Economics Bulletin

Volume 33, Issue 2

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Citation: Maddalena Cavicchioli, (2013) "On asymptotic properties of the QLM estimators for GARCH models", *Economics Bulletin*, Vol. 33 No. 2 pp. 959-966.
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Submitted: October 30, 2012.   **Published:** April 05, 2013.

Economics Bulletin, 2013, Vol. 33 No. 2 pp. 959-966

Economics Bulletin

Submission Number:EB-12-00766

ON ASYMPTOTIC PROPERTIES OF THE QML ESTIMATORS FOR GARCH MODELS

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This note can be considered as a continuation of a nice paper from Francq and Zakoian (2012) concerning with strict stationarity testing and estimation of GARCH models. We compute the asymptotic variances of the quasi-maximum likelihood estimators for stationary GARCH models.

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1. Main results

We use notations and results from Francq and Zakoïan (2012). Let us consider the GARCH$(1,1)$ model

$$\begin{aligned} \epsilon_t &= \sigma_t(\theta) u_t \\ \sigma_t^2(\theta) &= \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2(\theta) \end{aligned} \tag{1}$$

with $\theta = (\omega\ \alpha\ \beta)'$ in a compact parameter set $\Theta \subset (0, +\infty)^3$. The process (u_t) is a sequence of i.i.d. variables such that $E(u_t) = 0$, $E(u_t^2) = 1$ and $E(u_t^4) = k_u \in (1, +\infty)$. The true parameter, denoted by $\theta_0 = (\omega_0\ \alpha_0\ \beta_0)'$, belongs to the interior of Θ. The process (1) is *covariance stationary* if and only if $\alpha_0 + \beta_0 < 1$. This is a sufficient but non necessary condition for strict stationarity. Nelson (1990) proved that $\sigma_t^2 < \infty$ almost surely and $\{\epsilon_t, \sigma_t^2\}$ *strictly stationary* if and only if $\gamma_0 = E[\ln(\alpha_0 u_t^2 + \beta_0)] < 0$. For simplicity, we treat covariance stationary GARCH even if many results can be suitable extended to the general case. The quasi–maximum likelihood estimator (in short, QMLE) $\widehat{\theta}_T = (\widehat{\omega}_T\ \widehat{\alpha}_T\ \widehat{\beta}_T)'$ is any measurable solution of

$$\widehat{\theta}_T = \text{argmin}_{\theta \in \Theta}\ \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta) \tag{2}$$

where

$$\ell_t(\theta) = \frac{\epsilon_t^2}{\sigma_t^2(\theta)} + \ln \sigma_t^2(\theta).$$

For any asymptotically stationary process $(X_t)_{t \geq 0}$, let

$$E_\infty(X_t) = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} X_t$$

provided this limit exists. For instance, for the process (ϵ_t), we have

$$E_\infty(\epsilon_t^2) = E_\infty(\sigma_t^2) = \omega_0 (1 - \alpha_0 - \beta_0)^{-1} \tag{3}$$

when $\alpha_0 + \beta_0 < 1$. In the stationary case ($\gamma_0 < 0$), it is well–known the consistency and the asymptotic normality of the QMLE $\widehat{\theta}_T$ as follows.

Theorem 1. *Suppose* $\gamma_0 < 0$. *For* $\Theta \subset (0, +\infty)^3$ *such that for every* $\theta \in \Theta$, $\beta < 1$, *then*

$$\lim_{T \to +\infty} \widehat{\theta}_T = \theta_0 \qquad (a.s.)$$

and

$$\sqrt{T}(\widehat{\theta}_T - \theta_0) \underset{T \to +\infty}{\longrightarrow} \mathcal{N}\left(0, (k_u - 1)\mathcal{I}^{-1}\right)$$

where

$$\mathcal{I} = E_\infty \left(\frac{\partial \ln \sigma_t^2}{\partial \theta}|_{\theta=\theta_0} \frac{\partial \ln \sigma_t^2}{\partial \theta'}|_{\theta=\theta_0} \right)$$

is a positive definite constant symmetric matrix.

