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Additivity and Uncertainty

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Abstract

We provide necessary and sufficient conditions for an alpha-MEU preference order to coincide with an SEU order.

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1 Introduction

This note concerns the representation of preferences over *acts*: mappings from states to outcomes. The leading example is *subjective expected utility (SEU)*, but numerous alternatives to SEU have been described in the literature. These representations typically involve a utility function over outcomes and a real-valued function defined on utility-acts (i.e., mappings from states to utilities). For SEU, this latter function has the convenient property of linearity; but this convenience does not extend beyond SEU, at least not without further restrictions on the domain.

In an α-MEU (or Arrow-Hurwicz) representation there is a closed and convex set, C, of probability measures on the state space. Utility-acts are evaluated by computing a weighted average of the maximum and minimum expected utility over C. This evaluation is not, in general, linear in utility-acts, even when the maximum and minimum expected utilities are weighted equally. We provide necessary and sufficient conditions for such preferences to be consistent with SEU behaviour.

2 The main result

Let Ω be a finite state space. A *utility-act* is a function from Ω to $\mathbb{R}$. Since Ω is finite, it is convenient to let $\Omega = \{1, 2, ..., n\}$ and to think of utility-acts as vectors in $\mathbb{R}^n$. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be the unit simplex in $\mathbb{R}^n$. We interpret elements of $\mathcal{P}$ as probabilities on Ω. Finally, let $\mathcal{C}$ denote the class of non-empty, closed and convex subsets of $\mathcal{P}$.

Given $C \in \mathcal{C}$ and $\alpha \in [0, 1]$, define the function $K_C^\alpha : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$K_C^\alpha(x) \;=\; \alpha \min_{p \in C} \; p \cdot x \;+\; (1-\alpha) \max_{p \in C} \; p \cdot x.$$

This is an α-MEU function for evaluating utility-acts. We say that K_C^α *coincides with an SEU ordering* (or is *SEU-consistent*) if there exists some $q \in \mathcal{P}$ such that

$$K_C^\alpha(x) \;=\; q \cdot x$$

for all $x \in \mathbb{R}^n$. The purpose of this note is to establish necessary and sufficient conditions for K_C^α to coincide with an SEU ordering. This is equivalent to linear homogeneity and additivity of K_C^α.

Definition 1 *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* linearly homogeneous *if* $f(\lambda x) = \lambda f(x)$ *for any* $\lambda \in \mathbb{R}$ *and any* $x \in \mathbb{R}^n$.

Definition 2 *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* additive *if* $f(x+y) = f(x) + f(y)$ *for any* $x, y \in \mathbb{R}^n$.

The following is well-known but we include a proof for the sake of completeness.

Lemma 2.1 *The function K_C^α is SEU-consistent iff it is both linearly homogeneous and additive.*

Proof. If K_C^α is SEU-consistent then it is obvious that K_C^α is linearly homogeneous and additive. Conversely, let K_C^α be linearly homogeneous and additive. Let δ^i be the i^{th} standard basis vector for $\mathbb{R}^n$. That is, δ^i is a vector with one for its i^{th} component and zeroes elsewhere. Let $q_i = K_C^\alpha(\delta^i)$. Then

$$K_C^\alpha(x) = \sum_{i=1}^{n} x_i K_C^\alpha(\delta^i) = q \cdot x,$$

where $q = (q_1, ..., q_n) \in \mathbb{R}^n$. $\square$

We first establish necessary and sufficient conditions for linear homogeneity.

Lemma 2.2 *The following are equivalent:*

(1) The function K_C^α is linearly homogeneous.

(2) The set C is a singleton or $\alpha = 1/2$.

Proof. It is obvious that K_C^α is linearly homogeneous when C is a singleton. It is also clear that $K_C^\alpha(\lambda x) = \lambda K_C^\alpha(x)$ for any $\alpha \in [0,1]$, any $C \in \mathcal{C}$, any $x \in \mathbb{R}^n$ and any $\lambda \geq 0$. If $\lambda < 0$:

$$\begin{aligned} K_C^\alpha(\lambda x) &= \alpha \min_{p\in C} \left(-|\lambda|(p\cdot x)\right) + (1-\alpha)\max_{p\in C}\left(-|\lambda|(p\cdot x)\right) \\ &= -\left[\alpha \max_{p\in C}\left(|\lambda|(p\cdot x)\right) + (1-\alpha)\min_{p\in C}\left(|\lambda|(p\cdot x)\right)\right] \\ &= -|\lambda|\left[\alpha\max_{p\in C} p\cdot x + (1-\alpha)\min_{p\in C} p\cdot x\right] \\ &= \lambda K_C^{1-\alpha}(x). \end{aligned}$$

Note that $K_C^{1-\alpha}(x) = K_C^\alpha(x)$ iff $\alpha = 1/2$ or

$$\max_{p\in C} p\cdot x = \min_{p\in C} p\cdot x.$$

Unless C is a singleton, we can always find some $x \in \mathbb{R}^n$ for which

$$\max_{p\in C} p\cdot x \neq \min_{p\in C} p\cdot x.$$

Therefore, (1) holds iff (2). □

Lemma 2.2 implies that a necessary condition for K_C^α to coincide with an SEU ordering is that C is a singleton or $\alpha = 1/2$. Obviously, the former condition is also sufficient. When $\alpha = 1/2$, a necessary and sufficient condition is that C is centrally symmetric.

