Economics Bulletin

Volume 33, Issue 2

Bargaining with uncertain value distributions

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Abstract

This paper studies a bargaining model in which the seller is uncertain about which distribution the buyer's values are drawn from. The distribution of the buyer's values is fixed across periods, while the buyer's values are drawn independently from the distribution each period. In the classical model of repeated bargaining where the buyer's value is drawn from a commonly known distribution and fixed across periods, the high-value buyer has a strong incentive to conceal his value, and the seller loses most of her bargaining power. An important question is whether adding a layer of uncertainty makes the high-value buyer more willing to accept high-price offers and improves the seller's revenue. We find this to be the case as long as the seller's ex ante beliefs are sufficiently optimistic.

I am indebted to Andreas Blume for his guidance and support. I am grateful for helpful comments from the editor, an anonymous referee, Oliver Board, John Duffy, James Feigenbaum, Robert Lemke, Oksana Loginova, Utku Unver, Lise Vesterlund, and Roberto Weber. Thanks are also due to the seminar participants in University of Pittsburgh, Concordia University, City University of Hong Kong, Fudan University and Game Theory Festival in Stony Brook 2006.
Citation: Huan Xie, (2013) "Bargaining with uncertain value distributions", *Economics Bulletin*, Vol. 33 No. 2 pp. 1047-1066.
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Submitted: November 11, 2012. **Published:** April 18, 2013.

1. Introduction

One important question in the literature of repeated bargaining is how players' private information is revealed over time, and related to that, how economic surplus is distributed between the bargaining parties. In order to examine these questions, we focus on a framework commonly used in the previous literature: a buyer (denoted as he) and a seller (denoted as she) bargain over multiple periods, with the buyer having private information; in each period, the seller proposes a take-it-or-leave-it offer and the buyer decides whether to accept or reject the offer. In the classical models a common assumption is that the buyer's private information is his value, which is fixed across periods, and the distribution of the buyer's value is common knowledge. Under this assumption, the seller has a large disadvantage and loses most of her monopoly power. This stark result is established in both cases when the seller sells or rents a durable good.

The literature on the Coase conjecture finds that if the durable-good monopolist sells over time and can quickly lower prices, the seller can hardly achieve profits greater than the lowest buyer valuation and the buyer obtains the entire surplus from trade in excess of his lowest valuation (Coase 1972, Fudenberg et al. 1985).[1] When the monopolist bargains over renting the durable good to a buyer with private value, Hart and Tirole (1988) show that the seller always offers a low price until the end of the game if the horizon is long enough.[2] Intuitively, when the time horizon is long, the high-value buyer type will not accept any price rejected by the low-value buyer type, in order to avoid being charged with a high price in all later periods. So the seller is not able to price discriminate and she charges a low price to both low-value and high-value types, until close to the end of the horizon. Therefore, if the durable-good monopolist rents the durable good, the seller is again caught in an unfavorable position.

In this paper, we examine a two-period rental model (equivalent to the case that a seller repeatedly charges to sell a perishable good or provide service to a buyer) where the buyer has private information not only about his valuation when each period comes about but also about the distribution from which his values are drawn. Our model is different from Hart and Tirole's rental model in two ways. First, we introduce an additional layer of uncertainty on the buyer's value distribution. The distribution may be either good or bad. Both distributions can draw high value or low value, with the good distribution generating a high value with a higher probability. The buyer privately observes the distribution at the beginning of the game. But the seller only knows the ex ante probability of the two distributions. Second, the buyer's value is drawn from one of the two distributions independently across periods at the beginning of each period. Since the seller does not know which distribution the buyer's values are drawn from, the buyer's value is correlated across time periods from the seller's perspective.

The purpose of the paper is to ask whether the seller can improve her standing when there exists this second layer of uncertainty about the distribution of the buyer's values. On one hand, our

[1] This result holds under the assumption that the seller's marginal cost is lower than the buyer's lowest value, which is called the "gap" case in the literature. Failures of the Coase conjecture are found when the lowest buyer valuation does not exceed the seller's cost, which is referred as the "no-gap" case in the literature (Gul et al. 1986, Ausubel and Deneckere 1989).

[2] Hart and Tirole (1988) examine the sale model and the rental model in three cases: (1) where the parties can commit themselves to a contract once and for all; (2) where the parties can only write short-term contracts which rule within a period, but cannot commit themselves between periods; (3) where parties can write a long-term contract which rules across periods, but cannot commit themselves not to renegotiate this contract by mutual agreement. The rental model without commitment as in this paper is part of the analysis in Hart and Tirole.

model maintains the buyer's strategic considerations across periods, which makes the problem still interesting and close to many real life examples where the bargaining parties are involved in a long-term relationship. On the other hand, we are able to examine whether allowing the buyer's value to be redrawn provides a leeway to solve the problem of the durable-good monopolist, without assuming the buyer is anonymous.

The main result we find is that the seller is indeed better off when she has sufficiently optimistic ex ante beliefs about the favorable distribution, compared to a two-period version of Hart and Tirole' (1988) model with the same ex ante probability of a high-value buyer type. Sufficient conditions for the seller to be better off are provided. We also find that in equilibrium the seller will not be able to learn the buyer's value distribution for sure.

Two other papers also examine a rental model in which a non-anonymous buyer's value randomly changes over time. Kennan (2001) analyzes infinitely repeated contract negotiations where the buyer's value is assumed to change according to a two-state Markov chain. He focuses on the cyclic screening equilibria in which several pooling offers in sequence are followed by a screening offer. Loginova and Taylor (2008) investigate a two-period model where the monopolist employs price experimentations to learn the permanent demand parameter of the buyer, which is a continuous random variable distributed on $[0,1]$. In this paper, we assume that the value distribution may either be favorable or unfavorable. We keep our model simple so that we can completely characterize the equilibria and compare the seller's revenue with that in Hart and Tirole (1988).

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 presents the results and fully characterizes the set of equilibria. Section 4 compares the seller's revenue in this model with that in Hart and Tirole (1988). Section 5 concludes. All proofs are in Appendix A. Appendix B provides discussion on equilibrium concept.

2. The Model

One buyer and one seller bargain over renting a durable good in two periods $t = 1,2$. The seller's cost is assumed to be 0. The buyer has a positive value, v_t, in consuming the good in each period t. The buyer's value v_t is drawn from one of two distributions in each period: the bad distribution B or the good distribution G. For a given distribution d, v_t equals h with probability q^d and equals l with probability $1-q^d$. Assume that $0 < q^B < q^G < 1$ and $0 < l < h$, i.e., the good distribution G has a higher probability of generating a high value h. The buyer knows which one of the two distributions his values are actually drawn from as well as his current and former values. However, the seller only knows that the buyer's value is drawn each period from one of these two distributions. The ex ante probability is α for the G distribution and $1-\alpha$ for the B distribution.

At the beginning of the game, the buyer privately observes the realization of distribution d, which will be fixed throughout the game. At the beginning of each period t, the buyer's valuation v_t is drawn from the realized distribution independently across time periods. After the buyer privately observes v_t, the seller offers a price $p_t \in \mathbb{R}$, and then the buyer chooses an action $a_t \in \{0,1\}$, where $a_t = 1$ means acceptance and $a_t = 0$ means rejection.

Both the seller and the buyer are assumed to be risk-neutral. If the buyer accepts the seller's offer in period t, the buyer's payoff is $v_t - p_t$ and the seller's payoff is p_t in period t. They both gain nothing in period t if p_t is rejected. The two players share a common discount factor δ, and both of them maximize the discounted present value of expected payoffs.

Let $\theta_1 = (d, v_1)$ denote the buyer's type in period 1 and $\theta_2 = (d, v_1, v_2)$ the buyer's type in period 2. Since we will focus on the buyer's first period behavior later, it is helpful to notice that there are four buyer types in period 1: (G,l), (B,l), (G,h), and (B,h). Denote h_t^s as the history observed by the seller before she announces p_t and h_t^b as the history observed by the buyer before he chooses a_t. Specifically, $h_1^s = \varnothing$, $h_1^b = (\theta_1, p_1)$, $h_2^s = (p_1, a_1)$ and $h_2^b = (\theta_2, p_1, a_1, p_2)$. A behavioral strategy for the seller, σ^s, assigns probability (or density) $\sigma^s(p_t \mid h_t^s)$ to p_t given any history h_t^s for $t = 1,2$. A behavioral strategy for the buyer, σ^b, assigns probability $\sigma^b(a_t \mid h_t^b)$ to a_t given any history h_t^b for $t = 1,2$. For convenience, let $\sigma^b(h_t^b) \equiv \sigma^b(a_t = 1 \mid h_t^b)$ denote the probability that the buyer accepts p_t given history h_t^b, since the buyer can only choose to accept or reject an offer. Finally, let $\gamma(h_t^s)$ denote the probability that the seller's belief assigns to the G distribution at the beginning of period t given history h_t^s. Notice that $\gamma(p_1, 0)$ and $\gamma(p_1, 1)$ denote the seller's belief of $d = G$ given that p_1 is rejected and accepted respectively.

