Economics Bulletin

Volume 32, Issue 4

A note on the axiomatization of Wang premium principle by means of continuity considerations

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Abstract

The so called "Wang premium" is the well known principle of premium calculation expressed by means of the Choquet integral with respect to a (concave) distorted probability. In this paper we present a simple axiomatization of a sublinear Wang premium which is based on considerations related to the uniform continuity of a comonotone subadditive and monotone premium functional on the space of all bounded risks on a probability space.

Citation: Gianni Bosi and Magalì Zuanon, (2012) "A note on the axiomatization of Wang premium principle by means of continuity considerations", *Economics Bulletin*, Vol. 32 No. 4 pp. 3158-3165.
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Submitted: September 01, 2012. **Published:** November 19, 2012.

1 Introduction

In this paper a *premium functional* $\mathbb{P}$ is intended as a functional from the space $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ of all the nonnegative *risks* (i.e., nonnegative bounded random variables) on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ into $\mathbb{R}^+$. For a risk $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$, $X(\omega)$ is obviously interpreted as the loss that the insurance company incurs if the outcome is $\omega \in \Omega$.

The representability of an (extended) real-valued function on a space of risks by means of a *Choquet integral* was studied by several authors (see e.g. Chateauneuf (1996), Young (1998), Wang et al. (1997), Wirch and Hardy (2001), Wu and Wang (2003), Dhaene et al. (2006a), Wu and Zhou (2006) and Bosi and Zuanon (2010)). More recently, very deep results in this direction have been presented by Song and Yan (2006, 2009).

Denote by $S_X(t) = 1 - F_X(t) = \mathcal{P}(\{\omega \in \Omega : X(\omega) > t\})$ the *decumulative distribution function* of $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$. There exists a representation of this kind if there is a *probability distortion* g such that, for every $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$,

$$\mathbb{P}(X) = \int X dg \circ \mathcal{P} = \int_0^{+\infty} g(S_X(t))dt,$$

(i.e., $\mathbb{P}(X)$ is the *Choquet integral* of X with respect to the distorted probability $\mu = g \circ \mathbb{P}$). In this case, $\mathbb{P}$ is said to be a *Wang's premium principle* in the actuarial literature (see e.g. Wu and Wang (2003)).

We characterize the representability of a sublinear premium functional $\mathbb{P}$ on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ as the Choquet integral with respect to a concave probability distortion. To this aim, we use results presented by Parker (1996), who provided a characterization of comonotone additivity of a monotone and positive homogeneous functional on $L^\infty_+(\Omega, \mathcal{F})$ by means of a property that is referred to as *wide translation invariance* in the present paper. Such a condition, which generalizes the concept of a "comonotone additive" functional, is used in connection with very impressive results which have been recently presented by Song and Yan (2006, 2009). In particular we show that a monotone sublinear premium functional is a Wang's premium if and only if it satisfies the assumption of no unjustified loading and it is widely translation invariant and monotone with respect to stop loss order. The above condition of wide translation invariance can be replaced by uniform continuity with respect to the norm topology together with additivity with respect to the indicator functions of the members of any linearly ordered finite chain in $\mathcal{F}$.

We recall that continuity properties of risk measures have been recently studied by Frittelli and Rosazza Gianin (2011), while in a slightly different context the existence of a sublinear and continuous utility function for a not necessarily total preorder on a topological vector space was characterized in

Bosi and Zuanon (2003) and Bosi et al. (2007).

2 Notation and preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a *probability space*, and denote by $\mathbb{1}_F$ the *indicator function* of any subset F of Ω. In the sequel, we shall denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. For the sake of brevity, for every $c \in \mathbb{R}^+$ we identify c with the constant function $c\mathbb{1}_\Omega$.

Denote by $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ the space of all the nonnegative bounded *random variables* on $(\Omega, \mathcal{F}, \mathcal{P})$.

If for two $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ we have that $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, then we shall simply write $X \leq Y$. We have that $\leq$ is a preorder (i.e., $\leq$ is a *reflexive* and *transitive* binary relation).