We prove the following results:

Theorem 2. *Under the assumptions of Theorem 1, the asymptotic score matrix is given by*

$$E_\infty \left(\frac{\partial^2 \ell_t(\theta)}{\partial \theta \partial \theta'}|_{\theta=\theta_0} \right) = \mathcal{I}.$$

Theorem 3. *If $\beta_0 = 0$ and $\alpha_0 < 1$, then $\mathcal{I}$ has the form*

$$\mathcal{I} = \begin{pmatrix} \frac{4k_u\alpha_0^3 - 3\alpha_0 + 1}{\omega_0^2(1-\alpha_0)(1-k_u\alpha_0^2)} & \frac{1-2k_u\alpha_0^2 - k_u\alpha_0}{\omega_0(1-\alpha_0)(1-k_u\alpha_0^2)} \\ \frac{1-2k_u\alpha_0^2 - k_u\alpha_0}{\omega_0(1-\alpha_0)(1-k_u\alpha_0^2)} & \frac{k_u(1+\alpha_0)}{(1-\alpha_0)(1-k_u\alpha_0^2)} \end{pmatrix}$$

This provides an explicit form of the asymptotic variance matrix of the QML estimators $(\widehat{\omega}_T, \widehat{\alpha}_T)$ for an ARCH(1):

$$\mathrm{var}_a(\widehat{\theta}_T) = \begin{pmatrix} k_u\omega_0^2(1-\alpha_0^2) & \omega_0(1-\alpha_0)(2k_u\alpha_0^2 + k_u\alpha_0 - 1) \\ \omega_0(1-\alpha_0)(2k_u\alpha_0^2 + k_u\alpha_0 - 1) & (1-\alpha_0)(4k_u\alpha_0^3 - 3\alpha_0 + 1) \end{pmatrix}.$$

In Section 4 we compute the asymptotic score matrix $\mathcal{I}$ for the general GARCH(1,1). This result together with Theorem 1 provides an explicit form for the asymptotic variance matrix of the QMLE of GARCH(1,1).

2. Consistency and Asymptotic Normality

To make the reading self–contained, we give a proof of Theorem 1. The FOC is given by

$$\frac{1}{T}\sum_{t=1}^T \left\{ \frac{1}{\sigma_t^2(\widehat{\theta}_T)} \frac{\partial \sigma_t^2}{\partial \theta}|_{\theta=\widehat{\theta}_T} - \frac{\epsilon_t^2}{[\sigma_t^2(\widehat{\theta}_T)]^2} \frac{\partial \sigma_t^2}{\partial \theta}|_{\theta=\widehat{\theta}_T} \right\} = 0$$

hence

$$\frac{1}{T}\sum_{t=1}^T \left[1 - \frac{\epsilon_t^2}{\sigma_t^2(\widehat{\theta}_T)} \right] \frac{\partial \ln \sigma_t^2}{\partial \theta}|_{\theta=\widehat{\theta}_T} = 0.$$

Taking the 1st Taylor expansions around θ_0 of $[\sigma_t^2(\widehat{\theta}_T)]^{-1}$ and $\frac{\partial \ln \sigma_t^2}{\partial \theta}|_{\theta=\widehat{\theta}_T}$ and using $\epsilon_t^2 = \sigma_t^2(\theta_0)u_t^2$, we get

$$\mathcal{I}_T(\widehat{\theta}_T - \theta_0) + O_p(1) = -\frac{1}{T}\sum_{t=1}^T (1-u_t^2) \frac{\partial \ln \sigma_t^2}{\partial \theta}|_{\theta=\theta_0} \quad (4)$$

where

$$\mathcal{I}_T = \frac{1}{T}\sum_{t=1}^{T}\left\{(1-u_t^2)\frac{\partial^2 \ln\sigma_t^2}{\partial\theta\partial\theta'}|_{\theta=\theta_0} + u_t^2\frac{\partial \ln\sigma_t^2}{\partial\theta}|_{\theta=\theta_0}\frac{\partial \ln\sigma_t^2}{\partial\theta'}|_{\theta=\theta_0}\right\}.$$

