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Economic Modeling for Optimal Trading of Financial Asset in Volatile Market

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Abstract

We build an optimal trading model for submitting market orders in volatile market. We show some analytical properties of our computational solution. We conduct numerical simulations to investigate the model performance. In comparison with other two alternative models, the simulation results show that the performance of our model is generally superior, particularly when the market turns to be extremely bullish or bearish.

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1 Introduction

Large market orders placed by institutional investors have market impact on asset price. When the market is volatile, these orders will strengthen market volatility and increase overall transaction costs. Chiyachantana et al. (2004) point out the underlying market condition is a major determinant of the price impact and there exists the asymmetry between price impacts of institutional buy and sell orders. In order to reduce the market impact, institutional traders often to split the large order into smaller pieces and subsequently submit them throughout a predetermined trading period. The challenge is how to optimally distribute the entire order to several single submissions in the market so that the overall market impact can be minimized. Bertsimas and Lo (1998), Almgren and Chriss (2000), Almgren (2003), Obizhaeva and Wang (2013), Almgren and Lorenz (2007), Schöneborn and Schied (2009), Alfonis et al. (2010), and Sun et al. (2013) have addressed problems of this type. Financial institutions develop algorithmic trading techniques to manage the order and reduce the market impact with respect to optimal trade execution. Several studies provide the computational algorithms for solving the optimal strategy, see Sun et al. (2013) and references therein.

We consider the problem of optimal trading (execution) problem in a limit-order-book market where the trade occurs when buy and sell orders match (see Bertsimas and Lo (1998), Almgren and Chriss (1999, 2000), and Obizhaeva and Wang (2013)). In our model, we assume the underlying price dynamics follows a geometric Brownian motion instead of using a Brownian motion for the price dynamics by Obizhaeva and Wang (2013) and martingale by Alfonsi et al. (2010). The advantage of using geometric Brownian motion is that we can model jumps in the underlying asset prices with a variable for drift. In this way, we can model both bullish and bearish markets by assuming (or estimating) a significant value of positive or negative drift respectively. It makes the model more flexible, particularly for the volatile market to capture trading activities of driving down and moving up the market after some large trades. In addition, a geometric Brownian motion guarantees there are no negative asset prices (see Alfonsi et al. 2010). We show that it is important to incorporate jumps (i.e., unexpected price changes) of market when the institutional investors looking for the optimal trading strategy. The higher the altitude market moves to one direction (upward or downward), the more important the drift variable considered in our model turns to be. We show that the market volatility (measured by the variance of underlying price changes) has no significant influence on our optimal trading model.

2 The Model

A representative trader seeks to execute an order with size $X_0 (X_0 > 0)$ for a security during a given trading period $[0, T] (T > 0)$. This order is a market order for buying or selling certain amount of securities. In this paper, we focus only on buying order, since this model can be easily adopted for selling order. In this model, the trader is only allowed to trade at discrete time.

The trader is only allowed to buy certain number of securities at time point $N+1$ during the trading period, which are equidistantly distributed and not in between of them. $N \in \{1, 2, \ldots\}$ stands for the trading frequency. $t_i (i \in 0, \ldots, N)$ are time points starting at $t_0 = 0$ and ending in $t_N = T$. We then write $t_i = i\tau$, where $\tau = T/N$ is the length of the duration between two successive time points being able to trade. We define x_{t_n} as the number of securities we buy at time point t_n. Then we have $X_0 = \sum_{n=0}^{N} x_{t_n}$. We define $X_{t_n} = X_0 - \sum_{i=0}^{n-1} x_{t_i}$ as the remaining order to be executed at the time t before trading at t_n. The trader is not allowed to sell securities before the buying task is completed, i.e., $x_{t_n} \geq 0$. The space of feasible strategies is then defined as follows:

$$\Phi = \left\{ \{x_{t_0}, \ldots, x_{t_N}\} : x_{t_n} \geq 0 \; \forall \, n \in \{0, \ldots, N\} \, ; \sum_{n=0}^{N} x_{t_n} = X_0 \right\}.$$