The equilibrium concept used is strong Perfect Bayesian equilibrium.[3] Bayes' rule is used to update the seller's belief conditional on reaching any price p_1, even if p_1 is off the equilibrium path. We also employ a refinement which is a variant of criterion D_1 in the signalling game (Cho and Kreps 1987, Banks and Sobel 1987). In Appendix B, we formally define criterion D_1.

3. Results

We start with the analysis from the second (last) period. The equilibrium strategies in the last period are simple, as described in Lemma 1, where $\gamma^* = (l/h - q^B)/(q^G - q^B)$ is the cutoff belief that makes the seller indifferent between offering $p_2 = l$ and h.

Lemma 1 *In any PBE, the buyer accepts p_2 if $p_2 \leq v_2$ and rejects p_2 if $p_2 > v_2$. The seller offers $p_2 = l$ if $\gamma(h_2^s) < \gamma^*$, offers $p_2 = h$ if $\gamma(h_2^s) > \gamma^*$, and randomizes between $p_2 = l$ and $p_2 = h$ if $\gamma(h_2^s) = \gamma^*$.*

Let $x(h_2^s)$ denote the probability that the seller offers $p_2 = l$ following history h_2^s, so $x(p_1, 1)$ and $x(p_1, 0)$ is the probability for the seller to offer $p_2 = l$ after acceptance and rejection of p_1 respectively. Therefore, the expected payoff of the buyer type (d, v_1) from accepting p_1 is $v_1 - p_1 + \delta q^d x(p_1, 1)(h - l)$ and from rejecting p_1 is $\delta q^d x(p_1, 0)(h - l)$. Define $C(d, v_1) \equiv v_1 + \delta q^d [x(p_1, 1) - x(p_1, 0)](h - l)$ as the *Cutoff Value* for buyer type $\theta_1 = (d, v_1)$ given $x(p_1, 0)$ and $x(p_1, 1)$, then the buyer of type (d, v_1) accepts $p_1 < C(d, v_1)$, rejects $p_1 > C(d, v_1)$, and randomizes at $p_1 = C(d, v_1)$.

Buyer types with $v_1 = l$ always have a smaller cutoff value than types with $v_1 = h$, regardless of the value distribution d and the seller's strategy in the second period. However, the cutoff values of type (B, v_1) and (G, v_1) depend on the seller's strategy in the second period. If the seller offers $x(p_1, 0) < x(p_1, 1))$, then $C(G, v_1) > C(B, v_1)$. On contrast, if the seller offers $x(p_1, 0) > x(p_1, 1)$, then $C(G, v_1) < C(B, v_1)$. The basic intuition is that, type (G, v_1) has a larger probability of generating an h value in the second period and has a larger expected payoff if $p_2 = l$ than type (B, v_1), so the former is more willing to take the action that will induce the seller to offer $p_2 = l$. By the same intuition, however, in equilibrium the seller will not be able to separate one buy type from all other types. Consider if p_1 is only accepted by type (G, h), then the seller will become extremely optimistic and offer $p_2 = h$ after acceptance of p_1, which reaches a contradiction to $x(p_1, 0) < x(p_1, 1)$.

[3]For the consideration of efficiency, we require the buyer's strategy be left continuous at the cutoff prices where the buyer is indifferent between two actions, that is, the behavioral strategy following the cutoff price p_1 is the same as the behavioral strategy following $p_1 - \varepsilon$.

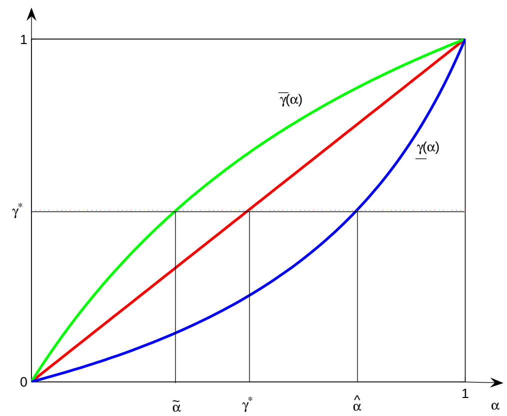

Figure 1: $\overline{\gamma}(\alpha)$ and $\underline{\gamma}(\alpha)$ with $q^B = 0.4$, $q^G = 0.8$ and $l/h = 0.6$

Therefore, in equilibrium the seller is not able to learn the buyer's distribution for sure. However, the seller will still update her belief when p_1 separates h-value types from l-value types. Given that the G distribution has a higher probability of generating an h value, the seller must be more optimistic after acceptance than rejection of p_1. Different from Hart and Tirole, in which the buyer's value is the private information, in this model the seller's belief about the buyer's value distribution changes gradually even if she learns the buyer's first-period value. Therefore, the seller's posterior beliefs conditional on acceptance and rejection of p_1 may not be different enough for her to offer a different p_2.

Define functions

$$\overline{\gamma}(\alpha) \equiv \frac{\alpha q^G}{\alpha q^G + (1-\alpha)q^B},$$

and

$$\underline{\gamma}(\alpha) \equiv \frac{\alpha(1-q^G)}{\alpha(1-q^G)+(1-\alpha)(1-q^B)}.$$

Based on Bayes' rule, $\overline{\gamma}(\alpha)$ and $\underline{\gamma}(\alpha)$ are the seller's posterior beliefs of the G distribution conditional on $v_1 = h$ and $v_1 = l$ respectively. Define $\widetilde{\alpha} \equiv \overline{\gamma}^{-1}(\gamma^*)$ and $\widehat{\alpha} \equiv \underline{\gamma}^{-1}(\gamma^*)$.[4] Figure 1 plots $\overline{\gamma}(\alpha)$ and $\underline{\gamma}(\alpha)$ as functions of the seller's ex ante belief α, choosing $q^B = 0.4$, $q^G = 0.8$, and $l/h = 0.6$.

Lemma 2 *If p_1 is both accepted and rejected with a positive probability in the continuation game, then we have*
(i) $x(p_1,0) > x(p_1,1) \Rightarrow \alpha \in [\widetilde{\alpha},\widehat{\alpha}]$;
(ii) $\alpha \in (0,\widetilde{\alpha}) \cup (\widehat{\alpha},1) \Rightarrow x(p_1,0) = x(p_1,1)$.

According to the seller's ex ante belief of the G distribution, we define a seller *Pessimistic* if $0 < \alpha < \widetilde{\alpha}$, *Moderately Pessimistic* if $\widetilde{\alpha} < \alpha < \gamma^*$, *Moderately Optimistic* if $\gamma^* < \alpha < \widehat{\alpha}$, and *Optimistic* if $\widehat{\alpha} < \alpha < 1$. As in the previous literature, the knife-edge cases are omitted.

[4]Both $\overline{\gamma}(\alpha)$ and $\underline{\gamma}(\alpha)$ are continuous and increasing in α; $\underline{\gamma}(\alpha) < \alpha < \overline{\gamma}(\alpha)$ for $\alpha \in (0,1)$; $\underline{\gamma}(\alpha) = \overline{\gamma}(\alpha) = \alpha$ for $\alpha \in \{0,1\}$. $\widetilde{\alpha}$ and $\widehat{\alpha}$ are well-defined and $\widetilde{\alpha} < \gamma^* < \widehat{\alpha}$.

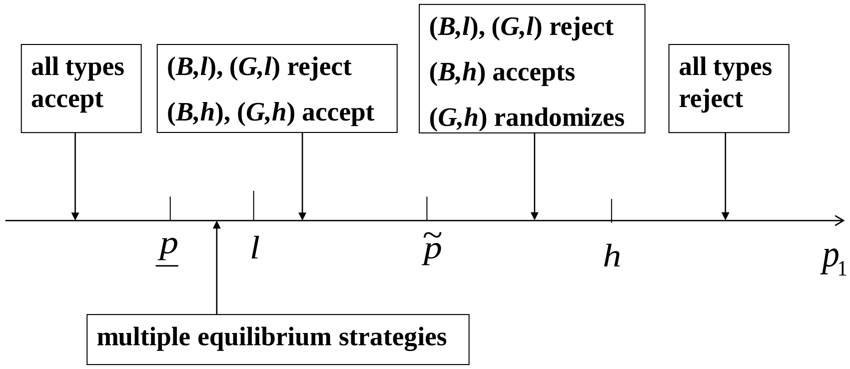

Figure 2: Buyer's Equilibrium Strategy when $\widetilde{\alpha} < \alpha < \widehat{\alpha}$

3.1 Seller with Extreme Ex Ante Beliefs

Given the second part of Lemma 2, when the seller has an extreme prior, she will offer the same price in the second period regardless whether p_1 is accepted or rejected. Anticipating that, all buyer types truthfully reveal their value and accept p_1 if $p_1 \leq v_1$.

