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No-envy and dominant strategy implementability in non-excludable public good economies with quasi-linear preferences

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Abstract

This paper studies the relationship between no-envy (Foley, D. (1967) "Resource allocation and the public sector," Yale Economics Essays 7, pp.45-98) and dominant strategy implementability in non-excludable public good economies with quasi-linear preferences. The main result shows that the combination of non-bossiness (Satterthwaite, M. A. and H. Sonnenschein (1981) "Strategy-proof allocation mechanisms at differentiable points," Review of Economic Studies 48, pp.587-597) and equal treatment of equals is equivalent to no-envy under strategy-proof social choice functions in the economies which are incompatible with strict monotonic closedness (Fleurbaey, M. and F. Maniquet (1997) "Implementability and horizontal equity imply no-envy," Econometrica 65, pp.1215-1219).

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1 Introduction

This paper studies the relationship between no-envy (Foley, 1967) and dominant strategy implementability in non-excludable public good economies with quasi-linear preferences. **No-envy** requires that each agent does not strictly prefer other agent's consumption bundle. One of necessary conditions for dominant strategy implementability is **strategy-proofness** which requires that truthful revelation is a weakly dominant strategy for each agent. [1] In non-excludable public good economies with classical preferences, Moulin (1994), Serizawa (1999), and others studied strategy-proof social choice functions in terms of equity. This paper considers quasi-linear preferences and studies the social choice functions in non-excludable public good economies. Proposition 1 in this paper shows that the combination of strategy-proofness, non-bossiness, and equal treatment of equals implies no-envy. **Non-bossiness** (Satterthwaite and Sonnenschein, 1981) requires that each agent cannot change the outcome by the revelation while maintaining the agent's consumption bundle. This property is a necessary condition for group strategy-proofness in non-excludable public good economies, as shown by Serizawa (1994). [2] **Equal treatment of equals** requires that any two agents with the same preference are treated equally in terms of their utility levels. In general, this property is weaker than no-envy.

The relationship in Proposition 1 was also presented by Moulin (1993) and Fleurbaey and Maniquet (1997) in different environments from those of this paper. In the problems of allocating private goods, Moulin (1993) showed the relationship under group strategy-proof social choice functions if the domain satisfies monotonic closedness (Dasgupta, Hammond, and Maskin, 1979). In the problems including those of Moulin (1993), Fleurbaey and Maniquet (1997) showed the relationship under strategy-proof and non-bossy social choice functions if the domain satisfies strict monotonic closedness (Fleurbaey and Maniquet, 1997). [3] Although this paper considers a model similar to those of Moulin (1993), the domain is a set of quasi-linear utility functions and does not satisfy monotonic closedness, as shown by Dasgupta, Hammond, and Maskin (1979). In addition, the domain does not satisfy strict monotonic closedness, as pointed by Fleurbaey and Maniquet (1997). These imply that the relationship in Proposition 1 does not follow the results of Moulin (1993) and Fleurbaey and Maniquet (1997) straightforwardly.

In addition, Proposition 2 in this paper shows that no-envy implies non-bossiness in non-excludable public good economies with quasi-linear preferences. Together with the relationship in Proposition 1, this implies the main result of this paper as follows: the combination of non-bossiness and equal treatment of equals is equivalent to no-envy under strategy-proof

[1] See Mizukami and Wakayama (2007) and Saijo, Sjöström, and Yamato (2007) for dominant strategy implementable social choice functions.

[2] See Mizukami and Wakayama (2009) for a relationship between non-bossiness and Nash implementability (Maskin, 1977).

[3] See Fleurbaey and Maniquet (1997) for the relationship between monotonic closedness and strict monotonic closedness.

social choice functions in the environments.

This paper is organized as follows. Section 2 introduces notation and definitions. Section 3 shows the results. Section 4 concludes this paper.

2 Notation and Definitions

Let $I \equiv \{1,\dots,n\}$ be the set of **agents** and $Y \subseteq \mathbb{R}$ be the set of **production levels of the public good**. Let $y \in Y$ be **consumption of the public good for each agent**. For each $i \in I$, let $x_i \in \mathbb{R}$ be a **cost share of the public good for agent *i*** and $(y,x_i) \in Y \times \mathbb{R}$ be a **consumption bundle for agent *i***. Let $x \equiv (x_i)_{i\in I} \in \mathbb{R}^n$ be a profile of cost shares of the public good and $(y,x) \in Y \times \mathbb{R}^n$ be an **allocation**.

