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Acceptability prioritized preferences and equilibrium existence

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Abstract

Economic agents may sometimes find themselves in circumstances in which they are effectively endowed with preferences that place a priority on first meeting specific "acceptability standards" before attention can be placed on optimizing their "aspirational objective." We introduce the notion of acceptability prioritized preferences to model such situations and show that not only are these preferences generally discontinuous, but they can also fail to be characterizable by utility functions of any kind. Despite this fact, we establish a pure strategy Nash equilibrium existence result for interactive environments whose populations include agents endowed with acceptability prioritized preferences.

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1. Introduction

The modeling and analysis of strategic environments generally assumes that the preferences held by decision makers are characterizable by continuous utility functions. Notable exceptions include papers such as Dasgupta and Maskin (1986), Simon and Zame (1990), and Reny (1999), which carry out equilibrium analysis in the context of agent preferences that are characterized by discontinuous utility functions. Some settings of practical interest, however, involve economic agents whose actions are directed by preferences that are not only discontinuous, but also fail to be characterizable by any utility function whatsoever. Examples include government bureaucracies that allocate resources according to rigid priorities; health care providers whose decisions are guided by priority criteria such as age, degree of genetic match, and time spent in queue; nonprofit organizations who face a primary obligation to satisfy regulatory standards and maintain solvency but are driven by a mission to provide charitable service once the primary obligation is assured; profit maximizing firms who face sanctions and penalties for violation of regulatory standards; or individuals who strive to minimize the stigma attached to violation of social norms but seek to optimize their "worldly desires" if these norms are satisfied.

We generalize multi-objective preferences such as those described with the notion of *acceptability prioritized preferences* - where the realization of certain "acceptability standards" are of primary importance, but once these standards are met, decisions are guided by a desire to maximize some given aspirational objective. We show that when acceptability and aspirational objectives are sufficiently independent, such preferences are necessarily discontinuous. Furthermore, we confirm that some such preferences cannot be represented by any real-valued utility function, continuous or not. Examples of such include, but are not limited to, the class of lexicographic preferences. Traditional equilibrium existence results of course do not apply in contexts where utility functions fail to be defined. Indeed, the classical formulation of strategic form games itself requires the specification of each player's payoff function. Despite this fact, our main result establishes a pure strategy Nash equilibrium existence theorem for strategic environments populated by agents who hold acceptability prioritized preferences – even those that fail to be utility function characterizable.

2. Acceptability Prioritized Preferences

A feature of the effective preferences confronted by some economic decision makers is that they are endowed with a primary desire to satisfy a set of acceptability standards and a secondary desire to maximize some "aspirational" objective. These acceptability standards could involve achieving profit sufficient to retain credit access and sustain operations, meeting regulatory standards or contractual obligations, satisfying religious or moral codes of conduct to avoid social stigmatization, or any of a myriad of other fixed or variable standards of performance/behavior. An animal rescue center, for instance, may have a primary obligation to fulfill various regulatory standards and to generate revenues sufficient to sustain operations. However, provided that these goals are met, their mission may be to rescue and rehabilitate as many domestic animals as possible. This section's goal will be to formalize a general framework for characterizing such preferences and to examine the extent to which this framework is or is not consistent with a traditional utility function approach.

We begin by introducing basic notation and structural assumptions maintained throughout our analysis. Let N denote the finite set of players corresponding to a given interactive environment. For each $i \in N$, let S_i denote the strategy space of player i, which is assumed to be a compact and convex subset of a finite dimensional Euclidean space. Let $S = \times_{i \in N} S_i$ denote the space of all strategy profiles and for each $A \subseteq S$, let the complement of A be denoted by A^c and let $cl(A)$ denote the closure of the set A. For each $i \in N$, $s' \in S$, and $s_i \in S_i$, $s' \backslash s_i$ will denote the strategy profile in which each player $j \neq i$ pursues strategy s_j' and player i pursues strategy s_i.

DEFINITION 2.1 An *acceptability standard* is a real valued function $f_0 : S \to \mathbb{R}$ and we shall say that $s \in S$ meets the acceptability standard f_0 if $f_0(s) \geq 0$. An *aspirational utility* is a real valued function $f_1 : S \to \mathbb{R}$.