Proposition 1 *[Extreme Seller] When the seller is pessimistic (optimistic), there is a unique D_1 equilibrium outcome in which the seller offers $p_t = l$ ($p_t = h$) and all buyer types accept $p_t \leq v_t$ for $t = 1, 2$.*

3.2 Seller with Moderate Ex Ante Beliefs

When the seller's ex ante belief is moderate, she offers $p_2 = h$ conditional on $v_1 = h$ and $p_2 = l$ conditional on $v_1 = l$. This gives the l-value buyer types an incentive to reject p_1 less than but close to l in order to be distinguished from the h-value types. The lowest price the l-value types may reject is $\underline{p} \equiv l - \delta q^B(h-l)$. For $p_1 \in (\underline{p}, l]$ there are multiple equilibrium strategies. One equilibrium strategy, as described above, is for the l-value buyer types to reject p_1 and for the h-value buyer types to accept p_1. It is also possible for all buyer types to accept $p_1 \in (\underline{p}, l]$, if the seller's second-period strategy is independent of whether p_1 is accepted or rejected. Finally, if the seller adopts a mixed strategy in the second period, there also exists an equilibrium strategy in which the buyer type (B,l) plays a mixed strategy. When p_1 exceeds l, the l-value types will reject the offer and the h-value types will accept it if p_1 is relatively low. However, when p_1 approaches h, the h-value types do not accept p_1 with probability one, since the gain in the first period cannot compensate the loss from being offered with a high price in the second period. Similar to the strategy of the l-value types for $p_1 \in (\underline{p}, l]$, the h-value types have an incentive to conceal their current value. In particular, buyer type (B,h) accepts p_1 and buyer type (G,h) plays a mixed strategy. The highest p_1 both h-value types accept with probability one is $\widetilde{p} \equiv h - \delta q^G(h-l)$. Finally, for $p_1 > h$, all buyer types reject p_1. Figure 2 and Lemma 3 summarize the buyer's strategy.

Lemma 3 *When the seller has a moderate prior belief ($\widetilde{\alpha} < \alpha < \widehat{\alpha}$), the buyer's strategy in a D_1 equilibrium is as follows:*

- *if* $p_1 \leq \underline{p}$*, all buyer types accept* p_1*;*
- *if* $\underline{p} < p_1 \leq l$*, there exist multiple equilibrium strategies: (1) all buyer types accept* p_1*; (2) type* (B,l) *and* (G,l) *reject* p_1 *and type* (B,h) *and* (G,h) *accept* p_1*; (3) type* (G,l) *rejects* p_1*, type* (B,l) *randomizes and type* (G,h) *and* (B,h) *accept* p_1*;*
- *if* $l < p_1 \leq \widetilde{p}$*, type* (B,l) *and* (G,l) *reject* p_1 *and type* (B,h) *and* (G,h) *accept* p_1*;*
- *if* $\widetilde{p} < p_1 \leq h$*, type* (B,l) *and* (G,l) *reject* p_1*, type* (G,h) *randomizes and type* (B,h) *accepts* p_1*;*
- *if* $p_1 > h$*, all buyer types reject* p_1*.*

Next we discuss the seller's optimal p_1 and conclude by describing the equilibria of the game.[5] Given the buyer's strategy described in Lemma 3, it is sufficient to consider the seller's payoff at the cutoff prices $p_1 \in \{\underline{p}, l, \widetilde{p}, h\}$. For moderately pessimistic seller, denote U_1 as the seller's payoff from offering $p_1 = \underline{p}$, U_2 as the payoff from offering $p_1 = l$ and all buyer types accept p_1, U_3 as the payoff from offering $p_1 = l$ and type (B,l) randomizes, U_4 as the payoff from offering $p_1 = \widetilde{p}$, and finally U_5 as the payoff from offering $p_1 = h$. Similarly, we use V_1, V_2, V_3, V_4, V_5 to denote the seller's payoff for moderately optimistic seller.[6]

Proposition 2 *[MP Seller: Pooling Equilibria] When the seller is moderately pessimistic, there always exists a pooling* D_1 *equilibrium with* $p_1 = l$*.*

(i) If $U_1 > \max\{U_4, U_5\}$*, any* $p_1 \in [\underline{p}, l]$ *can arise in a pooling equilibrium;*

(ii) If $U_1 < \max\{U_4, U_5\}$*, any* $p_1 \in [p', l]$*, with* $\underline{p} < p' < l$*, can arise in a pooling equilibrium.*

Proposition 3 *[MP Seller: Semi-separating Equilibria] When the seller is moderately pessimistic, the semi-separating* D_1 *equilibria are characterized as follows.*

(i) If $U_1 > \max\{U_3, U_4, U_5\}$*, no semi-separating equilibrium exists;*

(ii) If $U_1 < \max\{U_3, U_4, U_5\} = \max\{U_4, U_5\}$*, semi-separating equilibria exist and the path is unique, with* $p_1 = \widetilde{p}$ *or* $p_1 = h$*;*

(iii) If $U_1 < \max\{U_3, U_4, U_5\} = U_3$*, any* $p_1 \in [p'', l]$*, with* $\underline{p} < p'' < l$*, can arise in a semi-separating equilibrium, so does* $p_1 = \widetilde{p}$ *or* $p_1 = h$ *if* $\max\{U_4, U_5\} > U_1$*.*

When the seller is moderately optimistic, the pooling and the semi-separating D_1 equilibria are similar as for moderately pessimistic seller and so are omitted here to avoid repetition. The only difference is that, for moderately pessimistic seller, $U_2 > \max\{U_4, U_5\}$, so there always exists a pooling D_1 equilibrium with $p_1 = l$. While for moderately optimistic seller, when $V_2 < \max\{V_4, V_5\}$, no pooling equilibrium exists.

4. Comparison of Expected Revenue

[5] All cases presented below in Proposition 2-3 arise for a non-negligible set of parameters, checked by a Mathmatica program.

[6] The explicit expression of $U_1, \cdots, U_5$ and $V_1, \cdots, V_5$ is delegated to the proof of Proposition 2 and 3.

The most important question that this paper is concerned with is whether the seller improves her revenue and gains more monopoly power with the uncertainty about the buyer's value distribution. In this section, we address this issue by comparing the seller's expected revenue in our model with that in the two-period version of Hart and Tirole's (1988) rental model, where the buyer's value distribution is common knowledge.

The two-period version of Hart and Tirole's (1988) rental model is as follows. The buyer has private information about his value, which can be either high or low. The buyer's value is drawn at the beginning of the game and is fixed once realized. In each period $t = 1$ or 2, the seller offers a rental price and the buyer decides whether to accept or reject the offer. Let μ denote the seller's ex ante belief that she is facing a high-value buyer. In order to make a fair comparison, we require that the ex ante probabilities of the high-value buyer in both models be equal, that is, $\mu = \alpha q^G + (1-\alpha)q^B$. The following proposition compares the expected revenues in the equilibria of the two models for any ex ante belief the seller may have.

Proposition 4 *[Revenue Comparison] If the ex ante probability of high value buyer type in the two-period version of Hart and Tirole's (1988) rental model is the same as in this model, then*

(i) for an optimistic seller, the seller's revenue is higher than in Hart and Tirole;

(ii) for a moderately optimistic seller, if q^B is small enough and q^G is big enough, there exists $\overline{\alpha} \in (\gamma^*, \widehat{\alpha})$ such that, for all $\alpha \in (\overline{\alpha}, \widehat{\alpha})$, the seller's revenue is higher than in Hart and Tirole;

(iii) for a pessimistic and moderately pessimistic seller, there always exists an equilibrium in this model which yields the same revenue as in Hart and Tirole.

From Proposition 4 we conclude that, when the seller has sufficiently optimistic ex ante beliefs, the seller is better off compared to the case that the distribution of the buyer's value is common knowledge.

5. Conclusion

In this paper we have considered a two-period repeated bargaining model where the seller offers a price to rent a durable good in each period. The buyer's value of consuming the durable good is drawn from a fixed distribution in each period. The buyer has private information not only about his value in each period, but also about the distribution which his values are drawn from.

We find that the seller will not learn the buyer's value distribution in equilibrium but may still learn the buyer's value. We compare the seller's expected revenue in our model with that in the two-period version of Hart and Tirole's (1988) rental model where the distribution of the buyer's value is common knowledge, under the assumption that the ex ante probabilities of high value buyer types are the same in the two models. We find that the seller is better off with the additional layer of uncertainty about the buyer's value distribution when she has sufficiently optimistic ex ante beliefs.

Appendix A: Proofs

Lemma 4 *In any PBE equilibrium, there does not exist p_1 that can screen one buyer type from the other three types.*

Proof. [Proof of Lemma 4]

Step 1: Suppose $x(p_1,0) > x(p_1,1)$. Then $C(G,l) < C(B,l) < l < C(G,h) < C(B,h) < h$. A price p_1 can screen one type from the other types only if $p_1 \in [C(G,l),C(B,l)]$ or $p_1 \in [C(G,h),C(B,h)]$.

If $p_1 \in [C(G,l),C(B,l)]$ and only type (G,l) rejects p_1, then $x(p_1,0) = 0$, so it contradicts with $x(p_1,0) > x(p_1,1)$.