Then

$$\mathcal{I} = \lim_{T\to+\infty}\mathcal{I}_T = E_\infty\left(\frac{\partial \ln\sigma_t^2}{\partial\theta}|_{\theta=\theta_0}\frac{\partial \ln\sigma_t^2}{\partial\theta'}|_{\theta=\theta_0}\right)$$

as $E_\infty(u_t^2) = 1$ and u_t^2 is independent of $\sigma_t^2(\theta_0)$ (and its ln derivatives). Since $\mathcal{I} < \infty$ by assumption, taking the limit for $T \to +\infty$ in (4) gives the consistency of $\widehat{\theta}_T$. For the asymptotic variance of $\widehat{\theta}_T$, we have

$$\text{var}_\infty(\widehat{\theta}_T) = \mathcal{I}^{-1}E_\infty\left((1-u_t^2)^2\frac{\partial \ln\sigma_t^2}{\partial\theta}|_{\theta=\theta_0}\frac{\partial \ln\sigma_t^2}{\partial\theta'}|_{\theta=\theta_0}\right)\mathcal{I}^{-1} = (k_u-1)\mathcal{I}^{-1}$$

as $E_\infty((1-u_t^2)^2) = k_u - 1$. Theorem 2 says that $\widehat{\theta}_T$ is a minimizer of the objective function for T sufficiently large. In fact, we have

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2\ell_t(\theta)}{\partial\theta\partial\theta'}|_{\theta=\widehat{\theta}_T} = &-\frac{1}{T}\sum_{t=1}^{T}\frac{[\sigma_t^2(\widehat{\theta}_T)-2\epsilon_t^2]}{[\sigma_t^2(\widehat{\theta}_T)]^3}\frac{\partial\sigma_t^2}{\partial\theta}|_{\theta=\widehat{\theta}_T}\frac{\partial\sigma_t^2}{\partial\theta'}|_{\theta=\widehat{\theta}_T}\\ &+\frac{1}{T}\sum_{t=1}^{T}\frac{[\sigma_t^2(\widehat{\theta}_T)-\epsilon_t^2]}{[\sigma_t^2(\widehat{\theta}_T)]^2}\frac{\partial^2\sigma_t^2}{\partial\theta\partial\theta'}|_{\theta=\widehat{\theta}_T}.\end{aligned}$$

Taking the 1st Taylor expansion of $\sigma_t^2(\widehat{\theta}_T)$ around θ_0 and using $\epsilon_t^2 = \sigma_t^2(\theta_0)u_t^2$ gives

$$\begin{aligned}&\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2\ell_t(\theta)}{\partial\theta\partial\theta'}|_{\theta=\widehat{\theta}_T} =\\ &-\frac{1}{T}\sum_{t=1}^{T}\frac{[(1-2u_t^2)\sigma_t^2(\theta_0)+\frac{\partial\sigma_t^2}{\partial\theta'}|_{\theta=\theta_0}(\widehat{\theta}_T-\theta_0)]}{\sigma_t^2(\widehat{\theta}_T)}\frac{\partial\ln\sigma_t^2}{\partial\theta}|_{\theta=\widehat{\theta}_T}\frac{\partial\ln\sigma_t^2}{\partial\theta'}|_{\theta=\widehat{\theta}_T}\\ &+\frac{1}{T}\sum_{t=1}^{T}\frac{[(1-u_t^2)\sigma_t^2(\theta_0)+\frac{\partial\sigma_t^2}{\partial\theta'}|_{\theta=\theta_0}(\widehat{\theta}_T-\theta_0)]}{[\sigma_t^2(\widehat{\theta}_T)]^2}\frac{\partial^2\sigma_t^2}{\partial\theta\partial\theta'}|_{\theta=\widehat{\theta}_T}.\end{aligned}$$