The identification of an acceptability standard with a real valued function is considerably more general than it may superficially appear. Indeed, note that acceptability standards are functions of the full strategy profile and thus may depend on a player's own actions as well as those of others, thereby allowing acceptability standards to represent either absolute or relative standards. That the right hand side of the inequality $f_0(s) \geq 0$ is fixed at zero does not restrict the range of standards as an equality in the form of $f_0(s) \geq g(s)$ can simply be rewritten in the form $h(s) = f_0(s) - g(s) \geq 0$. Note also that the single inequality $f_0(s) \geq 0$ can actually embody the satisfaction of multiple performance standards. For instance, the satisfaction of $f_k(s) \geq 0$ for each $k = 1, \ldots, K$ can be more concisely represented by defining $f_0(s) = min\{f_1(s), \ldots, f_K(s)\}$ and specifying the entire collection of standards as $f_0(s) \geq 0$.

Acceptability standards emerge in a wide array of economic applications as well as in more purely game theoretic examples. For instance, one can envision economic agents who effectively operate under mandates to maintain nonnegative profits, limit effluent discharge, limit CO_2 emissions, satisfy in-state enrollment targets, service emergency medical cases, achieve an ε-best reply with respect to a given objective function, or satisfy any of a variety of other performance standards. We next formalize the family of acceptability prioritized preferences that are supported by a given acceptability standard f_0 and aspirational utility f_1.

DEFINITION 2.2 The preference relation $\succsim$ defined over S is *supported* by the acceptability standard $f_0 : S \to \mathbb{R}$ and the aspirational utility $f_1 : S \to \mathbb{R}$ if for all $s, s' \in S$ it follows that:

a) $\{f_0(s') < 0$ and $f_0(s) > f_0(s')\}$ implies $s \succ s'$,

b) $\{f_0(s') \geq 0$ and $f_0(s) \geq 0\}$ implies $\{f_1(s) \geq f_1(s')$ iff $s \succsim s'\}$.

The preference relation $\succsim$ is said to be an *acceptability prioritized preference relation* if there exist f_0 and f_1 such that $\succsim$ is supported by the acceptability standard f_0 and aspirational utility f_1.

Condition a) captures the behavioral feature that the desire to satisfy acceptability is of primary importance as any outcome failing this standard is strictly less preferred to any outcome that is strictly closer to meeting the standard. Condition b) dictates that once acceptability is reached, maximization of aspirational utility becomes the agent's driving force.

In should be noted that Definition 2.2 also encompasses preferences that are utility function characterizable and continuous. Indeed, if $\succsim$ is characterized by the continuous utility function u, then $\succsim$ is supported by the acceptability standard $f_0 = u$ and the aspirational utility $f_1 = u$. It should be further noted that this definition leaves considerable flexibility in regards to how preferences are specified in the "unacceptable" region. In particular, if s' and s are strategy profiles for which $f_0(s')=f_0(s)<0$, then Definition 2.2 leaves unspecified whether s' is strictly preferred to s, s is strictly preferred to s', or if indifference prevails. This flexibility allows for lexicographic preferences as well as a wide range of nonlexicographic possibilities in which "ties" may be left unbroken, or may be broken by functional relationships that vary by the f_0-level surface on which they lie. For more on the use of lexicographic orderings in economic applications, see Fishburn (1974) and Blume et. al. (1991).

DEFINITION 2.3 Given the acceptability standard f_0, $A(f_0)=\{s|f_0(s)\geq 0\}$ is the *acceptable set* of strategy profiles and $AF(f_0)=A(f_0)\cap cl(A(f_0)^c)$ is the *acceptability frontier* of f_0. The acceptability standard f_0 and the aspirational utility f_1 are *frontier independent* if f_1 is nonconstant on $AF(f_0)$.

