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On the bargaining set of three-player games

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Abstract

The description and the characterization of the (Aumann-Maschler) bargaining set are known, but its determination is still hard given a generic transferable utility cooperative game. We provide here an exhaustive determination of the bargaining set of any three-player game, balanced or not, superadditive or not.

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1. Introduction

Among solutions to transferable utility cooperative games, the bargaining set $\mathcal{M}_1^i$ selects imputations that are stable via a certain bargaining possibilities of the players. First formulated by Aumann and Maschler (1964), the bargaining $\mathcal{M}_1^i$ set was explicitly introduced by Davis and Maschler (1967). Given an arbitrary transferable utility cooperative game with a nonempty set of imputations, the bargaining set $\mathcal{M}_1^i$ is nonempty for any coalition structure (see Davis and Maschler (1967); Peleg (1967)). Of course been nonempty constitutes an unquestionable attractiveness of the bargaining set $\mathcal{M}_1^i$.

Unlike the core which has a complete description as a single set of linear inequalities, the bargaining set is, as shown by Maschler (1966), a finite union of compact convex polyhedra. But, as mentioned by Maschler (1992, p. 601), this description does not "provide an easy way to compute the bargaining set for arbitrary generic games".

Recent findings and impovments by Solymosi (2002) reveal some surprising and hidden properties of the bargaining set: the bargaining set for any balanced game with less than five players and for arbitrary coalition structure coincides with the core. We provide here a complete determination of the bargaining set for three player games. All games, balanced or not, superadditive or not, are considered with geometric illustrations. It is shown that the bargaining set for any three player game is a single polytope, possibly reduced to a unique point.

The paper is organized as follows. Section 2 is devoted to some definitions and some notations. Section 3 provides an exhaustive determination of the bargaining set of any three player games. Finally Section 4 concludes the paper.

2. Notations and definitions

A transferable utility cooperative game, simply a game, is a pair (N, v) such that N is a finite set and v is a real-valued function from the set $2^N \setminus \{\emptyset\}$, the set of nonempty subsets of N. The set N and the function v are respectively called the set of players and the coalitional function of the game (N, v). A coalition in the game (N, v) is any nonempty subset of N. Given a payoff allocation $x \in \mathbb{R}^N$ and a coalition $S \subseteq N$, the total payoff to S is $x(S) = \sum_{i \in S} x_i$ and the indicator vector of S is the $\left(2^{|N|} - 1\right)$-dimensional vector a^S defined by $a_i^S = 1$ if $i \in S$ and $a_i^S = 0$ if $i \notin S$. The excess $e_x(v, S) = v(S) - x(S)$ of S at x measures the gain (if positive) and the loss (if negative) to S, if its member depart from the proposed payoff allocation x in order to form their own coalition. In order to simplify notations, we shall sometimes ignore braces and commas when describing coalitions, e.g., we shall write i instead of $\{i\}$ and $N \backslash ij$ instead of $N \setminus \{i, j\}$.

A coalition structure $\mathfrak{B}$ for the game (N, v) is a partition of N, that is a set of nonempty and disjoint coalitions, the union of which is N. Given a structure coalition

$\mathfrak{B}$ for (N, v) , each $B \in \mathfrak{B}$ is called a player class, two players belonging to the same player class are called partners and an imputation is a payoff allocation x such that

$$\begin{cases} e_x(v, S) = 0, \forall B \in \mathfrak{B} \\ e_x(v, i) \leq 0, \forall i \in N \end{cases} \tag{1}$$

Let $\mathfrak{X}(v, \mathfrak{B})$ be the set of all imputations given a coalition structure $\mathfrak{B}$. Then

$$\mathfrak{X}(v, \mathfrak{B}) \neq \emptyset \Longleftrightarrow \sum_{i \in B} v(i) \leq v(B), \forall B \in \mathfrak{B} \tag{2}$$

Definition 1 *The core, given a game (N, v) and a coalition structure $\mathfrak{B}$ is the set $C(v, \mathfrak{B})$ of all imputations x such that $e_x(v, S) \leq 0$ for any coalition S.*

Given a coalition structure, the game is balanced if it has a nonempty core.