We assume the asset price movement follows a geometric Brownian motion with the drift μ, variance σ, and initial value $F_0 = A_0$.

$$F_t = F_0 \exp\big((\mu - \frac{\sigma^2}{2})t + \sigma B_t\big), \tag{1}$$

which is the solution to

$$dF_t = \mu F_t dt + \sigma F_t dB_t,$$

where $\sigma > 0$ and $(B_t)_{t>0}$ is a Brownian motion defined on a given probability space (Ω, F, P) with the natural filtration $(\mathcal{F}_t)_{t>0}$ through B_t generated. We assume the bid-ask spread $s > 0$ is constant and symmetric around the asset price. The limit-order book is modeled by using a constant depth q, which means that we increase the price by 1 unit, if we execute a buy order of the size q. In general this translates into the price impact of an order x_{t_n} is x_{t_n}/q. The average price impact of the whole order is $x_{t_n}/2q$. We decompose the price impact into the permanent and temporary price impact. The price impact of an order x_{t_n} has two parts $x_{t_n}/q = \lambda x_{t_n} + \kappa x_{t_n}$, where $0 \leq \lambda \leq 1/q$ is the percentage of the permanent price impact and $\kappa = 1/q - \lambda$ is the percentage of the temporary price impact contributed respectively to the total price impact. We call the term λx_{t_n} with $0 \leq \lambda \leq 1/q$ the permanent price impact of the trade x_{t_n} and κx_{t_n} with $\kappa = 1/q - \lambda$ the temporary price impact. The temporary price impact of an order vanishes along with time and we use a resilience factor $\rho > 0$ to describe it. The part of the temporary price impact of the order x_{t_n} that remains until $t > t_n$ is $\kappa x_{t_n} e^{-\rho(t-t_n)}$, where $\rho > 0$ is the resilience factor.

The temporary price impact at time point t_n before we make a trade is defined as $D_{t_n} = \sum_{i=0}^{n-1} x_{t_i} \kappa e^{-\rho(t_n - t_i)}$. To simplify, we use the following proposition.

Proposition 1 *The temporary price impact D_{t_n} at t_n before we make a trade, satisfies the recursive equation $D_{t_n} = (D_{t_{n-1}} + \kappa x_{t_{n-1}})e^{-\rho\tau}$ with an initial condition $D_0 = 0$.*

Proof 1 *The starting condition $D_0 = 0$ is satisfied because we make no trades before $t_0 = 0$ and we are the only trader who create the temporary price impact. For $n = 1$ we use the definition of D_{t_n} to define that $D_{t_n} = x_{t_0}\kappa e^{-\rho(t_1-t_0)} = (0 + x_{t_0}\kappa)e^{-\rho\tau}$, which satisfies the recursive equation. Then we have*

$$\begin{aligned} D_{t_n} &= \sum_{i=0}^{n-1} x_{t_i}\kappa e^{-\rho(t_n-t_i)} \\ &= (\sum_{i=0}^{n-2} x_{t_i}\kappa e^{-\rho(t_{n-1}-t_i)} + x_{t_{n-1}}\kappa)e^{-\rho(t_n-t_{n-1})} \\ &= (D_{t_{n-1}} + x_{t_{n-1}}\kappa)e^{-\rho(t_n-t_{n-1})}. \end{aligned}$$

The objective function of our model is to minimizes the expected cost of the whole order, that is,

$$\min_{x_0,\dots,x_T} \Big(E\sum_{n=0}^{N} x_{t_n}\big(F_{t_n} + \frac{s}{2} + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{x_{t_n}}{2q})\Big).$$

3 Analytical Solution

We derive the following propositions to find the solution based on Obizhaeva and Wang (2013).