For each $i \in I$, let $u_i\colon Y \times \mathbb{R} \to \mathbb{R}$ be an **utility function for agent *i***, that is quasi-linear: there exists $v_i\colon Y \to \mathbb{R}$, called a **valuation function of the public good for agent *i***, such that for each $(y,x_i) \in Y \times \mathbb{R}$,

$$u_i(y,x_i) = v_i(y) - x_i.$$

For each $i \in I$, let V_i be the set of all valuation functions of the public good for agent i, that are continuous, strictly concave, and strictly increasing. For each $i \in I$, each $(y,x_i) \in Y \times \mathbb{R}$, and each $v_i \in V_i$, let $UC(y,x_i;v_i) \equiv \{(y_i',x_i') \in Y \times \mathbb{R} \mid v_i(y) - x_i \le v_i(y_i') - x_i'\}$ be the **upper contour set of v_i at (y,x_i)**, $M_i(y,x_i;v_i) \equiv \{v_i' \in V_i \mid UC(y,x_i;v_i') \subseteq UC(y,x_i;v_i)\}$ be the **set of monotonic transformations of v_i at (y,x_i)**, and

$$SM_i(y,x_i;v_i) \equiv \{v_i' \in M_i(y,x_i;v_i) \mid v_i(y) - x_i < v_i(y_i') - x_i' \text{ for each } (y_i',x_i') \in UC(y,x_i;v_i') \setminus \{(y,x_i)\}\}$$

be the **set of strict monotonic transformations of v_i at (y,x_i)**. Let $v \equiv (v_i)_{i\in I}$ be a profile of valuation functions of the public good and $V \equiv \prod_{i\in I} V_i$ be the set of profiles of valuation functions of the public good. For each $i \in I$, let $v_{-i} \equiv (v_k)_{k\in I\setminus\{i\}}$ be a profile of valuation functions of the public good other than agent i and $V_{-i} \equiv \prod_{k\in I\setminus\{i\}} V_k$ be the set of profiles of valuation functions of the public good other than agent i. For each $i,j \in I$, let $v_{-i,j} \equiv (v_k)_{k\in I\setminus\{i,j\}}$ be a profile of valuation functions of the public good other than agents i and j and $V_{-i,j} \equiv \prod_{k\in I\setminus\{i,j\}} V_k$ be the set of profiles of valuation functions of the public good other than agents i and j.

Let $f\colon V \to Y \times \mathbb{R}^n$ be a **social choice function**. [4] For each $v \in V$, let $(y(v),x(v)) \in f(V)$ be the allocation associated with the social choice function f at the profile of valuation functions of the public good v and $(y(v),x_i(v))$ be the consumption bundle for agent $i \in I$ at the allocation $(y(v),x(v))$.

Definition 1. The social choice function f satisfies **strategy-proofness** if and only if for each $v,v' \in V$ and each $i \in I$, $v_i(y(v_i,v_{-i})) - x_i(v_i,v_{-i}) \ge v_i(y(v_i',v_{-i})) - x_i(v_i',v_{-i})$.

[4] Note that the results of this paper do not depend on the assumptions of the cost function and the budget constraint.

Definition 2. The social choice function f satisfies **non-bossiness** if and only if for each $v, v' \in V$ and each $i \in I$, if $(y(v_i, v_{-i}), x_i(v_i, v_{-i})) = (y(v'_i, v_{-i}), x_i(v'_i, v_{-i}))$, then $(y(v_i, v_{-i}), x(v_i, v_{-i})) = (y(v'_i, v_{-i}), x(v'_i, v_{-i}))$.

Definition 3. The social choice function f satisfies **no-envy** if and only if for each $v \in V$ and each $i, j \in I$, $v_i(y(v)) - x_i(v) \geq v_i(y(v)) - x_j(v)$.

