Economics Bulletin

Volume 31, Issue 3

The mixed strategy equilibrium of the three-firm location game with discrete location choices

Zhiyong Huang
Department of Statistics, Hanken School of Economics

Abstract

In the paper, we derive a symmetric MSE for the three-firm location game on the discrete strategy space. Rather than being uniformly distributed, the MSE for the game has a multimodal distribution. Our theory is more convincing to predict equilibria of three-firm location games in the real world or controlled experiments, where players face finitely many choices.

I am very thankful to an anonymous referee for helpful comments and suggestions.
Citation: Zhiyong Huang, (2011) ''The mixed strategy equilibrium of the three-firm location game with discrete location choices'', *Economics Bulletin*, Vol. 31 no.3 pp. 2109-2116.
Submitted: Feb 09 2011. **Published:** July 18, 2011.

1. **Introduction**

Location games are initially investigated by Hotelling (1929). He assumes that players (customers and firms) are evenly distributed on a linear market. Each customer purchases one unit goods from the nearest firm. Hence firms choose optimal locations to maximize their expected profits. For such games, the case of three firms receives special interest, since Lerner and Singer (1937) show that no pure strategy equilibrium exists for the case. As for the mixed strategy equilibrium (hereafter MSE) of the three-firm location game, a prominent work is done by Shaked (1982). He present that, under the assumption of the continuous strategy space (i.e. locations set is a real interval, thus location choices are infinitely many), a symmetric MSE exists, and the MSE has a uniform probability function with the support between the first and third quartile of the interval.

Yet the game of three firms with discrete location choices has never been examined theoretically. A naive conjecture is that the MSE of the game is (discretely) uniformly distributed as well. Unfortunately it is not true. This is firstly noticed by Collins and Sherstyuk (2000). In their experiment of three-firm location game, the observed data can be poorly calibrated by a uniform distribution. Thus, a model capable of explaining the inconsistency between the conventional wisdom and the newly established experiment, becomes vital.

Our purpose is to derive the mixed strategy equilibrium for the three-firm game on the discrete strategy space. The condition of the discrete location set is justified by the fact that players often face finitely many choices in location games observed in both the real world and controlled experiments. Our result is striking. We show that, for the three-firm game on the discrete strategy space, an MSE defined between the first and the third quartile on the interval exists, but it is not uniformly distributed. Rather, the MSE has a multimodal probability distribution, and its p.m.f is sensitive to the number of available locations. The MSE is distinct from the notable result of Shaked (1982) for games on the continuous space, and it is more consistent with experimental observation of Collins and Sherstyuk (2000).

The rest of the paper contributes to the computation of the MSE under the assumption of discrete strategies confronted by players.

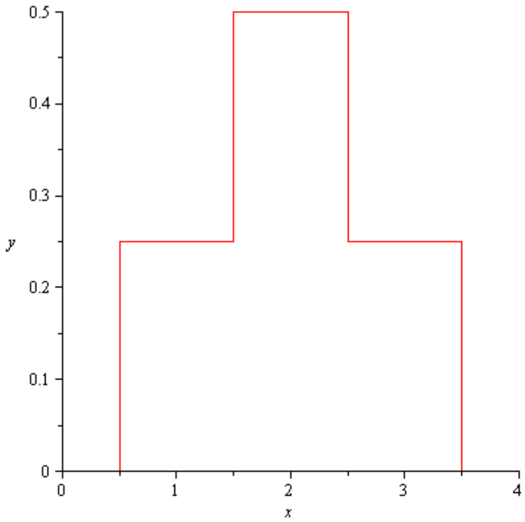
2. **Computations of Mixed Strategy Equilibrium**

Suppose firms choose locations from a discrete location set $S := \{0, 1, \dots, n\} \subset \mathbb{N}$. Without loss of generality, we assume per location there is one customer with one unit demand. Consumers purchase from the nearest firm(s) (if the number of the nearest firms is large than one, these firms equally share the market).