Denote by $S_X(t) = 1 - F_X(t) = \mathbb{P}(\{\omega \in \Omega : X(\omega) > t\})$ the *decumulative distribution function* of any random variable $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$.
A *premium functional* $\mathbb{P}$ is intended as a functional from the space $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ into $\mathbb{R}^+$. A premium functional is said to be

1. *Monotone* if $\mathbb{P}(X) \leq P(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $X \leq Y$;

2. *Monotone with respect to first order stochastic dominance* if $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $S_X(t) \leq S_Y(t)$ for all $t \in \mathbb{R}^+$;

3. *Monotone with respect to stop-loss order* if $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \in \mathbb{R}^+$;

4. *Translation Invariant* if $\mathbb{P}(X + c) = \mathbb{P}(X) + c$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ and $c \in R^+$;

5. *Widely Translation Invariant* if $\mathbb{P}(X + c\mathbb{1}_F) = \mathbb{P}(X) + c\mathbb{P}(\mathbb{1}_F)$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$, $c \in \mathbb{R}^+$ and $F \in \mathcal{F}$ such that $\{X > 0\} \subseteq F$;

6. *Comonotone Additive* if $\mathbb{P}(X + Y) = \mathbb{P}(X) + \mathbb{P}(Y)$ for all comonotone $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ (i.e., for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ for all $\omega_1, \omega_2 \in \Omega$);

7. *Comonotone Subadditive* if $\mathbb{P}(X+Y) \leq \mathbb{P}(X)+\mathbb{P}(Y)$ for all comonotone $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$

8. *Sublinear* if $\mathbb{P}$ is *positively homogeneous* (i.e., $\mathbb{P}(\gamma X) = \gamma\mathbb{P}(X)$ for every $\gamma \in \mathbb{R}^+$ and $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$) and *subadditive* (i.e., $\mathbb{P}(X + Y) \leq \mathbb{P}(X) + \mathbb{P}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$);

It is clear that if a premium functional $\mathbb{P}$ on $L^{\infty}_{+}(\Omega, \mathcal{F}, \mathcal{P})$ is widely translation invariant then it is translation invariant.

A premium functional $\mathbb{P}$ is said to satisfy the axiom of *no unjustified loading* if $\mathbb{P}(\mathbb{1}_{\Omega}) = 1$.

In the next section necessary and sufficient conditions are presented on a premium functional $\mathbb{P}$ on $L^{\infty}_{+}(\Omega, \mathcal{F}, \mathcal{P})$ under which there exists a *concave probability distortion* g (i.e., g is a real-valued, nondecreasing, nonnegative and concave function on $[0, 1]$ such that $g(0) = 0$ and $g(1) = 1$) such that, for every $X \in L^{\infty}_{+}(\Omega, \mathcal{F}, \mathcal{P})$,

$$\mathbb{P}(X) = \int X dg \circ \mathbb{P} = \int_{0}^{+\infty} g(S_X(t))dt.$$

In this case, $\mathbb{P}$ is the *Choquet integral* with respect to the *distorted probability* $\mu = g \circ \mathbb{P}$. If this happens, $\mathbb{P}$ is said to be a *sublinear Wang's premium principle*. It should be note that a sublinear Wang's premium principle is a *distortion risk measure*, in the sense that it is a *coherent risk measure* in the sense of Artzner et al. (1999), which in addition is expressed by means of a Choquet integral (see e.g. Balbàs et al. (2009)).

3 Choquet integral representation of premium functionals

In the following theorem we characterize the representability of a premium functional $\mathbb{P}$ on $L^{\infty}_{+}(\Omega, \mathcal{F}, \mathcal{P})$ as a Wang's premium principle with respect to a concave probability distortion. It is well known that if $\mathbb{P}$ is a Wang's premium functional on $L^{\infty}_{+}(\Omega, \mathcal{F}, \mathcal{P})$ with respect to a concave probability distortion g, then $\mathbb{P}$ is subadditive and in particular sublinear (see Denneberg (1994)).

We use the following nice result proved by Wirch and Hardy (2001).