Taking the limit for $T \to +\infty$ and using the consistency of $\widehat{\theta}_T$, we get

$$\begin{aligned}E_\infty\left(\frac{\partial^2\ell_t(\theta)}{\partial\theta\partial\theta'}|_{\theta=\theta_0}\right) &= -E_\infty\left[(1-2u_t^2)\frac{\partial\ln\sigma_t^2}{\partial\theta}|_{\theta=\theta_0}\frac{\partial\ln\sigma_t^2}{\partial\theta'}|_{\theta=\theta_0}\right]\\ &= -E_\infty(1-2u_t^2)\ \mathcal{I} = \mathcal{I}.\end{aligned}$$

3. Asymptotic Variance of the QMLE for ARCH(1)

Let us consider model (1) with $\beta = 0$. Since

$$\sigma_t^2(\theta) = \omega + \alpha\epsilon_{t-1}^2 = \omega(1 + \alpha\omega^{-1}\epsilon_{t-1}^2),$$

we have

$$\ln \sigma_t^2(\theta) = \ln \omega + \ln(1 + \alpha\omega^{-1}\epsilon_{t-1}^2) \sim \ln \omega + \alpha\omega^{-1}\epsilon_{t-1}^2$$

by using the 1st Taylor expansion of $\ln(1 + x)$ around zero. The first derivatives are given by

$$\frac{\partial \ln \sigma_t^2(\theta)}{\partial \omega} = \omega^{-1} - \alpha\omega^{-2}\epsilon_{t-1}^2 \qquad \frac{\partial \ln \sigma_t^2(\theta)}{\partial \alpha} = \omega^{-1}\epsilon_{t-1}^2.$$

So we get

$$A_t = \frac{\partial \ln \sigma_t^2(\theta)}{\partial \theta}|_{\theta=\theta_0} = \begin{pmatrix} \omega_0^{-1} - \alpha_0\omega_0^{-2}\epsilon_{t-1}^2 \\ \omega_0^{-1}\epsilon_{t-1}^2 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{I} &= E_\infty(A_t A_t') \\ &= \begin{pmatrix} \omega_0^{-2} + \alpha_0^2\omega_0^{-4}E_\infty(\epsilon_{t-1}^4) - 2\alpha_0\omega_0^{-3}E_\infty(\epsilon_{t-1}^2) & \omega_0^{-2}E_\infty(\epsilon_{t-1}^2) - \alpha_0\omega_0^{-3}E_\infty(\epsilon_{t-1}^4) \\ \omega_0^{-2}E_\infty(\epsilon_{t-1}^2) - \alpha_0\omega_0^{-3}E_\infty(\epsilon_{t-1}^4) & \omega_0^{-2}E_\infty(\epsilon_{t-1}^4) \end{pmatrix} \end{aligned}$$

Now Theorem 3 follows by using the formulae (see Rossi (2012), p.22):

$$E_\infty(\epsilon_t^2) = \omega_0(1 - \alpha_0)^{-1} \qquad E_\infty(\epsilon_t^4) = \frac{k_u\omega_0^2(1 + \alpha_0)}{(1 - \alpha_0)(1 - k_u\alpha_0^2)}.$$

Here we report the calculation of the last formula:

$$\begin{aligned} E_\infty(\epsilon_t^4) &= E_\infty(\sigma_t^4(\theta_0)u_t^4) = k_u E_\infty(\sigma_t^4(\theta_0)) \\ &= k_u(\omega_0^2 + \alpha_0^2 E_\infty(\epsilon_{t-1}^4) + 2\omega_0\alpha_0 E_\infty(\epsilon_{t-1}^2)) \\ &= k_u\alpha_0^2 E_\infty(\epsilon_t^4) + k_u\omega_0^2(1 + \alpha_0)(1 - \alpha_0)^{-1}. \end{aligned}$$