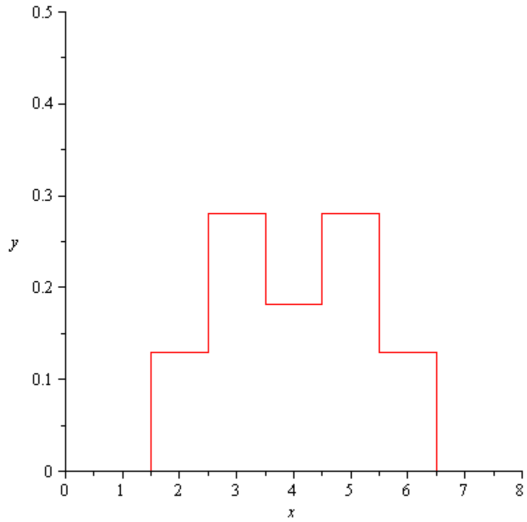
Since the calculation requires numerical methods, we first analyze simple games as $n = 4, 8, 12, 16$, then examine the game as $n = 100$ and compare the MSE with the experimental result of Collins and Sherstyuk (2000).

2.1. **Interval length $n = 4, 8, 12, 16$.** The p.m.f is illustrated in the following figures. See Appendix for the computation.

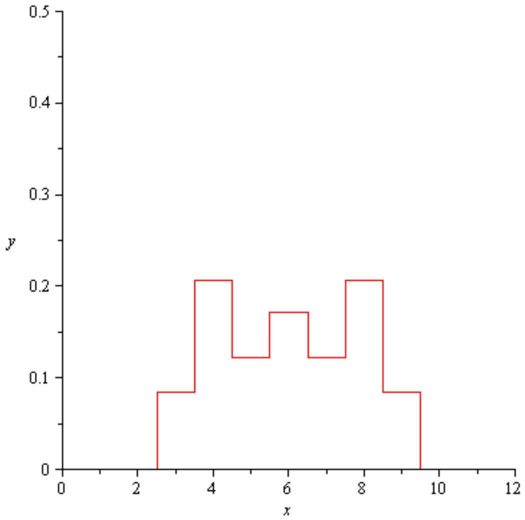
FIGURE 2.1. MSE of three-firm location game on discrete location space: $n = 4, 8, 12, 16$



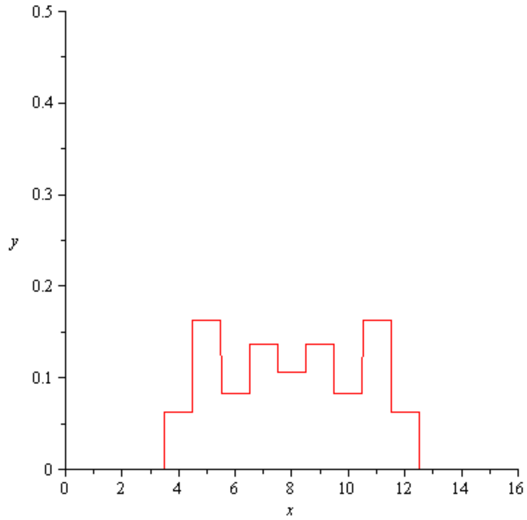
(A) n=4



(B) n=8



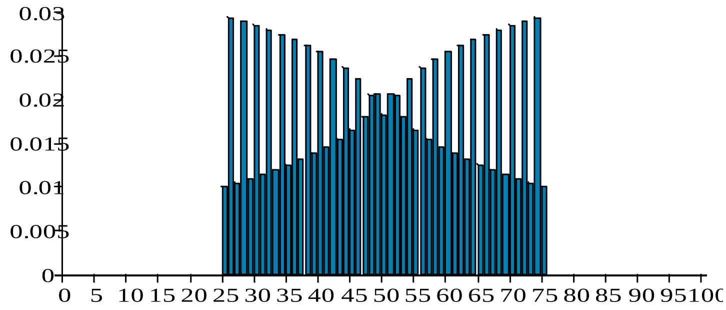
(C) n=12



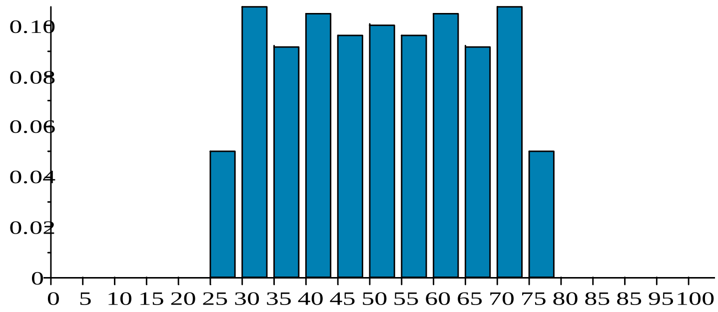
(D) n=16

2.2. **Interval length $n = 100$.** For games of $n = 100$, p.m.f of the mixed strategy equilibrium is depicted in the first figure. To compare with the result of Collins and Sherstyuk (2000), the second figure shows the p.m.f when location choices are grouped into 21 categories (with 3 locations in the first and the last category, and 5 locations in every other category).