If $p_1 \in [C(G,h),C(B,h)]$ and only type (B,h) accepts p_1, then $x(p_1,1) = 1$, so it contradicts with $x(p_1,0) > x(p_1,1)$.

Step 2: Suppose $x(p_1,0) < x(p_1,1)$. Then $l < C(B,l) < C(G,l) < h < C(B,h) < C(G,h)$. A price p_1 can screen one type from the other types only if $p_1 \in [C(B,l),C(G,l)]$ or $p_1 \in [C(B,h),C(G,h)]$.

If $p_1 \in [C(B,l),C(G,l)]$ and only type (B,l) rejects p_1, then $x(p_1,0) = 1$, so it contradicts with $x(p_1,0) < x(p_1,1)$.

If $p_1 \in [C(B,h),C(G,h)]$ and only type (G,h) accepts p_1, then $x(p_1,1) = 0$, so it contradicts with $x(p_1,0) < x(p_1,1)$.

Step 3: Suppose $x(p_1,0) = x(p_1,1)$. Then $C(B,l) = C(G,l) = l < C(B,h) = C(G,h) = h$.

If $p_1 \leq l$, all types accept p_1.

If $p_1 > h$, all types reject p_1.

If $l < p_1 \leq h$, both type (B,l) and (G,l) reject p_1 and both type (B,h) and (G,h) accept p_1.

In any case, screening one buyer type cannot happen in equilibrium. ■

Lemma 5 *Let $\Psi(p_1,a_1)$ denote the probability that action a_1 is taken in the continuation game following p_1. If $\Psi(p_1,1) \in (0,1)$ for a given p_1 in a PBE, then $x(p_1,0) \geq x(p_1,1)$.*

Proof. [Proof of Lemma 5] Suppose $\Psi(p_1,1) \in (0,1)$ and $x(p_1,0) < x(p_1,1)$ in a PBE. Then $l < C(B,l) < C(G,l) < h < C(B,h) < C(G,h)$, and $\Psi(p_1,1) \in (0,1)$ only if $p_1 \in [C(B,l),C(G,h)]$. We will show that it reaches a contradiction for any $p_1 \in [C(B,l),C(G,h)]$.

If $p_1 \in [C(B,l),C(G,l))$, then only type (B,l) rejects p_1 and $x(p_1,0) = 1 \geq x(p_1,1)$.

If $p_1 \in (C(B,h),C(G,h)]$, then only type (G,h) accepts p_1 and $x(p_1,1) = 0 \leq x(p_1,0)$.

If $p_1 \in (C(G,l),C(B,h))$, then $\gamma(p_1,0) = \underline{\gamma}(\alpha) < \alpha < \overline{\gamma}(\alpha) = \gamma(p_1,1)$ and $x(p_1,0) \geq x(p_1,1)$.

Denote X' as the probability for type $(G,\overline{l})$ to reject $p_1 = C(G,l)$ and Y' as the probability for type (B,h) to reject $p_1 = C(B,h)$.

If $p_1 = C(G,l)$,

$$\gamma(p_1,0) = \frac{\alpha X'(1-q^G)}{\alpha X'(1-q^G)+(1-\alpha)(1-q^B)} < \underline{\gamma}(\alpha)$$

and

$$\gamma(p_1,1) = \frac{\alpha q^G + \alpha(1-X')(1-q^G)}{\alpha q^G + (1-\alpha)q^B + \alpha(1-X')(1-q^G)} > \overline{\gamma}(\alpha),$$

so $x(p_1,0) \geq x(p_1,1)$.

If $p_1 = C(B,h)$,

$$\gamma(p_1,0) = \frac{\alpha(1-q^G)}{\alpha(1-q^G)+(1-\alpha)(1-q^B)+(1-\alpha)Y'q^B} < \underline{\gamma}(\alpha)$$

and

$$\gamma(p_1,1) = \frac{\alpha q^G}{\alpha q^G+(1-\alpha)(1-Y')q^B} > \overline{\gamma}(\alpha),$$

so $x(p_1,0) \geq x(p_1,1)$.

Since every case leads to a contradiction with $x(p_1,0) < x(p_1,1)$, the seller offers $x(p_1,0) \geq x(p_1,1)$ in a PBE if $\Psi(p_1,1) \in (0,1)$. ■

Proof. [Proof of Lemma 2] (i) $\Psi(p_1,1) \in (0,1)$ and $x(p_1,0) > x(p_1,1) \Rightarrow \alpha \in [\widetilde{\alpha},\widehat{\alpha}]$.

Suppose $\Psi(p_1,1) \in (0,1)$, $x(p_1,0) > x(p_1,1)$ and $\alpha \in (0,\widetilde{\alpha}) \cup (\widehat{\alpha},1)$. Then $C(G,l) < C(B,l) < l < C(G,h) < C(B,h) < h$. $\Psi(p_1,1) \in (0,1)$ only if $p_1 \in [C(G,l),C(B,h)]$. We will show that it reaches a contradiction for any $p_1 \in [C(G,l),C(B,h)]$.

If $p_1 \in [C(G,l),C(B,l))$, then only type (G,l) rejects p_1 and $x(p_1,0) = 0 \leq x(p_1,1)$.

If $p_1 \in (C(G,h),C(B,h)]$, then only type (B,h) accepts p_1 and $x(p_1,1) = 1 \geq x(p_1,0)$.

If $p_1 \in (C(B,l),C(G,h))$ and $\alpha < \widetilde{\alpha}$, then $\gamma(p_1,1) = \overline{\gamma}(\alpha) < \gamma^*$ and $x(p_1,1) = 1 \geq x(p_1,0)$.

If $p_1 \in (C(B,l),C(G,h))$ and $\alpha > \widehat{\alpha}$, then $\gamma(p_1,0) = \underline{\gamma}(\alpha) > \gamma^*$ and $x(p_1,0) = 0 \leq x(p_1,1)$.

Denote X as the probability for type (B,l) to reject $p_1 = C(B,l)$ and Y as the probability for type (G,h) to reject $p_1 = C(G,h)$.

If $p_1 = C(B,l)$,

$$\gamma(p_1,0) = \frac{\alpha(1-q^G)}{\alpha(1-q^G)+(1-\alpha)X(1-q^B)} > \underline{\gamma}(\alpha)$$

and

$$\gamma(p_1,1) = \frac{\alpha q^G}{\alpha q^G+(1-\alpha)q^B+(1-\alpha)(1-X)(1-q^B)} < \overline{\gamma}(\alpha).$$

When $\alpha < \widetilde{\alpha}$, $\gamma(p_1,1) < \overline{\gamma}(\alpha) < \gamma^*$ and $x(p_1,1) = 1 \geq x(p_1,0)$. When $\alpha > \widehat{\alpha}$, $\gamma(p_1,0) > \underline{\gamma}(\alpha) > \gamma^*$ and $x(p_1,0) = 0 \leq x(p_1,1)$.

If $p_1 = C(G,h)$,

$$\gamma(p_1,0) = \frac{\alpha Y q^G+\alpha(1-q^G)}{\alpha Y q^G+\alpha(1-q^G)+(1-\alpha)(1-q^B)} > \underline{\gamma}(\alpha)$$

and

$$\gamma(p_1,1) = \frac{\alpha(1-Y)q^G}{\alpha(1-Y)q^G+(1-\alpha)q^B} < \overline{\gamma}(\alpha).$$

When $\alpha < \widetilde{\alpha}$, $\gamma(p_1,1) < \overline{\gamma}(\alpha) < \gamma^*$ and $x(p_1,1) = 1 \geq x(p_1,0)$. When $\alpha > \widehat{\alpha}$, $\gamma(p_1,0) > \underline{\gamma}(\alpha) > \gamma^*$ and $x(p_1,0) = 0 \leq x(p_1,1)$.

Therefore, every case is contradictory to $x(p_1,0) > x(p_1,1)$.

(ii) It is directly derived from Lemma 5 and (i) of Lemma 2. ■

Proof. [Proof of Proposition 1]

Part 1: Pessimistic Seller

Step 1: We first show that it is the unique D_1 equilibrium strategy for the buyer to accept p_1 if and only if $p_1 \leq v_1$.

(1) Suppose $x(p_1,0) > x(p_1,1)$. Lemma 2 implies that either all buyer types accept p_1 or all buyer types reject p_1. Given $\alpha < \widetilde{\alpha}$ and $x(p_1,0) > x(p_1,1)$, it must be the case that all buyer types reject p_1, otherwise $x(p_1,1) = 1$. Since $x(p_1,0) > x(p_1,1)$ and all buyer types reject p_1, $p_1 > C(B,h)$, which is less than h. But a PBE cannot pass criterion D_1 if all buyer types reject $p_1 < h$. Therefore, $p_1 \geq h$ and all buyer types reject p_1.

(2) Suppose $x(p_1,0) < x(p_1,1)$. Given $\alpha < \widetilde{\alpha}$, Lemma 2 implies that $p_1 \leq C(B,l) = \min_{\theta_1}\{C(\theta_1)\}$, which is greater than l, and all buyer types accept p_1. But a PBE cannot pass criterion D_1 if all buyer types accept $p_1 > l$. Therefore, $p_1 \leq l$ and all buyer types accept p_1.