4. Asymptotic Score Matrix for GARCH(1,1)

For model (1) we have

$$\begin{aligned} \sigma_t^2(\theta) &= (1 - \beta L)^{-1}\omega + \alpha(1 - \beta L)^{-1}\epsilon_{t-1}^2 \\ &= \omega(1 - \beta)^{-1}\Big[1 + \alpha(1 - \beta)\omega^{-1}\sum_{i=0}^{\infty}\beta^i\epsilon_{t-i-1}^2\Big] \end{aligned}$$

hence

$$\ln \sigma_t^2(\theta) = \ln[\omega(1-\beta)^{-1}] + \ln[1 + \alpha(1-\beta)\omega^{-1}\sum_{i=0}^{\infty}\beta^i \epsilon_{t-i-1}^2]$$
$$\sim \ln[\omega(1-\beta)^{-1}] + \alpha(1-\beta)\omega^{-1}\sum_{i=0}^{\infty}\beta^i \epsilon_{t-i-1}^2$$

by using the 1st Taylor expansion of $\ln(1+x)$ around zero. The first derivatives of $\ln \sigma_t^2(\theta)$ are given by

$$\frac{\partial \ln \sigma_t^2(\theta)}{\partial \omega} = (1-\beta)\omega^{-1} - \alpha(1-\beta)\omega^{-2}\sum_{i=0}^{\infty}\beta^i \epsilon_{t-i-1}^2$$
$$\frac{\partial \ln \sigma_t^2(\theta)}{\partial \alpha} = (1-\beta)\omega^{-1}\sum_{i=0}^{\infty}\beta^i \epsilon_{t-i-1}^2$$
$$\frac{\partial \ln \sigma_t^2(\theta)}{\partial \beta} = (1-\beta)^{-1} - \alpha\omega^{-1}\sum_{i=0}^{\infty}\beta^i \epsilon_{t-i-1}^2 + \alpha(1-\beta)\omega^{-1}\sum_{i=1}^{\infty} i\beta^{i-1}\epsilon_{t-i-1}^2.$$

Then we have $\mathcal{I} = E_\infty(A_t A_t')$, where $A_t = \frac{\partial \ln \sigma_t^2(\theta)}{\partial \theta}|_{\theta=\theta_0}$. To compute $\mathcal{I}$ we need the moments of ϵ_t. Recall that $E_\infty(\epsilon_t^2)$ is given by (3). Now we determine $E_\infty(\epsilon_t^4)$ and $E_\infty(\epsilon_t^2\epsilon_{t+k}^2)$ for any $k \geq 1$:

$$E_\infty(\epsilon_t^4) = E_\infty(\sigma_t^4(\theta_0)u_t^4) = k_u E_\infty(\sigma_t^4(\theta_0))$$
$$E_\infty(\sigma_t^4(\theta_0)) = \omega_0^2 + \alpha_0^2 E_\infty(\epsilon_{t-1}^4) + \beta_0^2 E_\infty(\sigma_{t-1}^4(\theta_0))$$
$$+ 2\omega_0\alpha_0 E_\infty(\epsilon_{t-1}^2) + 2\omega_0\beta_0 E_\infty(\sigma_{t-1}^2(\theta_0)) + 2\alpha_0\beta_0 E_\infty(\epsilon_{t-1}^2\sigma_{t-1}^2(\theta_0))$$
$$= \frac{\omega_0^2(1+\alpha_0+\beta_0)}{1-\alpha_0-\beta_0} + \alpha_0^2 E_\infty(\epsilon_{t-1}^4) + (\beta_0^2 + 2\alpha_0\beta_0)E_\infty(\sigma_{t-1}^4(\theta_0))$$

hence

$$E_\infty(\epsilon_t^4) = \frac{k_u\omega_0^2(1+\alpha_0+\beta_0)}{1-\alpha_0-\beta_0} + k_u\alpha_0^2 E_\infty(\epsilon_{t-1}^4) + (\beta_0^2 + 2\alpha_0\beta_0)E_\infty(\epsilon_{t-1}^4)$$

that is

$$E_\infty(\epsilon_t^4) = \frac{k_u\omega_0^2(1+\alpha_0+\beta_0)}{(1-\alpha_0-\beta_0)(1-k_u\alpha_0^2-2\alpha_0\beta_0-\beta_0^2)}. \quad (5)$$