Definition 2 *For a game (N, v), let x be an imputation and $\{i, j\}$ a pair of partners for a structure coalition $\mathfrak{B}$. Given $S \subseteq N \backslash ij$ and y a payoff allocation to $S \cup i$, the pair $(S \cup i, y)$ is an objection of i via S against j at x if*

$$y(S \cup i) \leq v(S \cup i) \text{ and } y_k > x_k, \forall k \in S \cup i$$

Definition 3 *Given a game (N, v), a coalition structure $\mathfrak{B}$, an objection $(S \cup i, y)$ of i via S against j at x, $T \subseteq N \backslash ij$ and z a payoff allocation to $T \cup j$, the pair $(T \cup j, z)$ is a counter-objection of j to $(S \cup i, y)$ if*

$$z(T \cup j) \leq v(T \cup j); \;\; z_k \geq y_k, \forall k \in T \cap S \text{ and } z_k \geq x_k, \forall k \in T \cup j \backslash S$$

Definition 4 *The bargaining set $\mathcal{M}_1^i(v, \mathfrak{B})$ for a game (N, v) and a coalition structure $\mathfrak{B}$ is defined as the set of all imputations for $\mathfrak{B}$ at which every objection has at least one counter-objection.*

Peleg (1967) proved that for any coalition structure $\mathfrak{B}$, $\mathcal{M}_1^i(v, \mathfrak{B}) \neq \emptyset$ whenever $\mathfrak{X}(v, \mathfrak{B}) \neq \emptyset$. We determine here the bargaining set of three player games.

3. The bargaining set of three player games

3.1. Preliminaries

Suppose $N = \{i, j, k\}$. Then the coalition structures are: (i) $\mathfrak{B}_0 = \{\{i\}, \{j\}, \{k\}\}$ when each player stays alone; (ii) $\mathfrak{B}_1 = \{ij, k\}$ when two players are partners and the third player stays alone; and (iii) $\mathfrak{B}_2 = \{ijk\}$ when the three players are grouped in N. Note that for $\mathfrak{B} = \mathfrak{B}_0$, $\mathcal{M}_1^i(v, \mathfrak{B}_0) = \{(v(i), v(j), v(k))\}$. For $\mathfrak{B} \neq \mathfrak{B}_0$, the proposition below identifies the imputations that belong to the bargaining set in terms of (N, v) only.

Proposition 1 *Let $|N| = 3$ and $\mathfrak{B}$ be a coalition structure. Then an imputation x belongs to $\mathcal{M}_1^i(v, \mathfrak{B})$ if and only if for all couple (i, j) of partners with $N \backslash ij = \{k\}$,*

$$e_x(v, ik) \leq 0 \text{ or } e_x(v, j) = 0 \text{ or } e_x(v, jk) \geq e_x(v, ik) \tag{3}$$

Proof. Necessity. Let $x \in \mathcal{M}_1^i(v, \mathfrak{B})$. Suppose that (3) does not hold for some couple (i, j) of partners. Then $e_x(v, jk) < e_x(v, ik)$ and $e_x(v, ik) > 0$. As a consequence, there exists some $\varepsilon > 0$ such that $e_x(v, ik) - \varepsilon > 0$ and $e_x(v, ik) - \varepsilon > e_x(v, jk)$. Define a payoff allocation y to ik by $y_i = x_i + \varepsilon$ and $y_k = x_k + e_x(v, ik) - \varepsilon$. One can easily check that (ik, y) is an objection of i against j via k at x. Since $x \in \mathcal{M}_1^i(v, \mathfrak{B})$, there exists a counter-objection $(T \cup j, z)$ of j to (ik, y). Since (3) does not hold $e_x(v, j) < 0$. Thus $T \neq \emptyset$. That is $T = k$. Moreover $z_j \geq x_j$ and $z_k \leq v(jk) - z_j \leq v(jk) - x_j = x_k + e_x(v, jk) < x_k + e_x(v, ik) - \varepsilon = y_k$. Hence $z_k < y_k$ holds and is a contradiction to the fact that (jk, z) is a counter-objection to (ik, y).