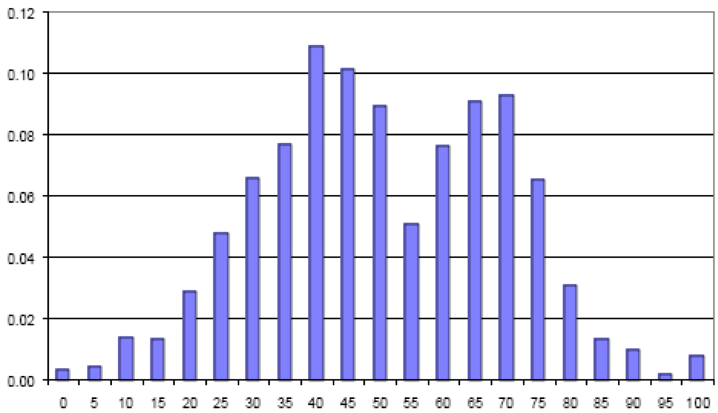
FIGURE 2.2. MSE of three-firm location game on discrete location space: $n = 100$



(A) n=100



(B) n=100, grouped into 21 categories



(C) Empirical distribution of Collins and Sherstyuk (2000)

3. CONCLUSION

In the paper, we derive a symmetric MSE for the three-firm location game on the discrete strategy space. Rather than being uniformly distributed, the MSE for the game has a multimodal distribution. Our theory is more convincing to predict equilibria of three-firm location games in the real world or controlled experiments such as Collins and Sherstyuk (2000), where players face finitely many choices.

APPENDIX

Inspired by Shaked (1982), we conjecture the symmetric mixed strategy equilibrium for three-firm discrete location problem is defined over points between the first and the third quartile of the interval, but without any presumptions imposed on the distribution form.

When $n = 8$, we assume the symmetric mixed strategy equilibrium has a probability mass function, given by:

$$d(x) = \begin{cases} a & x = \{2, 6\} \\ b & x = \{3, 5\} \\ c & x = \{4\} \\ 0 & otherwise \end{cases} \tag{1}$$

where $2a + 2b + c = 1$. The probability mass function is conjectured to be symmetric, with positive value defined over points between two middle quartiles, and zero value otherwise. Therefore, the support of $d(x)$ is $Z = \{2,\ 3,\ 4,\ 5,\ 6\}$ and its complement is $\overline{Z} = \{0,\ 1,\ 7,\ 8\}$.

For an equilibrium, payoff function of a firm - conditional on location choices with probability mass function of two other firms - should take a single value over support of $d(x)$ and a lower value outside the support.

Payoff function for a firm located at $z \in Z = \{2,\ 3,\ 4,\ 5,\ 6\}$ is:

$$\begin{aligned} F(z) = & \sum_{x=2}^{z-1} \left(d(x)^2 \left(n - z + 1 + quo(z - x - 1, 2) + \frac{rem(z - x - 1, 2)}{3} \right) \right) \\ & + 2 \sum_{x=3}^{z-1} \left(d(x) \sum_{t=2}^{x-1} d(t) \left(n - z + 1 + \frac{z - x - 1}{2} \right) \right) \\ & + 2d(z) \sum_{x=2}^{z-1} \left(d(x) \left(\frac{n - z + 1}{2} + \frac{quo(z - x - 1, 2)}{2} + \frac{rem(z - x - 1, 2)}{3} \right) \right) \\ & + 2 \sum_{x=2}^{z-1} \sum_{y=z+1}^{6} \left(d(x) d(y) \frac{y - x}{2} \right) \\ & + \sum_{x=z+1}^{6} \left(d(x)^2 \left(z + 1 + quo(x - z - 1, 2) + \frac{rem(x - z - 1, 2)}{3} \right) \right) \\ & + 2 \sum_{x=z+1}^{5} \left(d(x) \sum_{t=x+1}^{6} d(t) \left(z + 1 + \frac{x - z - 1}{2} \right) \right) \\ & + 2d(z) \sum_{x=z+1}^{6} \left(d(x) \left(\frac{z + 1}{2} + \frac{quo(x - z - 1, 2)}{2} + \frac{rem(x - z - 1, 2)}{3} \right) \right) \\ & + 3d(z)^2 \end{aligned} \tag{2}$$

where $quo(x,\ y)$ and $rem(x,\ y)$ are quotient and remainder of x/y. By definition, $\sum_{x=i}^{j} d(x) = 0$ if $j < i$.