The indifference sets of f_0 may have nonempty interiors, so identifying strategy profiles that are on the cusp of acceptability is not as simple as identifying those that meet, but do not exceed, the standard of acceptability. Instead, the acceptability frontier of f_0 is defined to be the set of acceptable points, if any, that are on the boundary of acceptability. Frontier independence describes the situation in which f_1 maps elements of this frontier to two or more values, thus implying f_0 and f_1 are not everywhere in agreement on what must be a nonempty frontier.

THEOREM 2.1 Let $\succsim$ be any acceptability prioritized preference relation supported by an acceptability standard f_0 and aspirational utility f_1 that are frontier independent. It follows that $\succsim$ cannot be represented by a continuous utility function.

Proof: Let $\succsim$, f_0, and f_1 satisfy the stated hypotheses. Frontier independence implies that there exists $s^0,s'\in AF(f_0)$ such that $f_1(s^0)<f_1(s')$. The fact that $s'\in AF(f_0)$ implies that there exists a sequence $\{s^n\}_{n=1,\ldots,\infty}\subseteq\{s\in S \mid f_0(s)<0\}$ such that $s^n\to s'$. Thus if u is a utility function representing $\succsim$, then $u(s^n)<u(s^0)<u(s')$ for all $n=1,\ldots,\infty$, yet $s^n\to s'$. We conclude that u must be discontinuous. *QED*

Frontier independence is a somewhat stronger condition than is necessary to establish discontinuity of acceptability prioritized preferences. However, our goal is not to establish precisely when such preferences are and are not "ill-behaved." Instead, the central goal of this section is to formally confirm that ill-behaved acceptability prioritized preferences of economic relevance do indeed exist. Theorem 2.1 establishes that if $\succsim$ is supported by an acceptability standard and an aspirational utility that are frontier independent, then any utility function that might be capable of characterizing $\succsim$ must also be discontinuous.

It has been well-known since at least Debreu (1954) that lexicographic preferences are not utility function characterizable. Theorem 2.2 reveals that a locality of lexicographic preferences can be supported without imposing universally orthogonal priorities. Note also that some ties can be left to stand or tie-breaking functions can be f_0-level dependent, leading to decidedly nonlexicographic preferences that also resist utility function representation. Failure of utility function characterization is thus far more general than even Theorem 2.2 may appear to suggest.

THEOREM 2.2 If f_0 and f_1 that are continuous and frontier independent then there exist preferences supported by the acceptability standard f_0 and aspirational utility f_1 such that these preferences are not utility function characterizable.

Proof: Let f_0 and f_1 satisfy the stated hypotheses and define $\succsim$ by the following conditions.

a) if $f_0(s') < 0 \leq f_0(s)$ then $s \succ s'$,

b) if $f_0(s), f_0(s') \geq 0$ then $s \succsim s'$ if and only if $f_1(s) \geq f_1(s')$,

c) if $f_0(s') < f_0(s) < 0$ then $s \succ s'$, and

d) if $f_0(s) = f_0(s') < 0$ then $s \succsim s'$ if and only if $f_1(s) \geq f_1(s')$.

Suppose that there exists a utility function u representing $\succsim$. (To anticipate where this proof is going, note the lexicographic nature of preferences from the acceptability frontier into the unacceptable region.) Frontier independence implies that there exists $s^0, s' \in AF(f_0)$ such that $f_1(s^0) < f_1(s')$. Continuity of f_1 implies there exist U^0 and U' neighborhoods of s^0 and s' such that $f_1(x) < f_1(y)$ for all $x \in U^0$ and $y \in U'$. Continuity of f_0 and the fact that $s^0, s' \in AF(f_0)$ implies there exists $a<0$ such that for every $z \in (a,0)$ there exists $s1(z) \in U^0$ and $s2(z) \in U'$ such that $f_0(s1(z)) = f_0(s2(z)) = z$, where by definition of U^0 and U' it also follows that $f_1(s1(z)) < f_1(s2(z))$. As u is assumed to represent $\succsim$, it must be the case that $u(s1(z)) < u(s2(z))$ and one can pick a rational number $r(z) \in (u(s1(z)), u(s2(z)))$. Further note that for every $z_1, z_2 \in (a,b)$ such that $z_1 < z_2$, the construction of $r(\bullet)$ implies that $u(s1(z_1)) < r(z_1) < u(s2(z_1)) < u(s1(z_2)) < r(z_2) < u(s2(z_2))$ and thus $r(z)$ is a strictly increasing function of z on $(a,0)$. It follows that $r(\bullet)$ assigns a unique rational number to each real number in the interval $(a,0)$, which is impossible and we conclude $\succsim$ is not utility function characterizable. *QED*