Definition 3 ***(Ewald 1996, p.23)*** *A set $C \in \mathcal{C}$ is* centrally symmetric *if there exists some $\hat{p} \in C$ (called the* centre *of C) such that, for any $p \in \mathcal{P}$,*

$$p \in C \quad \Rightarrow \quad \hat{p} - (p - \hat{p}) \in C.$$

Remark 1 *Note that we could write "⇔" instead of "⇒" without affecting Definition 3, since $q = \hat{p} - (p - \hat{p})$ implies $p = \hat{p} - (q - \hat{p})$.*

Theorem 2.1 *Let $C \in \mathcal{C}$ contain more than one element. Then K_C^α coincides with an SEU ordering iff $\alpha = 1/2$ and C is centrally symmetric.*

Proof. Suppose $\alpha = 1/2$ and C is centrally symmetric with centre q. Let $x \in \mathbb{R}^n$. Then

$$\min_{p \in C} \; p \cdot x \;=\; \min_{p \in C} \; (2q - p) \cdot x$$

by central symmetry.[1] Hence

$$\min_{p \in C} \; p \cdot x \;=\; 2q \cdot x \;-\; \max_{p \in C} \; p \cdot x$$

which implies $K_C^\alpha(x) = q \cdot x$ (recalling that $\alpha = 1/2$).

Conversely, suppose K_C^α coincides with an SEU ordering. That is, $K_C^\alpha(x) = q \cdot x$ for some $q \in \mathcal{P}$. Since C is not a singleton, we must have $\alpha = 1/2$ (Lemmas 2.1 and 2.2). We claim that C is centrally symmetric with centre q. If not, then there exists $\tilde{p} \in C$ such that $q - (\tilde{p} - q) \notin C$. Since C is closed and convex, the Separating Hyperplane Theorem implies the existence of some $x \in \mathbb{R}^n$ such that

$$\max_{p \in C} \; p \cdot x \;<\; [q - (\tilde{p} - q)] \cdot x.$$

But since $-\tilde{p} \cdot x \leq -\min_{p \in C} p \cdot x$, it follows that

$$\min_{p \in C} \; p \cdot x + \max_{p \in C} \; p \cdot x < 2\,(q \cdot x)\,,$$

which contradicts $K_C^\alpha(x) = q \cdot x$ for $\alpha = 1/2$. □

[1] See Remark 1 following Definition 3.

Remark 2 *It is not critical to the argument that the domain of* K_C^α *is* $\mathbb{R}^n$*. Suppose* $K_C^\alpha : X^n \to \mathbb{R}$ *for some* $X \subseteq \mathbb{R}$*. Provided* X *is convex and includes the origin in its interior, the same argument can be applied. In particular, there is a unique extension of* K_C^α *to* $\mathbb{R}^n$ *that preserves linear homogeneity.*

Further perspective on central symmetry may be gained as follows. Let

$$\bar{p}(x) = \arg\max_{p \in C} p \cdot x$$

$$\underline{p}(x) = \arg\min_{p \in C} p \cdot x$$

and define

$$\mathcal{P}_x = \frac{1}{2}\bar{p}(x) + \frac{1}{2}\underline{p}(x).$$

Thus, $p \in \mathcal{P}_x$ iff $p = \frac{1}{2}p' + \frac{1}{2}p''$ for some $p' \in \bar{p}(x)$ and some $p'' \in \underline{p}(x)$. If

$$\bigcap_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \mathcal{P}_x \neq \emptyset, \tag{1}$$

then it is clear that $K_C^{1/2}$ is SEU-consistent and hence C is centrally symmetric (Theorem 2.1). Conversely, if C is centrally symmetric with centre q, then $q \in \mathcal{P}_x$ for any x.[2] In other words, (1) is equivalent to central symmetry of C.

This gives an alternative geometric interpretation for Theorem 2.1. Consider Figure 1, in which C is a CS polytope. Say that two distinct vertices of C are *opposing* if there is some x such that one of the two vertices lies in $\underline{p}(x)$ and the other in $\bar{p}(x)$. For polytopes, $\mathcal{P}_x$ will be a singleton for all but a finite set of x vectors satisfying $\|x\| = 1$. Moreover, when $\mathcal{P}_x$ is not a singleton, there will be some x' (near x) such that $\|x'\| = 1$, $\mathcal{P}_{x'}$ is a singleton and $\mathcal{P}_{x'} \subseteq \mathcal{P}_x$. In other words, the central symmetry condition (1) is equivalent (for polytopes) to the requirement that the chords joining opposing vertices possess a common mid-point, as in Figure 1.