Sufficiency. Let x be an imputation that satisfies (3) for all couple (i, j) of partners. Suppose that there exists some couple (i, j) of partners and an objection $(S \cup i, y)$ of i at x against j. By definition of an objection, $x(S \cup i) \leq y(S \cup i) < v(S \cup i)$. Thus $e_x(v, S \cup i) > 0$. Since x is an imputation, $S = k$. Moreover $e_x(v, j) = 0$ or $e_x(v, jk) \geq e_x(v, ik)$ hold by (3). First note that when $e_x(v, j) = 0$, (j, x) is a trivial counter-objection of j to (ik, x). To see this, note that j can secure $v(j)$ alone. Now suppose that $e_x(v, jk) \geq e_x(v, ik)$. Define a payoff allocation z to jk by $z_j = x_j$ and $z_k = x_k + e_x(v, jk)$. Then $z(jk) = v(jk)$ and $z_k \geq x_k + e_x(v, ik)$. Recalling that $e_x(v, ik) = v(ik) - x(ik)$, it follows that $z_k \geq x_k + v(ik) - x(ik)$. But $y(ik) \leq v(ik)$. Then $z_k \geq x_k + y(ik) - x(ik) = y_k$. Therefore (jk, z) is a counter-objection to (ik, y). In both cases, at least a counter-objection to (ik, y) exists. Thus $x \in \mathcal{M}_1^i(v, \mathfrak{B})$. ■

Note that an equivalent form of proposition 1 can be obtained from Maschler (1966, theorem). Hereafter, condition (3) is denoted by $(i \to j)$. Moreover, $(i \to j)_1$ stands for $e_x(v, ik) \leq 0$, $(i \to j)_2$ for $e_x(v, j) = 0$ and $(i \to j)_3$ for $e_x(v, jk) \geq e_x(v, ik)$.

Given a game (N, v), $\alpha > 0$ and $\beta \in \mathbb{R}^N$, the game $(N, v_{\alpha,\beta})$ is defined by $v_{\alpha,\beta}(S) = \alpha v(S) + \beta(S)$ for all coalition S. It is known - see Maschler (1992) - that:

$$\mathcal{M}_1^i(v_{\alpha,\beta}, \mathfrak{B}) = \alpha \mathcal{M}_1^i(v, \mathfrak{B}) + \beta \tag{4}$$

Therefore, without no limitation, we pose $N = \{1, 2, 3\}$, $v(123) = \varepsilon$, $v(12) = a$, $v(13) = b$, $v(23) = c$ and $v(i) = 0$ for all $i \in N$. Moreover we assume that $a \leq b \leq c$.

3.2. When two players are partners and the third is alone

Here the coalition structure is $\mathfrak{B}_1 = \{ij, k\}$. Thus $\mathfrak{X}(v, \mathfrak{B}_1) \neq \emptyset$ if and only if $v(ij) \geq 0$.

Proposition 2 *Let $N = \{1, 2, 3\}$ and suppose that $\mathfrak{X}(v, \mathfrak{B}_1) \neq \emptyset$.*

1. *If $\mathfrak{B}_1 = \{12, 3\}$ then* $\mathcal{M}_1^i(v, \mathfrak{B}_1) = \begin{cases} \left\{\left(\frac{a+b-c}{2}, \frac{a-b+c}{2}, 0\right)\right\} & \text{if} \quad c \leq a + b \\ \{(0, a, 0)\} & \text{if} \quad c > a + b \end{cases}$

2. *If* $\mathfrak{B}_1 = \{13, 2\}$ *then* $\mathcal{M}_1^i(v, \mathfrak{B}_1) = \begin{cases} \left\{\left(\frac{a+b-c}{2}, 0, \frac{-a+b+c}{2}\right)\right\} & \text{if} \quad c \leq a+b \\ \{(0,0,b)\} & \text{if} \quad c > a+b \end{cases}$