(3) Suppose $x(p_1,0) = x(p_1,1)$. All buyer types accept p_1 if and only if $p_1 \leq v_1$ for any p_1.

Combining three cases above, it is the unique D_1 equilibrium strategy for the buyer to accept p_1 if and only if $p_1 \leq v_1$.

Step 2: Given the buyer's strategy, the seller offers $p_1 = l$ or $p_1 = h$, and always offers $p_2 = l$ on the equilibrium path. The respective payoffs for the seller is:

$$\begin{cases} \pi(l) = l + \delta l; \\ \pi(h) = \alpha h + \delta l. \end{cases}$$

Given $\alpha < \widetilde{\alpha} < \gamma^*$, it is optimal to offer $p_1 = l$.

Part 2: Optimistic Seller

Step 1: Similar to the pessimistic seller, we first show that it is the unique D_1 equilibrium strategy for the buyer to accept p_1 if and only if $p_1 \leq v_1$.

(1) Suppose $x(p_1,0) > x(p_1,1)$. Given $\alpha > \widehat{\alpha}$, Lemma 2 implies that $p_1 \leq C(G,l) = \min_{\theta_1}\{C(\theta_1)\} < l$, and all buyer types accept p_1. It passes criterion D_1.

(2) Suppose $x(p_1,0) < x(p_1,1)$. Given $\alpha > \widehat{\alpha}$, Lemma 2 implies that $p_1 > C(G,h) = \max_{\theta_1}\{C(\theta_1)\} > h$, and all buyer types reject p_1. It passes criterion D_1.

(3) Suppose $x(p_1,0) = x(p_1,1)$. All buyer types accept p_1 if and only if $p_1 \leq v_1$ for any p_1.

Combining three cases above, it is the unique D_1 equilibrium strategy for the buyer to accept p_1 if and only if $p_1 \leq v_1$.

Step 2: Given the buyer's strategy, the seller offers $p_1 = l$ or $p_1 = h$, and always offers $p_2 = h$ on the equilibrium path. The respective payoffs for the seller is:

$$\begin{cases} \pi(l) = l + \delta \alpha h; \\ \pi(h) = \alpha h + \delta \alpha h. \end{cases}$$

Since $\alpha > \widehat{\alpha} > \gamma^*$, it is optimal to offer $p_1 = h$. ■

Proof. [Proof of Lemma 3] We try to derive all the possible buyer's strategies following different second-period strategy of the seller.

(1) Suppose $x(p_1,0) < x(p_1,1)$.

We have $l < C(B,l) < C(G,l) < h < C(B,h) < C(G,h)$. Lemma 5 implies that p_1 must be accepted or rejected with probability one given $x(p_1,0) < x(p_1,1)$. Given $x(p_1,0) < x(p_1,1)$, when $\widetilde{\alpha} < \alpha < \gamma^*$, $p_1 \leq C(B,l)$ and all buyer types accept p_1. When $\gamma^* < \alpha < \widehat{\alpha}$, $p_1 > C(G,h)$

and all buyer types reject p_1. The equilibrium cannot pass criterion D_1 if all types accept $p_1 > l$. Thus, when $\widetilde{\alpha} < \alpha < \gamma^*$, $p_1 \leq l$ and all types accept p_1. When $\gamma^* < \alpha < \widehat{\alpha}$, $p_1 > C(G,h) > h$ and all buyer types reject p_1.

(2) Suppose $x(p_1,0) = x(p_1,1)$.

We have $C(G,l) = C(B,l) = l < C(G,h) = C(B,h) = h$. Therefore, for $p_1 \leq l$ all buyer types accept p_1, and for $p_1 > h$ all types reject p_1. For $p_1 \in (l,h]$, type (B,l) and (G,l) reject p_1 and type (B,h) and (G,h) accept p_1, so $x(p_1,0) = 1$ and $x(p_1,1) = 0$ given $\widetilde{\alpha} < \alpha < \widehat{\alpha}$, which leads to a contradiction.

(3) Suppose $x(p_1,0) > x(p_1,1)$.

We have $C(G,l) < C(B,l) < l < C(G,h) < C(B,h) < h$. Next we divide all the possibilities into three cases.

Case 1: p_1 is accepted or rejected with probability one, i.e., $\Psi(p_1,1) \in \{0,1\}$. So when $\widetilde{\alpha} < \alpha < \gamma^*$, $p_1 > C(B,h)$ and all buyer types reject p_1. When $\gamma^* < \alpha < \widehat{\alpha}$, $p_1 \leq C(G,l)$ and all buyer types accept p_1. The equilibrium cannot pass criterion D_1 if all buyer types reject $p_1 < h$. Thus, when $\widetilde{\alpha} < \alpha < \gamma^*$, $p_1 > h$ and all buyer types reject p_1. When $\gamma^* < \alpha < \widehat{\alpha}$, $p_1 \leq C(G,l) < l$ and all buyer types accept p_1.

Case 2: p_1 is accepted and rejected with a positive probability, i.e., $\Psi(p_1,1) \in (0,1)$, and the seller plays pure strategy in the second period, i.e., $x(p_1,0) = 1$ and $x(p_1,1) = 0$. Lemma 4 shows that no p_1 separates a single type from other types. So $p_1 \in (C(B,l), C(G,h)] = (\underline{p}, \widetilde{p}]$. Type (B,l) and (G,l) reject p_1 and type (B,h) and (G,h) accept p_1.

Case 3: p_1 is accepted and rejected with a positive probability, i.e., $\Psi(p_1,1) \in (0,1)$, and the seller plays mixed strategy in the second period, i.e., $0 < x(p_1,0) - x(p_1,1) < 1$. Then either $x(p_1,0) = 1$ and $x(p_1,1) \in (0,1)$ or $x(p_1,0) \in (0,1)$ and $x(p_1,1) = 0$, since the knife-edge condition $\alpha = \gamma^*$ is omitted. The former implies $\gamma(p_1,0) < \gamma^*$ and $\gamma(p_1,1) = \gamma^*$, and the latter implies $\gamma(p_1,0) = \gamma^*$ and $\gamma(p_1,1) > \gamma^*$. Therefore, $\gamma(p_1,1) = \gamma^*$ when $\widetilde{\alpha} < \alpha < \gamma^*$, and $\gamma(p_1,0) = \gamma^*$ when $\gamma^* < \alpha < \widehat{\alpha}$. From Lemma 4, it is not possible for type (G,l) or (B,h) to randomize, otherwise the seller can at least sometimes separate type (G,l) or (B,h) from other types. So only type (B,l) and (G,h) may play mixed strategy.

When $\widetilde{\alpha} < \alpha < \gamma^*$, type (B,l) randomizes to reject p_1 with probability X^*, (G,l) rejects p_1, and (G,h) and (B,h) accept p_1. Then

$$\gamma(p_1,1) = \frac{\alpha q^G}{\alpha q^G + (1-\alpha)q^B + (1-\alpha)(1-X^*)(1-q^B)} = \gamma^*.$$

Type (B,l) is indifferent from accepting and rejecting p_1, then

$$l - p_1 + \delta q^B x(p_1,1)(h-l) = \delta q^B (h-l).$$

So type (B,l) rejects $p_1 \in (\underline{p}, l]$ with probability $X^* = 1 + \frac{q^B}{1-q^B} - \frac{\alpha q^G (1-\gamma^*)}{(1-\alpha)(1-q^B)\gamma^*}$, and the seller offers $x(p_1,1) = 1 - \frac{l-p_1}{\delta q^B (h-l)}$, $x(p_1,0) = 1$.

When $\gamma^* < \alpha < \widehat{\alpha}$, type (B,l) randomizes to reject p_1 with probability X^{**}, (G,l) rejects p_1, and (G,h) and (B,h) accept p_1. Then[7]

$$\gamma(p_1,0) = \frac{\alpha(1-q^G)}{\alpha(1-q^G) + (1-\alpha)X^{**}(1-q^B)} = \gamma^*.$$

[7] Since the seller is more optimistic after acceptance of p_1 than rejection of p_1, her posterior belief $\gamma(p_1,1) > \gamma(p_1,0)$. Therefore, in a mixed strategy equilibrium, $\gamma(p_1,1) = \gamma^*$ if $\alpha < \gamma^*$ and $\gamma(p_1,0) = \gamma^*$ if $\alpha > \gamma^*$.

Type (B,l) is indifferent from accepting and rejecting p_1, then

$$l - p_1 = \delta q^B x(p_1,0)(h-l).$$

So type (B,l) rejects $p_1 \in (\underline{p},l]$ with probability $X^{**} = \frac{\alpha(1-q^G)(1-\gamma^*)}{(1-\alpha)(1-q^B)\gamma^*}$, and the seller offers $x(p_1,0) = \frac{l-p_1}{\delta q^B(h-l)}$ and $x(p_1,1) = 0$.

