}{da}>0$, $\frac{d\phi^c(a)}{da}<0$ (see the Appendix). Then, in the first period, the platform sets the value of a so as to maximize profits:[3]

$$Max_a\ V^c = 2\ a\ n(\phi^c(a)) \tag{2}$$

$$s.t.\quad p^c(a)\leq v-T(1),$$

$$\frac{dV^c}{da} = 2\left[n(\phi)+a\ n'(\phi)\frac{d\phi^c(a)}{da}\right] = 2\ n(\phi)\left[1-E_{n,\phi}\ E_{\phi,a}\right], \tag{3}$$

where $E_{\phi,a} = -\frac{\phi'(a)a}{\phi(a)}$. The following proposition states that the properties of the advertising technology determine, to a great extent, the pattern of price competition in the product market.

Proposition 1 *If $n(\phi)$ is IE, then $\frac{dV^c}{da}>0$ and product prices have a lower bound at the level in which the marginal potential consumer achieves zero utility, i.e. $p_l \geq v-T(1)$, $l=1,2$.*

A fundamental feature of the product market is that there is a linkage between the level of information and the degree of price competition between firms, and the question is how this link affects the behaviour of the TV operator. Clearly, a higher advertising fee increases the platform's advertising margin but, at the same time, it *directly* decreases the firms' demand for advertising and, therefore, their supply of information to potential consumers. However, low levels of information in the product market soften the degree of price competition between the firms, and the resulting higher potential profits lead sellers to demand more advertisements. As a result of this *indirect* "*informational differentiation effect*", an increase in the advertising fee only moderately decreases the firms advertising efforts, ϕ. The interesting point is that, under a simple property of the advertising technology, *increasing elasticity of* $n(\phi)$, for any $p\leq v-T(1)$, the platform will exploit the linkage information-price competition to charge a higher advertising fee, which yields high prices in the product market. The intuition of this results is as follows. Under the IE property, a lower ϕ decreases (or leaves constant) the sensitivity of the demand for ads to a change in the advertising reach. This decreasing elasticity provokes that, as long as all the informed consumers buy a product, $p^c(a)\leq v-T(1)$, the increase in the platform's profits obtained from a higher

[3] Obviously, the specialized medium could also advertise some goods of firms operating in other industries. For the sake of simplicity, in this paper we consider that the product market is *dominant* in the advertising business and any other advertising income is normalized to zero. However, as we later note, the intuition behind our basic results extends to the case where the operator sets the advertising fee considering advertising incomes from several industries.

margin *always* compensates for the loss suffered by the lower advertising demand induced by a higher a, so the platform always finds it optimal to raise the advertising fee. Therefore, problem (2) yields a *corner solution* where the operator sets the maximum fee which is consistent with a fully covered product market, $a = a^c = p_l^{-1}(v - T(1))$. This high advertising fee also reduces the degree of price competition in the product market, in such a way that firms can appropriate all the surplus of the marginal potential consumer ($x = 0, 1$) by charging a high price, $p_1 \geq v - T(1)$. It is worth noting that these results also apply to the case of a product market with more than two competitors.[4] Further, the intuition behind Proposition 1 also extends to the case that the operator advertises products from a number $k > 1$ of industries. To see this point, we note that, as long as the condition $p^c(a) \leq v - t$ holds in all of them, the function V^c is still monotonic, so the platform will find it optimal to charge a higher advertising fee, which implies that firms will set $p_{lk} \geq v - t$ in, *at least,* one of the k industries, whereas, if the model is symmetric, all the firms in the k markets will set a price $p_{lk} \geq v - t$. Therefore, our analysis suggests that, under quite general conditions, the strategic interaction between the platform and the firms yields high prices in the product market.

Having shed some light on the firms' optimal pricing strategies, we next study how the platform chooses the advertising fee, and the resulting optimal advertising efforts. To address this task, we consider that consumers have a sufficiently high valuation of the product so, in equilibrium, the market is always covered (all the informed consumers buy a product), i.e. $v \geq \overline{v} = Max\left\{T(1) + T'(1); T(1) + T'\left(\frac{1}{2}\right)\right\}$, that is, we focus the analysis on the most interesting market scenario, where the degree of product differentiation is low, relative to consumers' valuation of the product, so, in the full-information benchmark, the degree of price competition between firms is strong.