3. *If* $\mathfrak{B}_1 = \{1, 23\}$ *then* $\mathcal{M}_1^i(v, \mathfrak{B}_1) = \begin{cases} \left\{\left(0, \frac{a-b+c}{2}, \frac{-a+b+c}{2}\right)\right\} & \text{if} \quad c \leq a+b \\ \{(0, c-t, t) : b \leq t \leq c-a\} & \text{if} \quad c > a+b \end{cases}$

Proof. Let $\mathfrak{B}_1 = \{12, 3\}$ and $x \in \mathfrak{X}(v, \mathfrak{B}_1)$. By proposition 1, $x \in \mathcal{M}_1^i(v, \mathfrak{B}_1)$ if and only if x satisties $x_1 + x_2 = a$, $x_3 = 0$, $(1 \longrightarrow 2)$ and $(2 \longrightarrow 1)$. The only thing one has to do is to rule out redundant constraint to obtain the result. We present the proof only for $\mathfrak{B}_1 = \{12, 3\}$. Other cases are similar and then are left[1].

First suppose that $c > a+b$. Then observe that $e_x(v, 23) = c - x_2 \geq c - a > 0$ and $e_x(v, 23) - e_x(v, 13) = c + x_1 - b - x_2 \geq c - b - a > 0$. Therefore $(2 \longrightarrow 1)_1$ and $(2 \longrightarrow 1)_3$ do not hold. Thus $(2 \longrightarrow 1)$ is equivalent to $x_1 = 0$. Hence $x_2 = a$. Finally $(0, a, 0)$ satisfies $(1 \longrightarrow 2)_3$. This prove that $\mathcal{M}_1^i(v, \mathfrak{B}_1) = \{(0, a, 0)\}$.

Now suppose that $c \leq a+b$. **(a).** We claim that $(1 \longrightarrow 2)_3$ holds. Suppose the contrary. That is $e_x(v, 23) < e_x(v, 13)$. If $(1 \longrightarrow 2)_2$ holds, then $x_2 = 0$ and $x_1 = a \geq 0$. Thus $e_x(v, 23) < e_x(v, 13)$ yields $c < b - a \leq b$ which is a contradiction. If $(1 \longrightarrow 2)_1$ holds, then $x_1 \geq b$. Since $b \geq a$ and $x_1 \leq a$, then $a = b$ and $x_1 = a$. Hence $x_2 = 0$ and a contradiction arises as previously shown. **(b).** We also prove that $(2 \longrightarrow 1)_3$ holds. Suppose the contrary. That is $e_x(v, 13) < e_x(v, 23)$. If $(2 \longrightarrow 1)_2$ holds, then $x_1 = 0$ and $x_2 = a$. Thus $e_x(v, 13) < e_x(v, 23)$ becomes $b < c - a$ which is a contradiction as it is assumed that $c \leq a+b$. If $(1 \longrightarrow 2)_2$ holds, then $x_2 \geq c$. Since $c \geq a$ and $x_2 \leq a$, then $c = b$ and $x_2 = a$. Hence $x_2 = 0$ and a contradiction arises as it is already shown. In summary, $e_x(v, 23) \leq e_x(v, 13)$ and $e_x(v, 13) \geq e_x(v, 23)$. Hence $e_x(v, 13) = e_x(v, 23)$ and $\left(\frac{a+b-c}{2}, \frac{a-b+c}{2}, 0\right)$ and $\mathcal{M}_1^i(v, \mathfrak{B}_1) = \left\{\left(\frac{a+b-c}{2}, \frac{a-b+c}{2}, 0\right)\right\}$. ■

3.3. When the three players form the grand coalition

Here the coalition structure is $\mathfrak{B}_2 = \{ijk\}$. Thus $\mathfrak{X}(v, \mathfrak{B}_2) \neq \emptyset$ if and only if $v(ijk) = \varepsilon \geq 0$. Moreover, $\mathcal{M}_1^i(v, \mathfrak{B}_1) = \mathfrak{X}(v, \mathfrak{B}_2) = \{(0,0,0)\}$ whenever $\varepsilon = 0$. We then suppose $\varepsilon > 0$ and by (4), we set $v(ijk) = 1$ without no limitation.