Lemma 3.1 (Wirch and Hardy (2001), Theorem 2.2) *Let $\mathbb{P}$ be a Wang's premium functional on $L^{\infty}_{+}(\Omega, \mathcal{F}, \mathcal{P})$ with respect to a probability distortion g. If $\mathbb{P}$ is subadditive, then g is concave.*

The following lemma is also needed, which slightly improves Lemma 6 in Parker (1996). We recall that the *norm* on the space $L^{\infty}_{+}(\Omega, \mathcal{F})$ of all bounded measurable functions on the measurable space $(\Omega, \mathcal{F})$ is defined as follows:

$$|| X ||= \sup_{\omega \in \Omega} | X(\omega) | .$$

Obviously, the *norm topology* on $L^{\infty}_{+}(\Omega, \mathcal{F})$ is the topology corresponding to the norm above.

Lemma 3.2 *If $\mathbb{P}$ is a monotone, positively homogeneous and comonotone subadditive premium functional on $L^\infty_+(\Omega, \mathcal{F})$, then $\mathbb{P}$ is uniformly continuous with respect to the norm topology on $L^\infty_+(\Omega)$.*

Proof. Consider any two real-valued functions $X, Y \in L^\infty_+(\Omega, \mathcal{F})$ and let $\mathbb{P}$ be a premium functional with the indicated properties. Then we have that

$$\begin{aligned} \mathbb{P}(X) - \mathbb{P}(Y) &\leq \mathbb{P}(\| X - Y \| \mathbb{1}_\Omega + Y) - \mathbb{P}(Y) \\ &\leq \| X - Y \| \mathbb{P}(\mathbb{1}_\Omega) + \mathbb{P}(Y) - \mathbb{P}(Y) \\ &= \| X - Y \| \mathbb{P}(\mathbb{1}_\Omega). \end{aligned}$$

Analogously, it can be shown that $\mathbb{P}(Y) - \mathbb{P}(X) \leq \| X - Y \| \mathbb{P}(\mathbb{1}_\Omega)$. Hence, we have that $| \mathbb{P}(X) - \mathbb{P}(Y) | \leq \| X - Y \| \mathbb{P}(\mathbb{1}_\Omega)$. This consideration completes the proof. □

Theorem 3.3 *Let $\mathbb{P}$ be a positively homogeneous premium functional on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ and assume that $\mathbb{P}$ satisfies the axiom of no unjustified loading. Then the following conditions are equivalent:*

(i) *There exists a concave probability distortion g such that $\mathbb{P}(X) = \int X dg \circ \mathbb{P}$ for all $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$;*

(ii) $\mathbb{P}$ *is uniformly continuous in the norm topology on $L^\infty_+(\Omega, \mathcal{F})$, monotone with respect to stop loss order and for every finite chain $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ in $\mathcal{F}$ and for every $\gamma_1, ..., \gamma_n \in \mathbb{R}^+$,*

$$\mathbb{P}(\sum_{i=1}^n \gamma_i \mathbb{1}_{F_i}) = \sum_{i=1}^n \gamma_i \mathbb{P}(\mathbb{1}_{F_i}).$$

(iii) $\mathbb{P}$ *is uniformly continuous in the norm topology on $L^\infty_+(\Omega, \mathcal{F})$, monotone with respect to stop loss order and for every finite chain $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ in $\mathcal{F}$,*

$$\mathbb{P}(\sum_{i=1}^n \mathbb{1}_{F_i}) = \sum_{i=1}^n \mathbb{P}(\mathbb{1}_{F_i});$$

(iv) $\mathbb{P}$ *is monotone with respect to stop loss order and widely translation invariant.*

Proof. (i) $\Rightarrow$ (ii). Assume that condition (i) holds. From (Dhaene et al., 2006b, Theorem 8), we have that $\mathbb{P}$ is monotone with respect to stop-loss order. Since $\mathbb{P}$ is sublinear, it is in particular comonotone subadditive. Therefore from Lemma 3.2 we have that $\mathbb{P}$ is uniformly continuous in the norm topology on $L^\infty_+(\Omega, \mathcal{F})$. Since $\mathbb{P}$ is comonotone additive, it easily follows by induction that $\mathbb{P}(\sum_{i=1}^n \gamma_i \mathbb{1}_{F_i}) = \sum_{i=1}^n \gamma_i \mathbb{P}(\mathbb{1}_{F_i})$ for every finite chain