When $\widetilde{\alpha} < \alpha < \gamma^*$, type (G,h) randomizes to reject p_1 with probability Y^*, (B,l) and (G,l) reject p_1, and (B,h) accepts p_1. Then

$$\gamma(p_1,1) = \frac{\alpha(1-Y^*)q^G}{\alpha(1-Y^*)q^G + (1-\alpha)q^B} = \gamma^*.$$

Type (G,h) is indifferent from accepting and rejecting p_1, then

$$h - p_1 + \delta q^G x(p_1,1)(h-l) = \delta q^G (h-l).$$

So type (G,h) rejects $p_1 \in (\widetilde{p},h]$ with probability $Y^* = 1 - \frac{(1-\alpha)q^B\gamma^*}{\alpha q^G(1-\gamma^*)}$, and the seller offers $x(p_1,1) = 1 - \frac{h-p_1}{\delta q^G(h-l)}$ and $x(p_1,0) = 1$.

When $\gamma^* < \alpha < \widehat{\alpha}$, type (G,h) randomizes to reject p_1 with probability Y^{**}, (B,l) and (G,l) reject p_1, and (B,h) accepts p_1. Then

$$\gamma(p_1,0) = \frac{\alpha Y^{**}q^G + \alpha(1-q^G)}{\alpha Y^{**}q^G + \alpha(1-q^G) + (1-\alpha)(1-q^B)} = \gamma^*.$$

Type (G,h) is indifferent from accepting and rejecting p_1, then

$$h - p_1 = \delta q^G x(p_1,0)(h-l).$$

So type (G,h) rejects $p_1 \in (\widetilde{p},h]$ with probability $Y^{**} = \frac{(1-\alpha)(1-q^B)\gamma^*}{\alpha q^G(1-\gamma^*)} - \frac{1-q^G}{q^G}$ and the seller offers $x(p_1,0) = \frac{h-p_1}{\delta q^G(h-l)}$ and $x(p_1,1) = 0$.

Lemma 3 comes from the combination of three steps. ■

Proof. [Proof of Proposition 2 and 3]

Moderately Pessimistic Seller

(1) $p_1 = \underline{p}$: The seller offers $p_2 = l$, with p_1 and p_2 accepted by all buyer types.

$$U_1 = \underline{p} + \delta l;$$

(2) $p_1 = l$: Since there are multiple equilibrium strategies for the buyer following $p_1 \in (\underline{p},l]$, the seller's payoff from offering $p_1 = l$ depends on which strategy the buyer is using. Suppose that all buyer types choose to accept $p_1 = l$, then the seller offers $p_2 = p_1 = l$, with p_1 and p_2 accepted by all buyer types.

$$U_2 = l + \delta l;$$

If buyer type (G,l) rejects p_1, buyer type (B,l) randomizes, and buyer types (B,h) and (G,h) accept p_1, then the seller's payoff from offering $p_1 = l$ is

$$U_3 = [\alpha q^G + (1-\alpha)q^B + (1-\alpha)(1-q^B)(1-X^*)]l + \delta l;$$

Finally, if both l-value buyer types choose to reject and both h-value buyer types choose to accept $p_1 = l$, then offering $p_1 = l$ is dominated by offering $p_1 = \widetilde{p}$.

(3) $p_1 = \widetilde{p}$: Buyer types (B,l) and (G,l) reject p_1, buyer types (B,h) and (G,h) accept p_1, and the seller offers $p_2 = l$ if p_1 is rejected and $p_2 = h$ if p_1 is accepted.

$$\begin{aligned} U_4 &= [\alpha q^G + (1-\alpha)q^B]\widetilde{p} + \delta[\alpha(q^G)^2 + (1-\alpha)(q^B)^2]h \\ &\quad + \delta[\alpha(1-q^G) + (1-\alpha)(1-q^B)]l; \end{aligned}$$

(4) $p_1 = h$: The unique equilibrium strategy is for buyer type (G,h) to randomize, buyer type (B,h) to accept p_1, and buyer types (B,l) and (G,l) to reject p_1, so the seller's payoff is

$$U_5 = [\alpha q^G(1-Y^*) + (1-\alpha)q^B]h + \delta l.$$

Proof for pooling equilibria.

Step 1: First we show that $U_2 > \max\{U_4, U_5\}$ for $\widetilde{\alpha} < \alpha < \gamma^*$. Given this, there always exists a pooling equilibrium with $p_1 = l$ on the equilibrium path and all buyer types accepting $p_1 \in [\underline{p}, l]$.

$$\begin{aligned} & U_4 - U_2 \\ = \; & \delta(1-\alpha)q^B l(q^G - 1) + \delta\alpha q^G l(q^G - 1) \\ & + \delta(1-\alpha)q^B h(q^B - q^G) + [\alpha q^G h + (1-\alpha)q^B h - l] \\ < \; & 0 \end{aligned}$$

Each item on the right hand side of the equation is negative for $\widetilde{\alpha} < \alpha < \gamma^*$.

By plugging Y^* into the definition of U_5, $U_5 = \frac{1-\alpha}{1-\gamma^*}q^B h + \delta l$, which is decreasing in α. So

$$\begin{aligned} & U_5 - U_2 \\ < \; & \frac{1-\widetilde{\alpha}}{1-\gamma^*} q^B h - l \\ = \; & \frac{h}{q^G + q^B - l/h}(l/h - q^G)(l/h - q^B) < 0. \end{aligned}$$

Step 2: (i)If $U_1 > \max\{U_4, U_5\}$, for an arbitrary $p_1^* \in [\underline{p}, l]$, assume all buyer types accept $p_1 \in [\underline{p}, p_1^*]$, type (B,l) and (G,l) reject $p_1 \in (p_1^*, l]$, and type (B,h) and (G,h) accept $p_1 \in (p_1^*, l]$, then p_1^* is the optimal p_1.

(ii) Since $U_1 = \underline{p} + \delta l < \max\{U_4, U_5\} < U_2 = l + \delta l$, there exists $p' \in (\underline{p}, l)$ such that $u(p') = p' + \delta l = \max\{U_4, U_5\}$. For an arbitrary $p_1^* \in [p', l]$, assume all buyer types accept $p_1 \in [\underline{p}, p_1^*]$, type (B,l) and (G,l) reject $p_1 \in (p_1^*, l]$, and type (B,h) and (G,h) accept $p_1 \in (p_1^*, l]$. Then p_1^* is the optimal p_1 given $u(p') = \max\{U_4, U_5\}$.

Proof for semi-separating equilibria. (i) By definition U_3, U_4 and U_5 are the potential highest payoffs in a semi-separating equilibrium. If the lowest payoff from a pooling offer, U_1, is greater than $\max\{U_3, U_4, U_5\}$, there is no semi-separating equilibrium.

(ii) For all $p_1 \in (\underline{p}, l]$, the buyer can adopt two semi-separating equilibrium strategies: 1) types with $v_1 = l$ reject p_1 and types with $v_1 = h$ accept p_1, or 2) types with $v_1 = h$ accept p_1, type (G,l) rejects p_1 and type (B,l) randomizes. If the first strategy is adopted at $p_1 \in (\underline{p}, l]$, the seller's payoff by offering p_1 is less than U_4. If the second strategy is adopted, the payoff is weakly less than U_3, which is less than $\max\{U_4, U_5\}$. Therefore, when the buyer adopts either of these two strategies, given $U_1 < \max\{U_4, U_5\}$, a semi-separating equilibrium exists and the path is unique, with $p_1 = \widetilde{p}$ or $p_1 = h$, depending on whether U_4 or U_5 is larger.

(iii) Define $U(p_1, X^*) = [\alpha q^G + (1-\alpha)q^B + (1-\alpha)(1-q^B)(1-X^*)]p_1 + \delta l$, which is increasing in $p_1 \in (\underline{p}, l]$. First suppose $\max\{U_4, U_5\} < U_1 < U_3$. By definition $U(\underline{p}, X^*) < U_1 < U_3 = U(l, X^*)$. Therefore, there exists $p'' \in (\underline{p}, l)$ such that $U(p'', X^*) = U_1$. For any arbitrary $p_1^* \in [p'', l]$, assume the buyer uses the second strategy for $p_1^* \leq p''$ and uses the first strategy described in part (ii) for $p_1^* > p''$, then $p_1^* \in [p'', l]$ is the optimal p_1.

Then suppose $U_1 < \max\{U_4, U_5\} < U_3$. Since $U(\underline{p}, X^*) < U_1 < U_3 = U(l, X^*)$, $U(\underline{p}, X^*) < \max\{U_4, U_5\} < U(l, X^*)$. Define $p'' \in (\underline{p}, l)$ such that $U(p'', X^*) = \max\{U_4, U_5\}$. If for any arbitrary $p_1^* \in [p'', l]$, the buyer uses the second strategy described in part (ii) for $p_1^* \leq p''$ and uses the first strategy for $p_1^* > p''$, then $p_1^* \in [p'', l]$ is the optimal p_1. If for any $p_1 \in (\underline{p}, l]$, the buyer uses the first strategy, then $p_1 = \widetilde{p}$ or $p_1 = h$ is optimal, depending on whether U_4 or U_5 is larger.