Proposition 2 *For the geometric Brownian motion F_t defined by Equation (1), the following equations*

$$E_{t-1}[F_t] = F_{t-1}e^{\mu\tau} \tag{2}$$

and

$$E_{t-1}[F_t^2] = F_{t-1}^2 e^{(2\mu+\sigma^2)\times\tau} \tag{3}$$

are valid.

Proposition 3 *Given the model setting described in Section 2, the strategy $x_{t_N} = X_{t_N}$ and*

$$\begin{aligned} x_{t_n} &= -\frac{1}{2}\delta_{n+1}\big((-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa)X_{t_n} \\ &\quad +(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau})D_{t_n} \\ &\quad +(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a)F_{t_n}\big) \end{aligned} \tag{4}$$

for every $t_n \in \{t_0, \dots, t_{N-1}\}$ is optimal, if $(x_{t_0}, \dots, x_{t_N}) \in \Phi$. The optimal value function then has the form

$$\begin{aligned} J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n) &= (a^{N-n}F_n + \frac{s}{2})X_{t_n} + \lambda X_0 X_{t_n} + b_n X_{t_n}^2 \\ &+c_n D_{t_n}^2 + d_n F_{t_n}^2 + g_n X_{t_n} D_{t_n} + h_n X_{t_n} F_{t_n} + l_n D_{t_n} F_{t_n}, \end{aligned} \tag{5}$$

where $a = e^{\mu\tau}$, $m = e^{(2\mu+\sigma^2)\times\tau}$, *and the coefficients are given as follows:*

$$\begin{aligned}
b_n &= b_{n+1} - \frac{1}{4}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa)^2, \\
c_n &= c_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau})^2, \\
d_n &= d_{n+1}m - \frac{1}{4}\delta_{n+1}(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a)^2, \\
g_n &= g_{n+1}e^{-\rho\tau} - \frac{1}{2}\delta_{n+1}(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau}) \\
&\quad (-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa), \\
h_n &= h_{n+1}a - \frac{1}{2}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa) \\
&\quad (1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a), \\
l_n &= l_{n+1}e^{-\rho\tau}a - \frac{1}{2}\delta_{n+1}(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a) \\
&\quad (1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau}), \\
\delta_{n+1} &= (\frac{1}{2q} + b_{n+1} - g_{n+1}\kappa e^{-\rho\tau} + c_{n+1}\kappa^2 e^{-2\rho\tau})^{-1}, \qquad (6)
\end{aligned}$$

for every $t_n \in \{t_0, \ldots, t_{N-1}\}$*, and the terminal conditions given by*

$$b_N = \frac{1}{2q} - \lambda, c_N = 0, d_N = 0, g_N = 1, h_N = 0, \ and, \ l_N = 0. \qquad (7)$$

In proposition (3) the optimal strategy at time point t_N is given by $x_{t_N} = X_{t_N}$, because in the last period the trader has no better possibility but to complete the whole order immediately, so that the constraint $\sum_{n=0}^{N} x_{t_n} = X_0$ will not be violated. We can transform the last equation to obtain $x_{t_N} = X_{t_N} = X_0 - \sum_{n=0}^{N-1} x_{t_n}$, which means the trader has to buy the amount of remaining shares X_{t_n}.

From Equation (6) and (7) we can see that the variance σ^2 of the underlying price movement F_{t_n} only influences the parameter m, which in turn also appears in the recursive definition of the coefficient d_n. As the coefficient d_n has no impact on other coefficients, it will not directly influence the optimal strategy. Therefore, we see that the optimal strategy does not depend on the variance of underlying price. The reason for this observation is that the trader only focuses on completing the whole trade at the given trading period.