Proposition 3 *Let* $N = \{1, 2, 3\}$, $S = a + b + c$ *and suppose that* $\mathfrak{X}(v, \mathfrak{B}_2) \neq \emptyset$.

1. *If* $c \leq 1$ *and* $S \leq 2$ *then* $\mathcal{M}_1^i(v, \mathfrak{B}_2) = C(v, \{N\})$;
2. *Else if* $c \leq \frac{1+S-c}{2}$ *then* $\mathcal{M}_1^i(v, \mathfrak{B}_2) = \left\{\left(\frac{1+S-2c}{3}, \frac{1+S-2b}{3}, \frac{1+S-2a}{3}\right)\right\}$;
3. *Else if* $a + b \leq 1$ *then* $\mathcal{M}_1^i(v, \mathfrak{B}_2) = \{(0, \alpha, 1-\alpha), a \leq \alpha \leq 1-b\}$;
4. *Else if* $b \leq 1 + a$ *then* $\mathcal{M}_1^i(v, \mathfrak{B}_2) = \left\{\left(0, \frac{1+a-b}{2}, \frac{1-a+b}{2}\right)\right\}$;
5. *Otherwise* $b > 1 + a$, *and then* $\mathcal{M}_1^i(v, \mathfrak{B}_2) = \{(0, 0, 1)\}$.

[1]The entire proof is available from the author under simple request.

Proof. Once more, by proposition 1, the bargaining set is the collection of all imputations that satisfies (3) for any couple of partners. We only have to check which constraints are redundant and then rule such constraints out to derive the exact solution. One can easily check that the game is balanced when $c \leq 1$ and $S \leq 2$. In this case, the core and the bargaining set coincide. We provide here[2] the proof only when ($c > 1$ or $S > 2$) and $c \leq \frac{1+S-c}{2}$. Other cases can be proved in the same way.

Suppose that ($c > 1$ or $S > 2$) and $c \leq \frac{1+a+b}{2}$. Note that in this case, the game is not balanced. It is also assumed that $a \leq b \leq c$. Thus $b \leq \frac{1+a+c}{2}$ and $a \leq \frac{1+b+c}{2}$. In other words, $v(ij) \leq \frac{1+v(jk)+v(ik)}{2}$ for all couple (i,j) with $N = \{i,j,k\}$.

Sufficiency. Let $y = \left(\frac{1+S-2c}{3}, \frac{1+S-2b}{3}, \frac{1+S-2a}{3}\right)$. Then y is an imputation and y satisfies $(i \to j)_3$ for any couple (i,j) of partners. By proposition 1, $\{y\} \subseteq \mathcal{M}_1^i(v, \mathfrak{B}_2)$. Moreover, y is the unique imputation that satisfies $(i \to j)_3$ for any $\{i,j\} \subseteq N$.

Necessity. Now suppose x is an imputation such that $x \neq y$. Therefore $(i \to j)_3$ does not occur for at least a couple (i,j) of partners. Pose $N = \{i,j,k\}$. By proposition 1, $(i \to j)_1$ or $(i \to j)_2$.