Set $\Phi_t = \{\epsilon_t, \epsilon_{t-1}, \dots\}$. By the Law of Iterated Expectations, for any $k \geq 1$, we get

$$\gamma(k) = E_\infty(\epsilon_t^2\epsilon_{t+k}^2) = E_\infty(E_\infty(\epsilon_t^2\epsilon_{t+k}^2|\Phi_{t+k-1})) = E_\infty(\epsilon_t^2 E_\infty(\epsilon_{t+k}^2|\Phi_{t+k-1}))$$
$$= E_\infty(\epsilon_t^2 E_\infty(\sigma_{t+k}^2(\theta_0)|\Phi_{t+k-1})) = E_\infty(\epsilon_t^2\sigma_{t+k}^2(\theta_0))$$
$$= \omega_0 E_\infty(\epsilon_t^2) + \alpha_0 E_\infty(\epsilon_t^2\epsilon_{t+k-1}^2) + \beta_0 E_\infty(\epsilon_t^2\sigma_{t+k-1}^2(\theta_0))$$
$$= \omega_0^2(1-\alpha_0-\beta_0)^{-1} + (\alpha_0+\beta_0)\gamma(k-1).$$

By iteration, we obtain

$$\gamma(k) = \omega_0^2(1-\alpha_0-\beta_0)^{-1}\sum_{i=0}^{\infty}(\alpha_0+\beta_0)^i = \omega_0^2(1-\alpha_0-\beta_0)^{-2}. \qquad (6)$$

To determine $\mathcal{I}$, we now use (3), (5) and (6) and these relations:

$$E_\infty(\sum_{i=0}^{\infty}\beta_0^i\epsilon_{t-i-1}^2) = \sum_{i=0}^{\infty}\beta_0^i E_\infty(\epsilon_{t-i-1}^2)$$

$$E_\infty(\sum_{i=0}^{\infty}\beta_0^i\epsilon_{t-i-1}^2)^2 = \sum_{i=0}^{\infty}\beta_0^{2i}E_\infty(\epsilon_{t-i-1}^4) + 2\sum_{i<j}\beta_0^{i+j}E_\infty(\epsilon_{t-i-1}^2\epsilon_{t-j-1}^2)$$

$$E_\infty(\sum_{i=1}^{\infty}i\beta_0^{i-1}\epsilon_{t-i-1}^2) = \sum_{i=1}^{\infty}i\beta_0^{i-1}E_\infty(\epsilon_{t-i-1}^2)$$

$$E_\infty(\sum_{i=1}^{\infty}i\beta_0^{i-1}\epsilon_{t-i-1}^2)^2 = \sum_{i=1}^{\infty}i^2\beta_0^{2(i-1)}E_\infty(\epsilon_{t-i-1}^4) + 2\sum_{i<j}ij\beta_0^{i+j-2}E_\infty(\epsilon_{t-i-1}^2\epsilon_{t-j-1}^2)$$

$$E_\infty(\sum_{i,j}j\beta_0^{i+j-1}\epsilon_{t-i-1}^2\epsilon_{t-j-1}^2) = \sum_{i=1}^{\infty}i\beta_0^{2i-1}E_\infty(\epsilon_{t-i-1}^4) + \sum_{i\neq j}j\beta_0^{i+j-1}E_\infty(\epsilon_{t-i-1}^2\epsilon_{t-j-1}^2).$$

Finally we need the sums of these series. For $0 < x < 1$, we have:

$$\sum_{i=0}^{\infty}x^i = \frac{1}{1-x} \qquad \sum_{i<j}^{\infty}x^{i+j} = \frac{x}{(1-x)^2(1+x)} \qquad \sum_{i=1}^{\infty}ix^{i-1} = \frac{1}{(1-x)^2}$$

$$\sum_{i=1}^{\infty}ix^{2i-1} = \frac{x}{(1-x^2)^2} \qquad \sum_{i\neq j}^{\infty}jx^{i+j-1} = \frac{2x^2+x+1}{(1-x)^3(1+x)^2}$$

$$\sum_{i=1}^{\infty}i^2x^{2(i-1)} = \frac{1+x^2}{(1-x^2)^3} \qquad \sum_{i<j}^{\infty}ijx^{i+j-2} = \frac{x^3+x^2+2x}{(1-x)^4(1+x)^3}.$$

Putting together the above formulae gives the matrix $\mathcal{I}$.

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