4 Numerical Study

We investigate performance of our optimal trading model (S-K) with numerical simulations. We compare our model with that of Obizhaeva and Wang (2013) (O-W) and the naive trading (Naive). We use the volume weighted average price (VWAP) as the benchmark to decide the during-trading cost measure of these trading strategies (see, for example Werner (2003) and Goldstein et al. (2009)). Following Obizhaeva and Wang (2013), we predefine the variables that describe the limit order book as $q = 5,000$, $\lambda = 1/2q$, $\kappa = 1/q - \lambda$, $\rho = 2.2$, $N = 9$, $X_0 = 100,000$, $T = 1$, $\mu \in \{-2\%, -1\%, 0\%, 1\%, 2\%, 3\%\}$, and $\sigma \in \{1\%, 2\%, 3\%, 5\%, 10\%\}$, for this study.

Figure 1 illustrates the trading behavior of our model. We see that our strategy changes along with the underlying asset price driven by the geometric Brownian motion, which different from the optimal trading model suggested by Obizhaeva and Wang (2013). But we still observe the U-shape trading behavior suggested by Obizhaeva and Wang (2013).

We conduct the simulation for 100,000 runs in order to verify the consistence of statistical significance. We report the results in Table 1. We see that our model has the smallest value of VWAP and is preferred to the alternatives. We summarize the results by Figure 2. From Figure 2, we can see when $\mu \neq 0\%$, which means there exists either the bullish ($\mu > 0\%$) or the bearish ($\mu < 0\%$) market, our optimal trading model performs superior than both alternatives trading models, which means our optimal trading model has lowest trading cost. When $\mu = 0\%$, the performance of our trading model coincides with that of Obizhaeva and Wang (2013). We can see that our optimal trading model performs significantly better when the market turns to be extremely upward (bullish) or downward (bearish).

5 Conclusion

In this paper, we construct an optimal trading model based on the resilience models discussed in the literature. In our model, we allow the underlying price dynamics follow the geometric Brownian motion that makes the model much flexible to capture price jumps. We adopt the linear price impact function in our model, that is, a linear combination of the permanent and temporary price impact is characterized by the price impact function. We also focus on the discrete-time setting as other studies documented in literature. When using the VWAP as the trading cost measure, numerical results given by the simulations show that our optimal trading model performs generally better. In addition, our model significantly dominates the naive trading strategy and strategy suggested by Obizhaeve and Wang (2013) when the market experiences extreme bullish or bearish situation.

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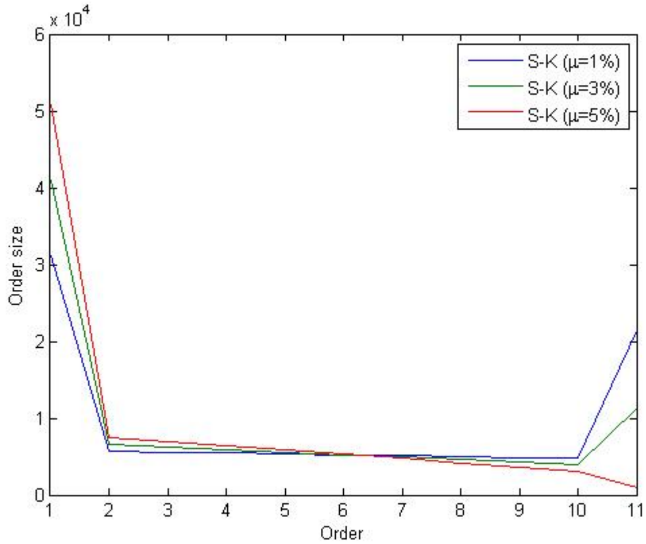

Figure 1: Optimal order behavior of the proposed model with different values of μ.