The first and the second summations are payoffs when the other two firms sit to the left of z at one or two locations respectively. The third summation represents payoff when one firm is located at z and the other to the left of z. The fourth summation corresponds to the case of one firm on each side of z. The fifth and sixth summations echo to the cases the two firms standing to the right of z at one or two locations respectively. The seventh summation evaluates payoff when one firm is located at z and the other to the right of z. The last summation is payoff when the other two firms standing at z as well.

For any point of the support Z, payoff takes a single value:

$$F(z \mid z \in Z = \{2,\ 3,\ 4,\ 5,\ 6\}) = F \tag{3}$$

where F is a constant.

Equations 1, 2 and 3 give:

$$a = 0.129,\ b = 0.280,\ c = 0.181,\ F = 3$$

To verify this is an mixed strategy equilibrium, we need to show the payoff outside the support of $d(x)$ has values lower than F.

Payoff for a firm situated at $z \in \overline{Z} = \{0,\ 1,\ 7,\ 8\}$ is:

$$\begin{aligned} F(z \mid z \in \{0,\ 1\}) = &\sum_{x=2}^{6} \left(d(x)^2 \left(z + 1 + quo(x-z-1,2) + \frac{rem(x-z-1,2)}{3} \right) \right) \\ &+ 2 \sum_{x=2}^{5} \left(d(x) \sum_{t=x+1}^{6} d(t) \left(z + 1 + \frac{x-z-1}{2} \right) \right) \end{aligned} \tag{4}$$

or

$$\begin{aligned} F(z \mid z \in \{7,\ 8\}) = &\sum_{x=2}^{6} \left(d(x)^2 \left(9 - z + quo(z-x-1,2) + \frac{rem(z-x-1,2)}{3} \right) \right) \\ &+ 2 \sum_{x=3}^{6} \left(d(x) \sum_{t=2}^{x-1} d(t) \left(9 - z + \frac{z-x-1}{2} \right) \right) \end{aligned} \tag{5}$$

Simple calculation shows:

$$F(1) = F(7) = 2.619$$

Since $F(z \mid z \in \{0,\ 1\})$ is increasing function and $F(z \mid z \in \{7,\ 8\})$ is decreasing function with respect to z, $F(z \mid z \in \{0,\ 1,\ 7,\ 8\}) < F$.

Hence:

$$d(x) = \begin{cases} 0.129 & x = \{2, 6\} \\ 0.280 & x = \{3, 5\} \\ 0.181 & x = \{4\} \\ 0 & otherwise \end{cases}$$

is one symmetric mixed strategy equilibrium for three- firm location problem when $n = 8$.
The same procedure reveals:

$$d(x) = \begin{cases} 0.25 & x = \{1, 3\} \\ 0.5 & x = \{2\} \\ 0 & otherwise \end{cases}$$

is one symmetric mixed strategy equilibrium for the location problem when $n = 4$,

$$d(x) = \begin{cases} 0.085 & x = \{3, 9\} \\ 0.207 & x = \{4,\ 8\} \\ 0.122 & x = \{5,\ 7\} \\ 0.172 & x = \{6\} \\ 0 & otherwise \end{cases}$$

is one symmetric mixed strategy equilibrium for the location problem when $n = 12$, and

$$d(x) = \begin{cases} 0.063 & x = \{4, 12\} \\ 0.163 & x = \{5,\ 11\} \\ 0.083 & x = \{6,\ 10\} \\ 0.137 & x = \{7,\ 9\} \\ 0.106 & x = \{8\} \\ 0 & otherwise \end{cases}$$

is one symmetric mixed strategy equilibrium for the location problem when $n = 16$.

References

Collins, R., Sherstyuk, K., January 2000. Spatial competition with three firms: An experimental study. Economic Inquiry 38 (1), 73–94.
URL http://ideas.repec.org/a/oup/ecinqu/v38y2000i1p73-94.html

Hotelling, H., 1929. Stability in competition. Economic Journal 39, 41–57.

Lerner, A., Singer, H., 1937. Some notes on duopoly and spatial competition. Journal of Political Economy 45, 145–186.

Shaked, A., 1982. Existence and computation of mixed strategy nash equilibrium for 3-firms location problem. The Journal of Industrial Economics 31 (1/2), 93–96.