Moderately Optimistic Seller

(1) $p_1 = \underline{p}$: $p_2 = h$, p_1 accepted by all buyer types and p_2 accepted by types with $v_2 = h$.

$$V_1 = \underline{p} + \delta[\alpha q^G + (1-\alpha)q^B]h;$$

(2) $p_1 = l$: $p_2 = h$, p_1 accepted by all buyer types and p_2 accepted by types with $v_2 = h$.

$$V_2 = l + \delta[\alpha q^G + (1-\alpha)q^B]h;$$

Payoff V_3 is the seller's payoff from offering $p_1 = l$, buyer type (G, l) rejects p_1, buyer type (B, l) randomizes, and buyer types (B, h) and (G, h) accept p_1.

$$V_3 = [\alpha q^G + (1-\alpha)q^B + (1-\alpha)(1-q^B)(1-X^{**})]l + \delta[\alpha q^G + (1-\alpha)q^B]h;$$

(3) $p_1 = \widetilde{p}$: Payoff $V_4 = U_4$ is the seller's payoff when buyer types (B, l) and (G, l) reject p_1, buyer types (B, h) and (G, h) accept p_1, and the seller offers $p_2 = l$ if p_1 is rejected and $p_2 = h$ if p_1 is accepted.

$$\begin{aligned} V_4 &= [\alpha q^G + (1-\alpha)q^B]\widetilde{p} + \delta[\alpha(q^G)^2 + (1-\alpha)(q^B)^2]h \\ &\quad + \delta[\alpha(1-q^G) + (1-\alpha)(1-q^B)]l; \end{aligned}$$

(4) $p_1 = h$: Payoff V_5 is the seller's payoff from offering $p_1 = h$, buyer type (G, h) randomizes, buyer type (B, h) accepts p_1, and buyer types (B, l) and (G, l) reject p_1.

$$V_5 = [\alpha q^G(1-Y^{**}) + (1-\alpha)q^B]h + \delta[\alpha q^G + (1-\alpha)q^B]h.$$

The proof of the pooling and semi-separating equilibria are similar to that for moderately pessimistic seller and are therefore omitted. ■

Proof. [Proof of Proposition 4]

Step 1: We first describe the equilibrium in the two-period version of Hart and Tirole's rental model.

In period 2, both types accept p_2 if and only if $p_2 \leq v_2$ and reject p_2 otherwise. In the first period, the l-type buyer accepts p_1 if and only if $p_1 \leq l$ and reject p_1 otherwise. If $\mu < l/h$, the

h-type buyer accepts $p_1 \leq h - \delta(h-l)$ and reject $p_1 > h - \delta(h-l)$. If $\mu > l/h$, the h-type buyer accepts $p_1 \leq h - \delta(h-l)$, randomizes to accept $p_1 \in (h - \delta(h-l), h]$ with probability $y^* = \frac{\mu h - l}{\mu(h-l)}$, and reject $p_1 > h$. Therefore, if $\mu < l/h$, the seller offers $p_1 = p_2 = l$; if $l/h < \mu < \frac{hl+\delta l(h-l)}{hl+\delta h(h-l)}$, the seller offers $p_1 = h - \delta(h-l)$, $p_2 = h$ if p_1 is accepted and $p_2 = l$ if p_1 is rejected; if $\mu > \frac{hl+\delta l(h-l)}{hl+\delta h(h-l)}$, the seller offers $p_1 = p_2 = h$. The seller's revenue in the equilibrium is as follows.

$$\pi = \begin{cases} l + \delta l, & \text{if } \mu < l/h; \\ \mu[h - \delta(h-l)] + \delta \mu h + \delta(1-\mu)l = \mu h + \delta l, & \text{if } l/h < \mu < \frac{hl+\delta l(h-l)}{hl+\delta h(h-l)}; \\ \mu y^* h + \delta \mu h = \frac{\mu h^2 - hl + \delta \mu h^2 - \delta \mu hl}{h-l} & \text{if } \mu > \frac{hl+\delta l(h-l)}{hl+\delta h(h-l)}. \end{cases}$$

Step 2: Next we compare the revenue in our model with that in Hart and Tirole, assuming that $\mu = \alpha q^G + (1-\alpha)q^B$. Notice that $\alpha > \gamma^*$ is equivalent to $\mu > l/h$. For convenience, denote $W_1 = \mu h + \delta l$ and $W_2 = \mu y^* h + \delta \mu h$.

(i) For an optimistic seller ($\alpha > \widehat{\alpha}$), there is a unique equilibrium outcome as shown in Proposition 2, and the seller's revenue in our model is

$$\begin{aligned} & (\alpha q^G + (1-\alpha)q^B)h + \delta(\alpha q^G + (1-\alpha)q^B)h \\ = \; & \mu h + \delta \mu h \\ > \; & \max\{W_1, W_2\}. \end{aligned}$$

So the seller's revenue in our model is higher than in Hart and Tirole's.

(ii) For a moderately optimistic seller ($\gamma^* < \alpha < \widehat{\alpha}$), it suffices to compare the potential optimal revenues W_1 and W_2 in Hart and Tirole with the potential optimal revenues V_2, V_4 and V_5 in our model. Our proof consists of the following results.

Result 1:

$$W_1 - V_2 = (1-\delta)(\mu h - l) > 0$$

Result 2:

$$W_1 - V_4 = \delta(1-\alpha)q^B(q^G - q^B)h + \delta\mu(1-q^G)l > 0.$$

Result 3:

$$V_5 > W_2 \;\text{ if }\; q^G > 1 - q^B + q^B(l/h).$$

$$\begin{aligned} & V_5 - W_2 \\ = \; & [\alpha q^G(1 - Y^{**}) + (1-\alpha)q^B]h - \mu y^* h \\ = \; & \alpha(q^G - q^B)\left(\frac{1-q^B}{q^G - l/h} - \frac{1}{1-l/h}\right)h \\ & + [q^B - \frac{(1-q^B)(l/h - q^B)}{q^G - l/h} + \frac{l/h - q^B}{1 - l/h}]h \end{aligned}$$

If $(1-q^B)(1-l/h) < q^G - l/h$, then $V_5 > W_2$ when

$$\alpha < \frac{l/h - q^B}{q^G - q^B} - \frac{q^B}{q^G - q^B} \frac{1}{\frac{1-q^B}{q^G - l/h} - \frac{1}{1-l/h}}.$$

The RHS of the inequality is decreasing in q^G and converges to 1 when $q^G \to 1$, therefore the RHS of the inequality is greater than 1, so the inequality always holds when $(1-q^B)(1-l/h) < q^G - l/h$.

Result 4: There exists $\overline{\alpha} \in (\gamma^*, \widehat{\alpha})$ such that, for $\alpha \in (\overline{\alpha}, \widehat{\alpha})$, $W_2 > W_1$ if $q^G > 1 - q^B + q^B(l/h)$ and $q^B < \frac{\delta(l/h)(1-l/h)}{l/h+\delta(1-l/h)}$.

$$\begin{aligned} & W_2 > W_1 \\ \Rightarrow \quad & \mu > \frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} \\ \Rightarrow \quad & \alpha > \frac{1}{q^G - q^B}\left[\frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} - q^B\right] \equiv \overline{\alpha} \end{aligned}$$

It is easy to show that $\overline{\alpha} > \gamma^*$. Next we need to show the conditions under which $\overline{\alpha} < \widehat{\alpha}$.

$$\begin{aligned} & \overline{\alpha} < \widehat{\alpha} \\ \Leftrightarrow \quad & \frac{1}{q^G - q^B}\left[\frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} - q^B\right] < \frac{1-q^B}{(1-q^B)-(q^G - l/h)} \cdot \frac{l/h - q^B}{q^G - q^B} \\ \Leftrightarrow \quad & \frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} - q^B < \frac{(1-q^B)(l/h - q^B)}{(1-q^B)-(q^G - l/h)} \end{aligned}$$

At the same time,

$$\begin{aligned} & q^G - l/h > (1-q^B)(1-l/h) \\ \Leftrightarrow \quad & (1-q^B) - (q^G - l/h) < (1-q^B) - (1-q^B)(1-l/h) \\ \Leftrightarrow \quad & (1-q^B)(l/h) > (1-q^B) - (q^G - l/h) \\ \Leftrightarrow \quad & \frac{l/h - q^B}{l/h} < \frac{(1-q^B)(l/h - q^B)}{(1-q^B)-(q^G - l/h)} \end{aligned}$$

To show $\overline{\alpha} < \widehat{\alpha}$, it is sufficient to show that

$$\frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} - q^B < \frac{l/h - q^B}{l/h},$$

which is satisfied when $q^B < \frac{\delta(l/h)(1-l/h)}{l/h+\delta(1-l/h)}$.