$F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ in $\mathcal{F}$ and for every $\gamma_1, ..., \gamma_n \in \mathbb{R}^+$. Indeed, $\mathbb{1}_{F_1}$ and $\mathbb{1}_{F_2}$ are comonotone whenever $F_1 \subseteq F_2$.
(ii) $\Rightarrow$ (iii). Obvious.
(iii) $\Rightarrow$ (iv). If condition (iii) holds, we have that $\mathbb{P}(X+c\mathbb{1}_F) = \mathbb{P}(X)+c\mathbb{P}(\mathbb{1}_F)$ for every simple function $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$, $c \in \mathbb{R}^+$ and $F \in \mathcal{F}$ such that $\{X > 0\} \subseteq F$ (see the proof of Lemma 7 in Parker (1996)). Then the condition of wide translation invariance follows in general by using the norm continuity of $\mathbb{P}$ together with the possibility of approximating any $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ by means a sequence $\{X_n\}_{n\in\mathbb{N}}$ of simple functions in $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ in such a way that $\{X_n\}_{n\in\mathbb{N}}$ is norm convergent to X and $\{\omega \in \Omega : X_n(\omega) > 0\} = F$ for any given $F \in \mathbb{F}$ (see Lemma 5 in Parker (1996)).
(iv) $\Rightarrow$ (i). Now assume that condition (iv) is verified. Since $\mathbb{P}$ is positively homogeneous, widely translation invariant and monotone, it is comonotone additive (see (Parker, 1996, Theorem 7)). Further, since $\mathbb{P}$ is comonotone additive and monotone with respect to stop-loss order, we have that $\mathbb{P}$ is subadditive from Wang and Dhaene [Corollary 8]. From (Parker, 1996, Theorem 9), since $\mathbb{P}(\mathbb{1}_\Omega) = 1$ we have that $\mathbb{P}$ is the Choquet integral with respect to the normalized set function v on $\mathcal{F}$ defined by $v(F) = \mathbb{P}(\mathbb{1}_F)$ for all $F \in \mathcal{F}$. Since in addition $\mathbb{P}$ is in particular monotone with respect to first order stochastic dominance, from (Song and Yan, 2009, Proposition 2.1 and Remark 2.2) there exists a distortion function g such that $\mathbb{P}$ is the Choquet integral of X with respect to the distorted probability $\mu = g \circ \mathcal{P}$. Since $\mathbb{P}$ is subadditive, we have that g is concave from Lemma 3.1. □

In the following corollary we consider the particular case when the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is atomless.

Corollary 3.4 *Let $\mathbb{P}$ be a positively homogeneous premium functional on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ and assume that $\mathbb{P}$ satisfies the axiom of no unjustified loading. If the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is atomless, then the following conditions are equivalent:*

(i) $\mathbb{P}$ *satisfies any of the equivalent conditions* (i) *through* (iv) *of Theorem 3.3;*

(ii) $\mathbb{P}$ *is monotone with respect to first order stochastic dominance, subadditive, translation invariant and for every finite chain $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ in $\mathcal{F}$,*

$$\mathbb{P}(\sum_{i=1}^{n} \mathbb{1}_{F_i}) = \sum_{i=1}^{n} \mathbb{P}(\mathbb{1}_{F_i}).$$

.

Proof. (i) $\Rightarrow$ (ii). If any of the equivalent conditions (i) through (iv) of Theorem 3.3 is verified, then $\mathbb{P}$ is sublinear, comonotone additive and monotone with respect to first order stochastic dominance. Therefore condition (ii) immediately follows.
(ii) $\Rightarrow$ (i). Since the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is atomless, and the premium functional $\mathbb{P}$ is sublinear, translation invariant and monotone with respect to first order stochastic dominance, we have that $\mathbb{P}$ is monotone with respect to stop loss order by (Song and Yan, 2009, Theorem 3.2). Further, $\mathbb{P}$ is uniformly continuous in the norm topology on $L^{\infty}_{+}(\Omega, \mathcal{F})$ by Lemma 3.2. Hence the thesis follows from Theorem 3.3, condition (iii). $\square$

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