Proposition 2 *Let us consider that the advertising technology is IE, and that* v *is sufficiently high,* $v \geq \overline{v}$. *Then,* $p_l = v - T(1)$, *the optimal advertising efforts,* ϕ, *are implicitly determined by*

$$[v - T(1)]\left(1 - \frac{\phi}{2}\right) - a\ n'(\phi) = 0, \tag{4}$$

where $a = Max\left\{a^c, a^k\right\}$, *with* $a^c = p^{-1}(v - T(1))$ *and* a^k *is implicitly determined by*

$$n(\phi)\,[v - T(1)] + 2\ a\ \left[n(\phi)\ n''(\phi) - n'\left(\phi\right)^2\right] = 0. \tag{5}$$

Equation (4) simply states that, given the rival strategy and the optimal advertising fee, a firm will advertise up to the point where the marginal benefit of an ad equals the marginal cost.

[4] We can replicate the results of Proposition 1 for the circular model of Grossman and Shapiro (1984) with $m > 2$ firms and linear transportation costs, $T(x) = tx$.

To understand how the platform decides the optimal advertising fee, we first note that, whenever $x_m(p_l, t) \leq 1$, if $v \geq \overline{v}$, the firms' optimal pricing strategy, *for any* $a \geq 0$, yields $p^1(a) \leq v - T(1)$ (see the Appendix), which implies that, in equilibrium, it holds that $x_m(p_l, t) \geq 1$. Further, if $x_m(p_l, t) \geq 1$, from Proposition 1 it follows that, under an IE advertising technology, $V(a)$ is *always* increasing, so problem (2) yields a *corner solution*, $a = a^c$, and $p^c(a^c) = v - T(1)$. Therefore, if $v \geq \overline{v}$, the optimal advertising fee has a lower bound at $a = a^c$, and the issue is under which conditions the equilibrium is at this corner, $a = a^c$, or when the platform will set a higher fee. If $a > a^c$, we have that $p^c(a) > v - T(1)$, i.e. $x_m(p_l, t) < 1$ and, at the same time, it holds that $p^1(a) < v - T(1)$, i.e. $x_m(p_l, t) > 1$, so the model generates a "kinked equilibrium" where firms do not compete in prices, $p_l = v - T(1)$. This provokes that $\phi(a)$ is more elastic[5] and $V(a)$ strictly concave, so the platform's maximization problem generates an *interior solution*, $a = a^k$, which is described by equation (5). In order to compare the corner solution with the interior solution, let us denote by $V^c(a)$ the platform's profits when $a \leq a^c = p^{-1}(v - T(1))$, and by $V^k(a)$ the profits when $a > p^{-1}(v - T(1))$. The key point is that, when $a = a^c$, the equilibrium prices in the corner solution and the interior solution are identical, $p_l = v - T(1)$, and the optimal advertising efforts also coincide, $\phi^c(a^c) \equiv \phi^1(a^c)$, so we have that $V^c(\phi^c(a^c)) \equiv V^k(\phi^1(a^c))$. Taking into account the concavity of $V^k(a)$, this explains that, if $a^c > a^k$ ($a^c < a^k$), the platform maximizes profits by making $a = a^c$ ($a = a^k$), that is, $a = Max\left\{a^c, a^k\right\}$, and the solution to the game depends on both the advertising technology properties and the product market conditions. In particular, it can be proved that, if $n(\phi) = \phi^\alpha$, then $a = a^c$, whereas, if $n(\phi) = -Log(1 - \phi)$, then, for $v - T(1)$ sufficiently low (high), the optimal advertising fee is $a = a^k > a^c$ ($a = a^c > a^k$). It is worth noting, that, regardless of whether $a = a^c$ or $a = a^k$, the equilibrium advertising fee is always sufficiently high so as to leave the marginal potential buyer in the product market with zero utility, $p_l = v - T(1)$.

3. Conclusion

This paper shows that the strategic relationship between a group of sellers and a subscription-free specialized communication media leads the platform to ask for high prices for its advertising services, which mitigates price competition between firms, in the sense that the potential marginal consumer will obtain zero utility, thus yielding a low surplus to consumers.