(a) Suppose $(i \to j)_1$. That is $e_x(v, ik) \leq 0$. Since $(i \to j)_3$ does not hold, we get $e_x(v, jk) < e_x(v, ik)$. But $e_x(v, ik) \leq 0$. Thus $e_x(v, jk) < 0$. The imputation x satisfies $e_x(v, S) \leq 0$ for coalition S except for $S = ij$. Since the game is not balanced, $e_x(v, ij) > 0$. Then $(i \to k)_1$ does not hold. Moreover, $e_x(v, ij) > 0$ and $e_x(v, ik) < 0$. Hence $e_x(v, ik) < e_x(v, ij)$. Then $(i \to k)_3$ does not occur. Since $(i \to k)_1$ and $(i \to k)_3$ do not hold, x should satisfy $(i \to k)_2$ by proposition 1. This amounts to $x_k = 0$. In summary, $e_x(v, ij) = v(ij) - x_i - x_j > 0$, $e_x(v, ik) = v(ik) - x_i \leq 0$ and $e_x(v, jk) = v(jk) - x_j < 0$. We then deduce $v(jk) + v(ik) < x_i + x_j = 1 < v(ij)$. Therefore $v(ij) > 1 > \frac{1+v(jk)+v(ik)}{2}$. A contradiction appears.

(b) Suppose *not* $(i \to j)_1$. Then $(i \to j)_2$ should be satisfied. That is $e_x(v, ik) > 0$ and $x_j = 0$. We then get $x_i + x_k = 1 < v(ik)$. By proposition 1, $(j \to k)_1$, $(j \to k)_2$ or $(j \to k)_3$ should be staisfied. (b-1): First suppose that $(j \to k)_1$ occurs. Then $e_x(v, ij) \leq 0$. That is $x_i \geq v(ij)$. Since $(i \to j)_3$ does not hold, we get $v(jk) < v(ik) - x_i$. Therefore $v(ij) \leq x_i < v(ik) - v(jk)$. Hence $v(ik) > v(ik) + v(ij)$. Since $v(ik) > 1$, then $v(ik) > \frac{1+v(jk)+v(ij)}{2}$. A contradiction appears. (b-2): Now suppose that $(j \to k)_3$ occurs. Then $v(ik) - 1 \geq v(jk) - x_i$. But $x_i < v(ik) - v(jk)$ since $(i \to j)_3$ does not hold. Hence $v(jk) - v(ik) + 1 \leq x_i < v(ik) - v(jk)$. Therefore $v(ik) > \frac{1+v(jk)+v(ij)}{2}$. A contradiction appears. (b-3) Finally suppose that none of $(j \to k)_1$ and $(j \to k)_3$ hold. Then $(j \to k)_2$ occurs. We get $v(ik) < v(ij)$ by *not* $(j \to k)_3$ and $1 + v(jk) < v(ik)$ by *not* $(i \to j)_3$. Hence $v(ij) > \frac{1+v(jk)+v(ik)}{2}$. A contradiction appears. In all possible cases, a contradiction holds. Thus $x \notin \mathcal{M}_1^i(v, \mathfrak{B}_2)$.

Clearly $\mathcal{M}_1^i(v, \mathfrak{B}_2) = \{y\}$. ■

3.4. Geometric illustrations

We sketch in Fig 1 and Fig 2 the bargaining set when only two players are partners and when the grand coalition is formed. Each player can therefore be aware about what

[2] The entire proof is available from the author under simple request.

he would gain or expect from each coalition structure. It clearly appears that player 1 has the least favorable position while player 3 has a leading position. Of course, these observations are due to the assumption we have made on the coalitional function.