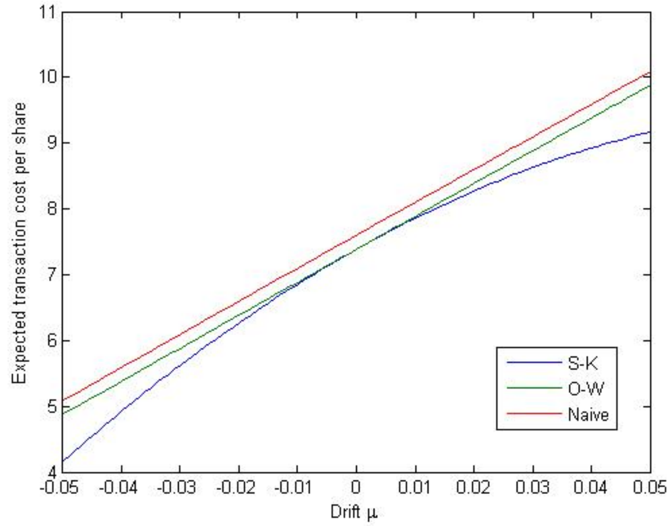

Figure 2: Comparison of expected transaction cost for different models.

Table 1: Comparison of different models using the expected value of VWAP and its variance (shown in parenthesis) for 100,000 runs of simulation for different combinations of parameters of the geometric Brownian motion.

		μ=-10%	μ=-5%	μ=-1%	μ=1%	μ=5%	μ=10%
	S-K	99.7665 (0.8944)	104.1680 (0.6446)	106.8575 (0.3349)	107.8566 (0.2189)	109.1661 (0.0667)	110.0000 (0.0000)
σ=1%	O-W	102.3428 (0.2843)	104.8722 (0.2775)	106.8836 (0.2717)	107.8846 (0.2706)	109.8751 (0.2678)	112.3470 (0.2649)
	Naive	102.5205 (0.3430)	105.0804 (0.3308)	107.0991 (0.3195)	108.0996 (0.3157)	110.0852 (0.3079)	112.5257 (0.2996)
	S-K	99.7616 (1.7799)	104.1683 (1.2946)	106.8600 (0.6761)	107.8591 (0.4357)	109.1654 (0.1350)	110.0000 (0.0000)
σ=2%	O-W	102.3452 (0.5693)	104.8731 (0.5587)	106.8849 (0.5474)	107.8877 (0.5420)	109.8751 (0.5359)	112.3465 (0.5244)
	Naive	102.5156 (0.6840)	105.0815 (0.6544)	107.0967 (0.6383)	108.0985 (0.6288)	110.0776 (0.6180)	112.5278 (0.5944)
	S-K	99.7636 (2.6755)	104.1652 (1.9306)	106.8549 (1.0053)	107.8584 (0.6580)	109.1673 (0.2010)	110.0000 (0.0000)
σ=3%	O-W	102.3412 (0.8549)	104.8751 (0.8341)	106.8842 (0.8205)	107.8837 (0.8168)	109.8774 (0.7947)	112.3462 (0.7962)
	Naive	102.5217 (1.0204)	105.0817 (0.9832)	107.1025 (0.9670)	108.0946 (0.9387)	110.0793 (0.9249)	112.5320 (0.8967)
	S-K	99.7642 (4.4405)	104.1635 (3.2364)	106.8564 (1.6824)	107.8575 (1.0920)	109.1612 (0.3347)	110.0000 (0.0000)
σ=5%	O-W	102.3369 (1.4307)	104.8707 (1.3905)	106.8775 (1.3697)	107.8805 (1.3625)	109.8777 (1.3371)	112.3448 (1.3075)
	Naive	102.5203 (1.7104)	105.0740 (1.6394)	107.0941 (1.5983)	108.0988 (1.5829)	110.0802 (1.5415)	112.5235 (1.4900)
	S-K	99.7716 (8.8818)	104.1672 (6.4790)	106.8683 (3.3227)	107.8617 (2.1836)	109.1643 (0.6683)	110.0000 (0.0000)
σ=10%	O-W	102.3365 (2.8652)	104.8667 (2.7721)	106.8834 (2.7429)	107.8790 (2.7345)	109.8881 (2.6655)	112.3422 (2.6464)
	Naive	102.5227 (3.4170)	105.0831 (3.2650)	107.0944 (3.2299)	108.0951 (3.1675)	110.0706 (3.0917)	112.5368 (2.9909)