[5] When $a = a^c$, we obtain $E_{\phi,a} = \frac{n'(\phi)}{\left[\frac{p}{2} + an''(\phi) - \frac{2-\phi}{2}\frac{1}{x'(t)\phi}\left(\frac{1}{2} - px'(t)\right)\right]}\frac{a}{\phi}$, whereas when $a > a^c$, $E_{\phi,a} = \frac{n'(\phi)}{\frac{p}{2} + an''(\phi)}\frac{a}{\phi}$.

References

[1] Dukes, A. (2004) "The Advertising Market in a Product Oligopoly" *The Journal of Industrial Economics*, vol. LII, 3, 327-348.

[2] Grossman G. and C. Shapiro (1984) "Informative Advertising with Differentiated Products" *Review of Economic Studies* 51, 63-82.

Appendix

Proof of Proposition 1: We first note that consumers can achieve a positive utility only if $x_m(p_l, t) > 1$. However, we next prove that, for any $x_m(p_l, t) \geq 1$, firms find it optimal to set a price $p_l \geq v - T(1)$, so the marginal potential consumer, $x = 1$, cannot achieve positive utility. Let us assume that $x_m(p_l, t) \geq 1$, so given the platform's strategy, a, in the second period firm 1 faces problem (1). The first order conditions (FOC's) are:

$$[(1-\phi_2) + \phi_2 \; x(p_2 - p_1, t)] + p_1 \phi_2 \frac{\partial x}{\partial p_1} = 0, \tag{6}$$

$$p_1 [(1-\phi_2) + \phi_2 \; x(p_2 - p_1, t)] - a \; n'(\phi_1) = 0, \tag{7}$$

whereas the second order conditions require that the corresponding Hessian matrix, H, be negative semidefinite. Applying the symmetry condition, $p_1 = p_2 = p$, $\phi_1 = \phi_2 = \phi$, we obtain:

$$\left(1 - \frac{\phi}{2}\right) - p \; \phi \; x'(t) = 0, \tag{8}$$

$$p \left(1 - \frac{\phi}{2}\right) - a \; n'(\phi) = 0. \tag{9}$$

Let us denote by $p^c(a)$ and $\phi^c(a)$ the solution of (8) and (9). From this system of equations we have that $\frac{d\phi^c}{da} = \frac{\phi \; x'(t) n'(\phi)}{\phi \; x'(t)\left[-\frac{p}{2} - a \; n''(\phi)\right] - \left(1 - \frac{\phi}{2}\right)\left[p x'(t) - \frac{1}{2}\right]} < 0$, and $\frac{dp}{da} = -\frac{\left[p x'(t) - \frac{1}{2}\right]}{\phi \; x'(t)} \frac{d\phi^c}{da} > 0$. Going back to the first stage of the game, the platform faces problem (2) and the FOC is $\frac{dV^c}{da} = 2n(\phi^c) + 2 \; a \; n'(\phi) \frac{d\phi^c}{da}$. Considering that $H > 0$, the sign of $\frac{dV^c}{da}$ equals

$$sign \left\{ x'(t) \left[-n(\phi) p - a \; \phi \; n(\phi) \; n''(\phi) + a \; \phi (n'(\phi))^2 \right] + \frac{n(\phi)}{2} \left(1 - \frac{\phi}{2}\right) \right\}.$$

Taking into account that, from (9) we have that $p = \frac{a \; n'(\phi)}{\left(1 - \frac{\phi}{2}\right)}$, we obtain:

$$sign \left\{ \frac{dV^c}{da} \right\} = -sign \left\{ -n(\phi) \left(\frac{a \; n'(\phi)}{\left(1 - \frac{\phi}{2}\right)} \right) - a \; \phi \; n(\phi) \; n''(\phi) + a \; \phi (n'(\phi))^2 \right\} =$$

$$sign \left\{ n(\phi) n'(\phi) \frac{2}{2-\phi} + \phi \left[n(\phi) n''(\phi) - (n'(\phi))^2 \right] \right\} = sign \left\{ n(\phi) n'(\phi) \frac{\phi}{2-\phi} + (n(\phi))^2 \frac{dE_{n,\phi}}{d\phi} \right\}.$$

Therefore, if the advertising technology is IE, it follows that $sign\left\{\frac{dV^c}{da}\right\} > 0$, so the solution of problem (2) is to charge the maximum advertising fee which is consistent with the restriction, $p^c(a) \leq v - T(1)$. Taking into account that $\frac{dp}{da} > 0$, this *corner solution* is $a = a^c = p^{-1}(v - T(1))$, so firms charge a price $p_l \geq v - T(1)$.