[2] If $p \in \overline{p}(x)$, then we must have $q - (p - q) \in \underline{p}(x)$. If there is some $\hat{p} \in C$ with

$$\hat{p} \cdot x \ < \ [q - (p - q)] \cdot x$$

then $[q - (\hat{p} - q)] \cdot x > p \cdot x$ which contradicts $p \in \overline{p}(x)$. Hence

$$q = \frac{1}{2}p \ + \ \frac{1}{2}[q - (p - q)] \in \mathcal{P}_x.$$

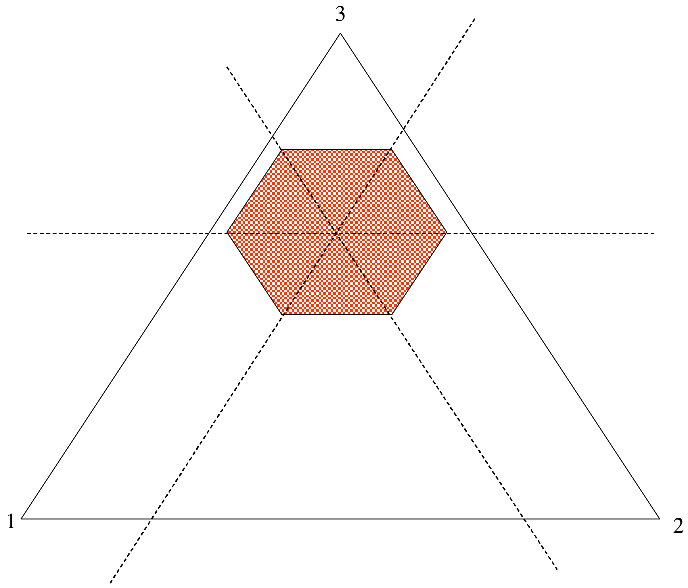

Figure 1: An example of a centrally symmetric set.

3 Discussion

Excluding trivial cases in which C is a singleton, we have shown that an α-MEU preference order is SEU-consistent iff $\alpha = 1/2$ and C is centrally symmetric. This result clarifies an issue about which there has been some confusion in the literature. Ghirardato, Klibanoff and Marinacci (1998; henceforth GKM) establish that an α-MEU preference order is SEU-consistent if $\alpha = 1/2$ and there exist $p, q \in \mathcal{P}$ such that

$$C = \{\lambda p + (1 - \lambda) q \mid \lambda \in [0, 1]\}. \tag{2}$$

It is obvious that such a set is centrally symmetric, with centre $(1/2)(p + q)$. GKM also hypothesise in passing (see their Remark 3) that $K_C^{1/2}$ is SEU-consistent *only* when C takes the form (2), though they subsequently realised that this hypothesis is false.[3] Theorem 2.1 provides a corrected necessary and sufficient condition.

Strictly speaking, this correction is already available in the published literature, albeit implicitly. Our Theorem 2.1 may be deduced as a corollary of Lemmas 1 and 3 in Siniscalchi (2009).[4] However, it seems useful to put a simple and direct proof into the public domain.[5]

[3]Paolo Ghirardato, private communication.

[4]We are indebted to Paolo Ghirardato for drawing our attention to this fact. Siniscalchi's results also demonstrate that Theorem 2.1 extends to infinite state spaces, with appropriate technical qualifications.

[5]We thank Sujoy Mukerji for encouraging us to do so. It was Sujoy who first alerted us to the GKM

Our result has recently been applied by Jewitt and Mukerji (2011) to study *relative act ambiguity* – a partial order which specifies when one act is more ambiguous than another.[6] This notion is defined in the context of a given class of preferences: different preference classes give rise to different partial orders over acts. Jewitt and Mukerji begin with a standard definition of *relative ambiguity aversion* – a partial order on preferences – from which they define the relative act ambiguity (for the given preference class) as follows: act f is *more ambiguous than* act g if every ambiguity neutral member of the given class is indifferent between f and g, but g is weakly preferred by every preference order (in the class) that is more ambiguity averse than some ambiguity neutral member of the class. This notion of relative act ambiguity is clearly vacuous if the given class of preferences does not include an ambiguity neutral member.

One of the preference classes that Jewitt and Mukerji consider is the class of α-MEU preferences for fixed C (i.e., the preferences obtained by considering all possible utility functions over outcomes and all $\alpha \in [0, 1]$). Our result establishes necessary and sufficient conditions for the notion of relative act ambiguity to be non-vacuous for such a class. In particular, an α-MEU preference is ambiguity neutral iff it is SEU-consistent, so relative act ambiguity is non-vacuous iff C is centrally symmetric.[7]

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hypothesis. We had developed the results in this paper somewhat earlier, for use in another project (Rogers and Ryan 2008), without appreciating their application to the GKM debate.

[6] In fact, they define two different notions of relative act ambiguity, but only one is relevant to the present discussion.

[7] Note that singletons are centrally symmetric.