The proof of Theorem 2.2 demonstrates that if the acceptability standard f_0 and aspirational utility f_1 are continuous and frontier independent, then there exists a preference relation supported by f_0 and f_1 that is essentially lexicographic in the values of f_0 and f_1 throughout a neighborhood of S. With this established, our proof makes a straightforward to appeal to the classical result on lexicographic preferences to establish the failure of utility characterizability.

Theorems 2.1 and 2.2 can thus be viewed as serving to establish just how badly behaved acceptability prioritized preferences can be in regards to the properties of continuity and utility characterizability. If one considers frontier independence as a condition that can generally be expected to hold, then Theorem 2.1 establishes that these acceptability prioritized preferences are generally discontinuous. Theorem 2.2 goes even further and establishes that these preferences may well even fail utility characterizability in applied settings. Despite the innate potential for discontinuity and utility characterizability failure that is thus harbored by acceptability prioritized preferences, the following section demonstrates that Nash equilibrium existence can nonetheless be assured in their presence.

3. Equilibrium existence with acceptability prioritized preferences.

The previous section of this paper has demonstrated that acceptability prioritized preferences are generally discontinuous and can even fail to be characterizable by any utility function. As a consequence, traditional Nash equilibrium existence results do not apply in the context of such preferences. Despite these apparently ill-behaved preferences, we will show that pure strategy Nash equilibrium existence can be assured throughout a broad class of economically relevant acceptability prioritized preferences. To formalize this insight, we begin by presenting standard notation, definitions, and results regarding correspondences, i.e., set-valued functions.

The *power set* of a set X is denoted by $P(X)$ and is defined to be the set of all subsets of X, including the empty set and X itself. A *correspondence* from a set X to a set Y is a set-valued function $\varphi:X\to P(Y)$.

DEFINITION 3.1 If X and Y are topological spaces and $\varphi:X\to P(Y)$ then φ is *upper hemicontinuous* at $x\in X$ if for every open set U such that $\varphi(x)\subseteq U$, there exists an open set V containing x such that $\varphi(x')\subseteq U$ for all $x'\in V$. φ is *lower hemicontinuous* at $x\in X$ if for every open set U such that $\varphi(x)\cap U\neq\varnothing$, there exists an open set V containing x such that $\varphi(x')\cap U\neq\varnothing$ for all $x'\in V$. φ is *upper hemicontinuous* (*lower hemicontinuous*) if φ is *upper hemicontinuous* (*lower hemicontinuous*) at every $x\in X$. If $\varphi:X\to P(X)$ and $x\in\varphi(x)$, then φ is said to have a *fixed point* at x.

The following two theorems are versions of the classic Berge's Maximum Theorem (Berge 1959) and Kakutani's Fixed Point Theorem (Kakutani 1941). Proofs of these versions can be found in Aliprantis and Border (2006, pp. 570-71) and Ichiishi (1982) respectively.

THEOREM 3.1 (Berge's Maximum Theorem) Let $\varphi:X\to P(Y)$ be a nonempty valued, compact valued upper and lower hemicontinuous correspondence between topological spaces and suppose that $f:X\times Y\to\mathbb{R}$ is continuous. It follows that the correspondence $\mu:X\to P(Y)$ defined by $\mu(x)=\arg\max_{y\in\varphi(x)} f(x,y)$ is nonempty valued, compact valued, and upper hemicontinuous.

THEOREM 3.2 (Kakutani's Fixed Point Theorem) Let X be a nonempty, convex, compact subset of a finite dimensional Euclidean space and let $\varphi:X\to P(X)$ be an upper hemicontinuous, nonempty valued, closed valued, and convex valued correspondence. Then φ has a fixed point.