Proof of Proposition 2: We first prove that, given $p_2 = v - T(1)$, if $n(\phi)$ is IE and $v \geq \overline{v}$, the best response of firm 1 is to charge $p_1 = v - T(1)$. If $n(\phi)$ is IE, Proposition 1 states that, in equilibrium, it holds that $p_l \geq v - T(1)$, i.e. $x_m(p_1, t) \leq 1$. Under this condition, firm 1 faces the problem:

$$Max_{p_1,\phi_1}\ \Pi_1 = p_1\phi_1\left[(1-\phi_2)\ x_m(p_l,t) + \phi_2\ x(p_2 - p_1, t)\right] - a\ n_1(\phi_1) \tag{10}$$
$$s.t. \qquad p_l \geq v - T(1), \quad l = 1,2$$

and the FOC corresponding to the price is:

$$\frac{\partial \Pi_1}{\partial p_1} = (1-\phi_2)\left[x_m(p_1,t) + p_1\frac{\partial x_m}{\partial p_1}\right] + \phi_2\left[x(p_2-p_1,t) + p_1\frac{\partial x}{\partial p_1}\right]. \tag{11}$$

From the definition of $x(p_2 - p_1, t)$, we obtain $\frac{\partial x}{\partial p_1} = \frac{1}{T'(x)+T'(1-x)}$. Further, if $v \geq T(1) + T'(1)$, we have that $x^m = 1$. If we evaluate equation (11) in $p_1 = p_2 = v - T(1)$, then $x(p_2 - p_1, t) = \frac{1}{2}$, so:

$$\frac{\partial \Pi_1}{\partial p_1}_{|p_1=p_2=v-T(1)} = (1-\phi_2)\left[x_m(p_1,t) + p_1\frac{\partial x_m}{\partial p_1}\right]_{|p_1=p_2=v-T(1)} + \phi_2\left[\frac{1}{2} - \frac{v-T(1)}{2T'\left(\frac{1}{2}\right)}\right]. \tag{12}$$

We note that, if $v \geq \overline{v}$, then $x_m = 1$ and equation (12) has a non-positive sign. This proves that, given $p_2 = v - T(1)$, when $x_m(p_1,t) \leq 1$, firm 1's best response is to set $p^1(a) \leq v - T(1)$ which, in turn, implies $x_m(p_1,t) \geq 1$. However, Proposition 1 states that, for any $x_m(p_1,t) \geq 1$, the optimal price is $p^c(a) \geq v - T(1)$, which implies $x_m(p_1,t) \leq 1$. This means that we obtain a *kinked equilibrium* in which the optimal price is $p_1 = v - T(1)$. In sum, if $n(\phi)$ is IE and $v \geq v$, then (i) price competition between the sellers yields a kinked equilibrium in which both firms set a price $p_l = v - T(1)$ and compete only in advertising efforts, and (ii) according to Proposition 1, the advertising fee has a lower bound at $a = a^c = p^{-1}(v - T(1))$. The issue is whether the equilibrium is at this corner, $a = a^c$, or whether the platform will set a higher fee. If $a \geq a^c$, firm 1's optimal advertising effort is obtained from the following optimization problem:

$$Max_{\phi_1}\ \Pi_1 = p_1\phi_1\left[(1-\phi_2) + \phi_2\ x(p_2 - p_1, t)\right] - a\ n_1(\phi_1) \tag{13}$$
$$s.t. \qquad p_l = v - T(1), \quad l = 1,2$$
$$\frac{\partial \Pi_1}{\partial \phi_1} = [v - T(1)]\left(1 - \frac{\phi_2}{2}\right) - a\ n_1'(\phi_1) = 0. \tag{14}$$