Fact. *Suppose that the social choice function f satisfies **no-envy**. For each $v \in V$ and each $i, j \in I$, $x_i(v) = x_j(v)$.*

Definition 4. The social choice function f satisfies **equal treatment of equals** if and only if for each $v \in V$ and each $i, j \in I$, if $v_i = v_j$, then $v_i(y(v)) - x_i(v) = v_j(y(v)) - x_j(v)$.

3 Results

Lemma 1 shows that each agent cannot change the agent's consumption of the public good by revealing any strict monotonic transformation of the valuation function at the consumption bundle if the social choice function satisfies strategy-proofness.

Lemma 1. *Suppose that the social choice function f satisfies **strategy-proofness**. For each $v, v' \in V$ and each $i \in I$, if $v'_i \in SM_i(y(v_i, v_{-i}), x_i(v_i, v_{-i}); v_i)$, then $y(v_i, v_{-i}) = y(v'_i, v_{-i})$.*

Proof. To the contrary, we suppose that there exist $v, v' \in V$ and $i \in I$ such that

$$v'_i \in SM_i(y(v_i, v_{-i}), x_i(v_i, v_{-i}); v_i); \tag{1}$$

$$y(v_i, v_{-i}) \neq y(v'_i, v_{-i}). \tag{2}$$

By (1), we know that

$$\begin{aligned} &v_i(y(v_i, v_{-i})) - x_i(v_i, v_{-i}) < v_i(y') - x'_i \\ &\qquad \text{for each } (y', x'_i) \in UC(y(v_i, v_{-i}), x_i(v_i, v_{-i}); v'_i) \setminus \{(y(v_i, v_{-i}), x_i(v_i, v_{-i}))\}. \end{aligned} \tag{3}$$

By **strategy-proofness**, we know that $(y(v'_i, v_{-i}), x_i(v'_i, v_{-i})) \in UC(y(v_i, v_{-i}), x_i(v_i, v_{-i}); v'_i)$. Together with (2), this implies that

$$(y(v'_i, v_{-i}), x_i(v'_i, v_{-i})) \in UC(y(v_i, v_{-i}), x_i(v_i, v_{-i}); v'_i) \setminus \{(y(v_i, v_{-i}), x_i(v_i, v_{-i}))\} \tag{4}$$

By (3) and (4), we find that $v_i(y(v_i, v_{-i})) - x_i(v_i, v_{-i}) < v_i(y(v'_i, v_{-i})) - x_i(v'_i, v_{-i})$. This contradicts **strategy-proofness**. □

Lemma 2 shows that each agent cannot change the agent's cost share of the public good while maintaining the agent's consumption of the public good if the social choice function satisfies strategy-proofness.

Lemma 2. *Suppose that the social choice function f satisfies* ***strategy-proofness****. For each $v, v' \in V$ and each $i \in I$, if $y(v_i, v_{-i}) = y(v'_i, v_{-i})$, then $x_i(v_i, v_{-i}) = x_i(v'_i, v_{-i})$.*

Proof. To the contrary, we suppose that there exist $v, v' \in V$ and $i \in I$ such that $y(v_i, v_{-i}) = y(v'_i, v_{-i})$ and $x_i(v_i, v_{-i}) \neq x_i(v'_i, v_{-i})$. If $x_i(v_i, v_{-i}) > x_i(v'_i, v_{-i})$, then we find that $v_i(y(v_i, v_{-i})) - x_i(v_i, v_{-i}) < v_i(y(v'_i, v_{-i})) - x_i(v'_i, v_{-i})$. This contradicts **strategy-proofness**. If $x_i(v_i, v_{-i}) < x_i(v'_i, v_{-i})$, then we find that $v'_i(y(v_i, v_{-i})) - x_i(v_i, v_{-i}) > v'_i(y(v'_i, v_{-i})) - x_i(v'_i, v_{-i})$. This contradicts **strategy-proofness**. □

By Lemma 2 and non-bossiness, we have the following corollary.