Combining Result 1, 2, 3, and 4, we have shown that, if $q^G > (1-q^B)(1-l/h) + l/h$ and $q^B < \frac{\delta(l/h)(1-l/h)}{l/h+\delta(1-l/h)}$, there exists $\overline{\alpha} \in (\gamma^*, \widehat{\alpha})$ such that, for $\alpha \in (\overline{\alpha}, \widehat{\alpha})$, $V_5 > W_2 > W_1 > \max\{V_2, V_4\}$. Therefore, V_5 is the optimal revenue in our model and it is higher than the optimal revenue in the two-period version of Hart and Tirole (1988).

(iii) For a pessimistic seller or moderately pessimistic seller ($\alpha < \gamma^*$), there always exists a pooling equilibrium in which the seller offers $p_1 = p_2 = l$ and all buyer types accept the offer as shown in Proposition 1 and 3. This equilibrium yields revenue $l + \delta l$, which is the same as in Hart and Tirole's (1988). ■

Appendix B: Criterion D_1

The following definition of Criterion D_1 is modified from Cho and Kreps (1987).

Consider a fixed equilibrium on the continuation of p_1, with action $a_1 \in \{0,1\}$ reached with zero probability. Suppose $x(p_1,1)$ and $x(p_1,0)$ is the seller's equilibrium strategy.

Step 1: Find the sets of all (mixed) responses ϕ by the seller that would cause type $\theta_1 = (d, v_1)$ to defect from the equilibrium and to be indifferent. If $a_1 = 0$ is the out-of-equilibrium action, form the sets

$$D_{\theta_1} \equiv \{\phi : (v_1 - p_1) + \delta q^d x(p_1,1)(h-l) < \delta q^d \phi(h-l), \phi \in [0,1]\},$$

$$D^0_{\theta_1} \equiv \{\phi : (v_1 - p_1) + \delta q^d x(p_1,1)(h-l) = \delta q^d \phi(h-l), \phi \in [0,1]\}.$$

If $a_1 = 1$ is the out-of-equilibrium action, form the sets

$$D_{\theta_1} \equiv \{\phi : (v_1 - p_1) + \delta q^d \phi(h-l) > \delta q^d x(p_1,0)(h-l), \phi \in [0,1]\},$$

$$D^0_{\theta_1} \equiv \{\phi : (v_1 - p_1) + \delta q^d \phi(h-l) = \delta q^d x(p_1,0)(h-l), \phi \in [0,1]\}.$$

Step 2: For a given out-of-equilibrium action a_1, if for some type θ_1 there exists a second type $\widetilde{\theta}_1$ with $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{\widetilde{\theta}_1}$, then the combination (θ_1, a_1) may be pruned from the continuation game following p_1.

Step 3: Check whether the fixed equilibrium is still sequentially rational given that the seller's belief is restricted to the buyer types who survive from Step 2. If not, then the equilibrium does not survive from D_1.

Given a PBE, if the corresponding equilibrium in all the continuation games following $p_1 \in \mathbb{R}$ survives from D_1, then we say that the PBE survives from D_1.

The effect of applying criterion D_1 in our model is summarized in the following lemma.

Lemma 6 *The equilibrium in the continuation game can not pass criterion D_1 if all buyer types accept $p_1 > l$ or all buyer types reject $p_1 < h$; The equilibrium in the continuation game passes criterion D_1 if all buyer types accept $p_1 \leq l$ or all buyer types reject $p_1 \geq h$.*

Proof. [Proof of Lemma 6] **Part 1:** Suppose all buyer types accept $p_1 > l$. Then $x(p_{1,}1) > x(p_1,0)$ and $x(p_{1,}1) = 1$ without considering the knife-edge case that $\alpha = \gamma^*$. Since $\max\{x(p_{1,}1) - x(p_1,0)\} = 1$ and all types accept p_1, $p_1 \leq \min_{(d,v_1)} \{v_1 + \delta q^d(h-l)\} = l + \delta q^B(h-l)$ by the definition of cutoff value.

Apply the definition of D_1 in the case that $a_1 = 0$ is the out-of-equilibrium message and form the sets D_{θ_1} and $D^0_{\theta_1}$ for each buyer type θ_1. So $D_{\theta_1} = \{\phi : \phi > x(p_1,1) + \frac{v_1 - p_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$ and $D^0_{\theta_1} = \{\phi : \phi = x(p_1,1) + \frac{v_1 - p_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$. Therefore, for $x(p_{1,}1) = 1$ and $p_1 \in (l, l + \delta q^B(h-l)]$, $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{(B,l)}$ for all $\theta_1 \neq (B,l)$. All the combinations $(\theta_1, a_1 = 0)$ with $\theta_1 \neq (B,l)$ are pruned from the game. Given the seller's belief is restricted on type (B,l) after rejection, $x(p_1,0) = 1$ and it is contradictory to $x(p_{1,}1) > x(p_1,0)$. So the equilibrium fails to pass criterion D_1.

Part 2: Suppose all buyer types accept $p_1 \leq l$. From Part 1, $D_{\theta_1} = \{\phi : \phi > x(p_1,1) + \frac{v_1 - p_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$ and $D^0_{\theta_1} = \{\phi : \phi = x(p_1,1) + \frac{v_1 - p_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$.

If $p_1 = l$ and $\alpha < \gamma^*$, $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for $\theta_1 \in \{(B,h),(G,h)\}$ and $D_{\theta_1} \cup D^0_{\theta_1} = \{1\}$ for $\theta_1 \in \{(B,l),(G,l)\}$.

If $p_1 = l$ and $\alpha > \gamma^*$, $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for $\theta_1 \in \{(B,h),(G,h)\}$ and $D_{\theta_1} \cup D^0_{\theta_1} = [0,1]$ for $\theta_1 \in \{(B,l),(G,l)\}$.

If $p_1 < l$ and $\alpha < \gamma^*$, then $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for all buyer types θ_1.

If $p_1 < l$ and $\alpha > \gamma^*$, then either $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for all buyer types θ_1 or $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{(G,l)}$. If the latter happens, the seller's belief is restricted on type (G,l) after rejection and she offers $x(p_1,0) = 0$. It is still sequential rational for all buyer types θ_1 to accept $p_1 < l$ given $x(p_1,1) = x(p_1,0) = 0$.

In all the cases above, the equilibrium passes criterion D_1.

Part 3: Suppose all buyer types reject $p_1 < h$. Then $x(p_{1,}0) > x(p_1,1)$ and $x(p_{1,}0) = 1$ without considering the knife-edge case that $\alpha = \gamma^*$. Since $\max\{x(p_{1,}0) - x(p_1,1)\} = 1$ and all types reject p_1, $p_1 \geq \max\limits_{(d,v_1)}\{v_1 - \delta q^d(h-l)\} = h - \delta q^B(h-l)$ by the definition of cutoff value.

Apply the definition of criterion D_1 in the case that $a_1 = 1$ is the out-of-equilibrium message. So $D_{\theta_1} = \{\phi : \phi > x(p_1,0) + \frac{p_1 - v_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$ and $D^0_{\theta_1} = \{\phi : \phi = x(p_1,0) + \frac{p_1 - v_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$. Then for $x(p_{1,}0) = 1$ and $p_1 \in [h - \delta q^B(h-l), h)$, $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{(B,h)}$ for all $\theta_1 \neq (B,h)$. All the combinations $(\theta_1, a_1 = 1)$ with $\theta_1 \neq (B,h)$ are pruned from the game. Given the seller's belief is restricted on type (B,h) after acceptance, $x(p_{1,}1) = 1$ and it is contradictory to $x(p_{1,}0) > x(p_1,1)$. So the equilibrium fails to pass Criterion D_1.

Part 4: Suppose all buyer types reject $p_1 \geq h$. From Part 3, $D_{\theta_1} = \{\phi : \phi > x(p_1,0) + \frac{p_1 - v_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$ and $D^0_{\theta_1} = \{\phi : \phi = x(p_1,0) + \frac{p_1 - v_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$.

If $p_1 = h$ and $\alpha < \gamma^*$, $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for $\theta_1 \in \{(B,l),(G,l)\}$ and $D_{\theta_1} \cup D^0_{\theta_1} = \{1\}$ for $\theta_1 \in \{(B,h),(G,h)\}$.

If $p_1 = h$ and $\alpha > \gamma^*$, $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for $\theta_1 \in \{(B,l),(G,l)\}$ and $D_{\theta_1} \cup D^0_{\theta_1} = [0,1]$ for $\theta_1 \in \{(B,h),(G,h)\}$.

If $p_1 > h$ and $\alpha < \gamma^*$, then $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for all buyer types θ_1.

If $p_1 > h$ and $\alpha > \gamma^*$, then either $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for all buyer types θ_1 or $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{(G,h)}$. If the latter case happens, the seller's belief is restricted on type (G,h) after acceptance and $x(p_1,1) = 0$. Then it is still sequential rational for all buyer types θ_1 to reject $p_1 > h$ given $x(p_1,1) = x(p_1,0) = 0$.

In all the cases above, the equilibrium passes criterion D_1. ■

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