Having introduced basic notation and classic results regarding correspondences, let us now return our attention specifically to acceptability prioritized preferences. Recall that under acceptability prioritized preferences a player's first priority is to select an action that meets the acceptability standard or, if the standard is unreachable, comes as close as possible to doing so. This leads to the notion of an "acceptable-as-can-be" reply correspondence that characterizes all actions that a player can take that are consistent with this priority. This correspondence is formalized below, where we again let $S=\times_{i\in N}S_i$ denote the space of strategy profiles.

DEFINITION 3.2 Given a continuous acceptability standard f_0 and $i\in N$ we define the *acceptable-as-can-be reply correspondence* for player i to be the correspondence $\alpha_i(\bullet|f_0):S\to P(S_i)$ defined by $\alpha_i(s|f_0)=\arg\max_{y_i\in S_i}(\min\{0,f_0(s\backslash y_i)\})$.

LEMMA 3.1 If S is a convex, compact subset of finite dimensional Euclidean space, $i\in N$ and $f_0:S\to\mathbb{R}$ is continuous and strictly quasi-concave in s_i, then $\alpha_i(\bullet|f_0)$ is nonempty valued, compact valued, convex valued, and both upper and lower hemicontinuous.

Proof: Let f_0 have the properties stated in the hypothesis and let $X=S$, $Y=S_i$, $\varphi(x)=Y$ for each $x\in X$, and $f(x,y)=\min\{0,f_0(x\backslash y)\}$ for each $x\in X$ and $y\in Y$. The hypotheses of Theorem 3.1 are satisfied for f and φ, thus $\mu:X\to P(Y)$ defined by $\mu(x)=\arg\max_{y\in\varphi(x)} f(x,y) = \alpha_i(x|f_0)$ is nonempty valued, compact valued, and upper hemicontinuous. That $\alpha_i(\bullet|f_0)$ is also convex valued follows immediately from the quasi-concavity of f_0.

For each $s\in S$, let $\beta_i(s|f_0)=\arg\max_{y_i\in S_i} f_0(s\backslash y_i)$. Define $g(x,y)= f_0(x\backslash y)$ for each $x\in X$ and $y\in Y$ and apply the argument presented in the paragraph above, with g playing the role of f. It follows that $\beta_i(\bullet|f_0)$ is nonempty valued and upper hemicontinuous. Strict quasi-concavity of f_0 also implies that $\beta_i(\bullet|f_0)$ is single valued and thus the upper hemicontinuity of this correspondence implies that it is also lower hemicontinuous. We establish lower hemicontinuity of $\alpha_i(\bullet|f_0)$ by considering each of two possible cases for an arbitrarily selected $s\in S$.

Case 1) $\alpha_i(s|f_0)= \beta_i(s|f_0)$. Since $\beta_i(\bullet|f_0)$ is everywhere lower hemicontinuous and by definition $\beta_i(s'|f_0)\subseteq \alpha_i(s'|f_0)$ for all $s'\in S$, it immediately follows that $\alpha_i(\bullet|f_0)$ is lower hemicontinuous at s.

Case 2) $\alpha_i(s|f_0)\neq \beta_i(s|f_0)$. Let y_i^* denote the sole element of $\beta_i(s|f_0)$. As $\beta_i(\bullet|f_0)$ is lower hemicontinuous, it follows that if U is an open neighborhood of y_i^* then there exists V an open neighborhood of s such that $\beta_i(s'|f_0)\cap U\neq\varnothing$ for all $s'\in V$. Since $\beta_i(s'|f_0)\subseteq \alpha_i(s'|f_0)$ by definition, it follows that $\alpha_i(s'|f_0)\cap U \neq\varnothing$ for all $s'\in V$. Now consider $y_i'\in \alpha_i(s|f_0)$ such that $y_i'\neq y_i^*$ and pick U an open neighborhood of y_i'. Note that $\alpha_i(s|f_0)= \beta_i(s|f_0)$ whenever $f_0(s\backslash y_i^*)\leq 0$, thus $\alpha_i(s|f_0)\neq \beta_i(s|f_0)$ and $y_i'\neq y_i^*$ implies that $0\leq f_0(s\backslash y_i')<f_0(s\backslash y_i^*)= \max_{y_i\in S_i} f_0(s\backslash y_i)$. Strict quasi-concavity implies that for all $\lambda\in(0,1)$ sufficiently small that $x_i=\lambda y_i^*+(1-\lambda)y_i'\in U$, it must also be true that $f_0(s\backslash x_i)>0$. Continuity of f_0 in turn implies that there exists V an open neighborhood of s such that $f_0(s'\backslash x_i)>0$ for all $s'\in V$, implying $x_i\in \alpha(s'|f_0)$ for all $s'\in V$ and thus $\alpha(s'|f_0)\cap U\neq\varnothing$ for all $s'\in V$. It follows that $\alpha_i(\bullet|f_0)$ is lower hemicontinuous at s.