Corollary. *Suppose that the social choice function f satisfies* ***strategy-proofness*** *and* ***non-bossiness****. For each $v, v' \in V$ and each $i \in I$, if $y(v_i, v_{-i}) = y(v'_i, v_{-i})$, then $(y(v_i, v_{-i}), x(v_i, v_{-i})) = (y(v'_i, v_{-i}), x(v'_i, v_{-i}))$.*

By Lemma 1 and the above corollary, we have the following relationship in non-excludable public good economies with quasi-linear preferences, similar to Moulin (1993) and Fleurbaey and Maniquet (1997).

Proposition 1. *If the social choice function f satisfies* ***strategy-proofness****,* ***non-bossiness****, and* ***equal treatment of equals****, then it satisfies* ***no-envy****.*

Proof. Let $v \in V$ and $i, j \in I$. It is sufficient to show that $x_i(v) \leq x_j(v)$. Let

$$v_0 \in SM_i(y(v), x_i(v); v_i) \cap SM_i(y(v), x_j(v); v_j). \tag{5}$$

Let $v'_i \in V_i$ be such that $v'_i = v_0$. Together with (5) and Lemma 1, this implies that

$$y(v_i, v_{-i}) = y(v'_i, v_{-i}). \tag{6}$$

Together with Corollary, this implies that

$$x(v_i, v_{-i}) = x(v'_i, v_{-i}). \tag{7}$$

By (5), (6), and (7), we find that

$$v_0 \in SM_i(y(v'_i, v_{-i}), x_i(v'_i, v_{-i}); v_i) \cap SM_i(y(v'_i, v_{-i}), x_j(v'_i, v_{-i}); v_j). \tag{8}$$

Let $v'_j \in V_j$ be such that $v'_j = v_0$. Together with (8) and Lemma 1, this implies that

$$y(v'_i, v_j, v_{-i,j}) = y(v'_i, v'_j, v_{-i,j}). \tag{9}$$

Together with Corollary, this implies that

$$x(v'_i, v_j, v_{-i,j}) = x(v'_i, v'_j, v_{-i,j}). \tag{10}$$

By **equal treatment of equals**, we know that $v'_i(y(v'_i, v'_j, v_{-i,j})) - x_i(v'_i, v'_j, v_{-i,j}) = v'_j(y(v'_i, v'_j, v_{-i,j})) - x_j(v'_i, v'_j, v_{-i,j})$ because $v'_i = v'_j$. This implies that

$$x_i(v'_i, v'_j, v_{-i,j}) = x_j(v'_i, v'_j, v_{-i,j}). \tag{11}$$

By (7), (10), and (11), we find that $x_i(v) = x_j(v)$. □

Remark. In the proof of Proposition 1, the existence of v_0 depends on the non-excludability of the public good. If the public good is excludable, then the existence is not guaranteed when preferences are quasi-linear.

The following relationship shows that non-bossiness is a necessary condition for no-envy.

Proposition 2. *If the social choice function f satisfies **no-envy**, then it satisfies **non-bossiness**.*

Proof. Let $v, v' \in V$ and $i \in I$ be such that $(y(v_i, v_{-i}), x_i(v_i, v_{-i})) = (y(v'_i, v_{-i}), x_i(v'_i, v_{-i}))$. It is sufficient to show that $x_j(v_i, v_{-i}) = x_j(v'_i, v_{-i})$ for each $j \in I \setminus \{i\}$. Let $j \in I \setminus \{i\}$. By Fact, we know that $x_i(v_i, v_{-i}) = x_j(v_i, v_{-i})$ and $x_i(v'_i, v_{-i}) = x_j(v'_i, v_{-i})$. These imply that $x_j(v_i, v_{-i}) = x_j(v'_i, v_{-i})$ because $x_i(v_i, v_{-i}) = x_i(v'_i, v_{-i})$. □

In general, no-envy implies equal treatment of equals. Together with this relationship and Propositions 1 and 2, we have the following theorem.

Theorem. *The social choice function satisfies **non-bossiness** and **equal treatment of equals** if and only if it satisfies **no-envy** when it satisfies **strategy-proofness**.*

4 Conclusion

The main result of this paper implies that no-envy is justified from group non-manipulability in non-excludable public good economies with quasi-linear preferences, similar to Moulin (1993) and Fleurbaey and Maniquet (1997). In addition, it implies that non-bossiness is justified from equity in the economies.

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