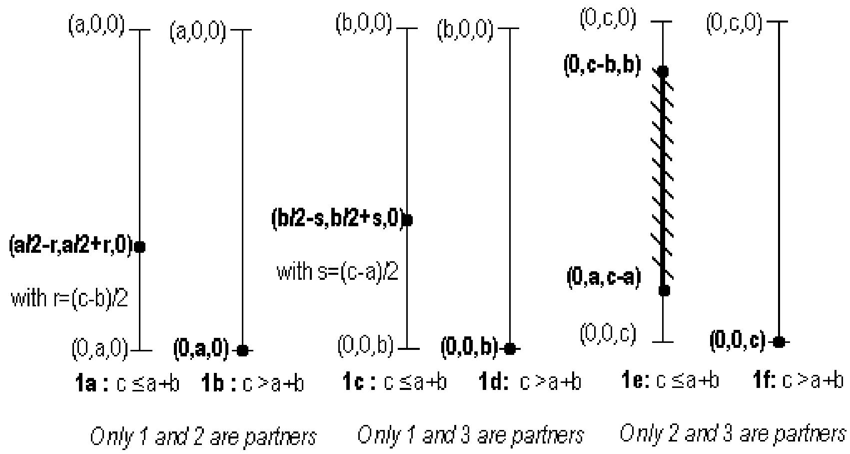

Fig 1. Three player bargaining set when only two players are partners

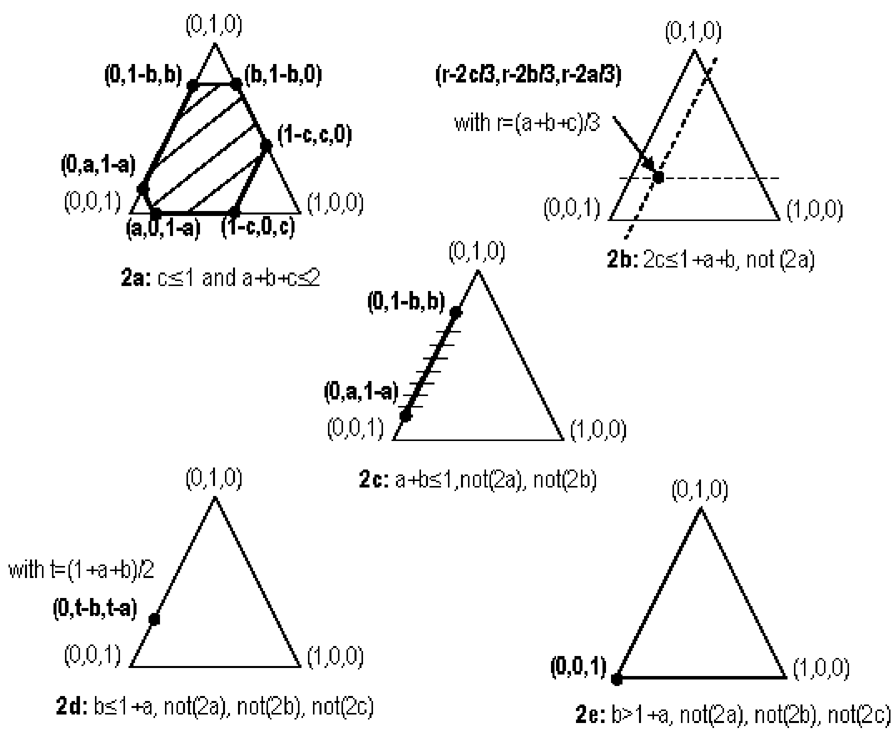

Fig 2. Three player bargaining set when the grand coalition is formed

4. Conclusion

Proposition 2 and proposition 3 combined with (4) provide an exhaustive determination of the bargaining set of any three player game for any coalition structure. A possible extension of the present work consists in a general setting to identify among polytopes that define the bargaining set, empty polytopes or to establish some families of set inclusion.

References

[1] Aumann, R.J. and M. Maschler (1964) "The bargaining set for cooperative games" In *Dresher M, Shapley LS, Tucker AW (eds.) Advances in Game Theory*, Princeton University Press, Princeton, 443–476.

[2] Davis, M. and M. Maschler (1967) "Existence of stable payoff configurations for cooperative games" *In Shubik M (ed.) Essays in Mathematical Economics in Honour of Oskar Morgenstern*, Princeton University Press, Princeton, 39–52.

[3] Maschler, M. (1966) "The inequalities that determine the bargaining set $\mathcal{M}_1^i$" *Israel Journal of Mathematics* **4**: 127–134.

[4] Peleg, B. (1967) "Existence theorem for the bargaining set $\mathcal{M}_1^i$" *in Shubik M (ed.) Essays in Mathematical Economics in Honour of Oskar Morgenstern,* Princeton University Press, Princeton, 53–56.

[5] Solymosi, T. (2002) "The bargaining set of four-person balanced games" *International Journal of Game Theory* **31**: 1–11.