Applying the symmetry condition, $\phi_1 = \phi_2 = \phi$, we obtain equation (4), which implicitly defines $\phi^k(a)$, with

$$\frac{d\phi^k(a)}{da} = -\frac{2n'(\phi)}{[v - T(1)] + 2an''(\phi)}. \tag{15}$$

Going back to the first stage of the game, the platform maximizes profits:

$$Max_a \; V^k = 2 \; a \; n(\phi^k(a)) \tag{16}$$
$$s.t. \quad a \geq p^{-1}(v - T(1))$$

To solve this problem, we formulate the corresponding Khun-Tucker conditions (with $\mu \geq 0$):

$$Max_{a,\mu_1} \quad L = 2 \; a \; n(\phi^k(a)) + \mu \left[\; a - p^{-1}(v - T(1)) \right]$$
$$\frac{\partial L}{\partial a} = 2n(\phi^k) + 2 \; a \; n'(\phi)\frac{d\phi^k}{da} + \mu = 0, \tag{17}$$
$$\mu \left(\frac{\partial L}{\partial \mu} \right) = \mu \left[\;\; a - p^{-1}(v - T(1)) \right] = 0. \tag{18}$$

We consider two cases: a) let us consider that $\mu > 0$, which implies $a = p^{-1}(v - T(1))$. Under this condition, equation (17) yields $2n(\phi) + 2 \; a \; n'(\phi)\frac{d\phi^k}{da} < 0$, which, taking into account (15), yields a sufficient condition under which the optimal solution is at the corner, $a = a^c$: $n(\phi)n'(\phi) + (2 - \phi)\left[n(\phi)n''(\phi) - (n'(\phi))^2\right] < 0$,

b) an interior solution, $a > p^{-1}(v - T(1))$, implies that $\mu = 0$, so the optimal advertising fee is determined by equation (17):

$$n(\phi) - \; a \; n'(\phi)\frac{2n'(\phi)}{[v - T(1)] + 2an''(\phi)} = 0 \Longrightarrow \tag{19}$$
$$a^k = \frac{[v - T(1)]}{2}\frac{n(\phi)}{(n'(\phi))^2 - n(\phi)n''(\phi)}. \tag{20}$$

The second order condition of this problem states that $V^k(a)$ is concave.

$$\frac{d^2V^k(a)}{da} = 2n'(\phi)\frac{d\phi^k}{da} + an''(\phi)\left(\frac{d\phi^k}{da}\right)^2 + an'(\phi)\frac{d^2\phi^k}{da^2}.$$

From equation (15) we obtain:

$$\frac{d^2\phi^k(a)}{da^2} = -\frac{2n''(\phi)\frac{d\phi^k}{da}\left[[v - T(1)] + 2an''(\phi)\right] - 4n''(\phi)n'(\phi) - 4an'''(\phi)\frac{d\phi^k}{da}n'(\phi)}{\left[[v - T(1)] + 2an''(\phi)\right]^2},$$

and some algebra yields:

$$\frac{d^2V^k(a)}{da} = \frac{2\left[n(\phi)n''(\phi) - (n'(\phi))^2\right]\left[2(n'(\phi))^3 - 3n(\phi)n'(\phi)n''(\phi) + (n(\phi))^2n'''(\phi)\right]}{(n'(\phi))^3\left[v - T(1)\right]}.$$

It is straightforward to check that, for the regular advertising technologies, $n\left(\phi\right) \in \{-Log(1 - \phi); \phi^\alpha\}$, this derivative has a negative sign, so it seems reasonable to assume that $V^k(a)$ is concave. The key point is that, when $a = a^c$, the optimal prices in the corner solution and in the kinked solution are the same, $p_l = v - T(1)$, so equations (9) and (14) coincide, which implies that the optimal advertising levels also coincide, $\phi^c(a^c) = \phi^k(a^c)$. This means that, when $a = a^c$, it holds that $V^c(a^c) = V^k(a^c)$. Given the concavity of $V^k(a)$, this implies that, if $a^c > a^k$, the optimal advertising fee is $a = a^c$, whilst, if $a^c < a^k$, the platform maximizes profits by making $a = a^k$, so $a = Max\left\{a^c, a^k\right\}$.