Since we have considered each $s\in S$, we conclude that $\alpha_i(\bullet|f_0)$ is everywhere lower hemicontinuous and our proof is complete. *QED*

Armed with Lemma 3.1, we are now prepared to establish a pure strategy Nash equilibrium existence result in the context of acceptability prioritized preference relations. It is important to note that this conclusion holds even when preferences fail utility function characterizability. (Strict quasi-concavity of f_{0i} ensures well-behaved optimization in cases where acceptability is unattainable. It does not guarantee utility function characterizability.)

THEOREM 3.3 If S is a convex, compact subset of finite dimensional Euclidean space then there exists a Nash equilibrium in pure strategies for any game $(S,(\succsim_i)_{i\in N})$ such that for each $i\in N$, $\succsim_i$ is an acceptability prioritized preference relation defined on S that is supported by an acceptability standard f_{0i} and aspirational utility function f_{1i} such that f_{0i} is continuous and strictly quasi-concave in s_i, and f_{1i} is continuous and quasi-concave in s_i.

Proof: For each $i\in N$ let $\alpha_i(\bullet|f_{0i})$ be the acceptable-as-can-be reply correspondence of player i. By Lemma 3.1, for each $i\in N$, the correspondence $\alpha_i(\bullet|f_{0i})$ is nonempty valued, compact valued, convex valued, and both upper and lower hemicontinuous. Taking as given $i\in N$, let $X=S$, $Y=S_i$, $\varphi=\alpha_i(\bullet|f_{0i})$, $f(s',s_i)= f_{1i}(s'\backslash s_i))$ for each $s'\in X$ and $s_i\in Y$, and $\beta_i(x)=\arg\max_{y\in\varphi(x)} f(x,y)$ for each $x\in X$. Berge's Maximum Theorem implies that β_i is nonempty valued, compact valued, and upper hemicontinuous. That it is also convex valued follows from the quasi-concavity of f_{1i} and convex valuedness of $\alpha_i(\bullet|f_{0i})$. Repeating this argument for each $i\in N$, it follows that β_i is nonempty valued, compact valued, convex valued, and upper hemicontinuous for each $i\in N$. Defining $\beta:S\rightarrow P(S)$ by $\beta(s)=(\beta_i(s))_{i\in N}$ for each $s\in S$, it follows that β is a nonempty valued, compact valued, convex valued, and upper hemicontinuous correspondence that maps S into itself. Kakutani's Fixed Point Theorem implies that β has a fixed point, which in turn represents a Nash equilibrium of the game $(S,(\succsim_i)_{i\in N})$. *QED*

4. Conclusion

Despite the fact that virtually all of game theoretic analysis is carried out in the context of games in which players are explicitly endowed with utility/payoff functions, some games of economic interest are populated by players who have multi-objective preferences that cannot be characterized by any utility function, even a discontinuous one. We have introduced the concept of acceptability prioritized preferences to model situations in which standards of acceptable performance must be met before the agent is effectively able to commit to a focus on its aspirational utility/mission. This paper provides important insights toward both the modeling of such multi-objective preferences and the analysis of environments populated by players who are motived by such preferences. In particular, we demonstrate that despite the fact that such preferences often cannot even be characterized by a utility function, existence of a pure strategy Nash equilibrium can nonetheless be assured.

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