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Minimum LM unit root test with one structural break

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Abstract

In this paper, we consider the minimum Lagrange Multiplier (LM) unit root test with one structural break in intercept and trend. This paper complements the earlier work of Lee and Strazicich (2003), who consider the minimum LM unit root test with two breaks. The asymptotic properties are derived, critical values are provided, and size and power properties are examined. The one-break minimum LM unit root test is valid in the presence of a break under the null and alternative hypotheses and is free of spurious rejections.

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1. Introduction

The importance of allowing for structural breaks in unit root tests is now well documented in the literature. Whereas Perron (1989) assumed that the break point was known *a priori*, or exogenously given, the subsequent literature has allowed for the break to be determined from the data. We refer to such tests as *endogenous break unit root tests*.

An important issue in many earlier endogenous break unit root tests is that they omit the possibility of a unit root with break. However, if a break occurs in the presence of a unit root then two undesirable outcomes can occur. First, in the presence of a break these earlier tests exhibit "spurious rejections" that increase with the magnitude of the break; see, for example, the discussion in Nunes, Newbold, and Kuan (1997), Lee and Strazicich (2001), and Byrne and Perman (2007). As a result, researchers may incorrectly conclude that a time series is stationary with break, or "trend-break stationary," when in fact the series is nonstationary with break. It is important to note that this nuisance parameter problem is restricted to the endogenous break unit root tests and does not occur in exogenous break unit root tests. For example, the asymptotic distribution of Perron's (1989) exogenous break unit root test does not depend on the magnitude of the break, even when the break occurs in the presence of a unit root.

A second consequence of utilizing many earlier endogenous break unit root tests is that the break point is incorrectly determined. As shown in Lee and Strazicich (2001), these earlier tests tend to identify the break one period prior to the true break where bias in estimating the persistence parameter is maximized and spurious rejections are the greatest. Moreover, this problem occurs under both the null and alternative hypotheses.[1] In the present paper, we utilize the theoretical findings in Lee and Strazicich (2003) and develop an endogenous one-break unit root test that is free of the above problems. Similar to the two-break minimum LM unit root test, the one-break test is free of bias and spurious rejections under the null and alternative hypotheses. Moreover, it is important to note that there can be many situations where a one break test is preferred since including unnecessary breaks can lead to loss of power.[2]

The remainder of the paper is organized as follows. In Section 2, we discuss properties of the minimum LM unit root test in the presence of a structural break. Section 3 describes the asymptotic properties of the one-break LM unit root test and derives invariance results. Section 4 provides simulations to examine finite sample properties of size and power. We summarize and conclude in Section 5.

[1] The problem of bias and spurious rejections in endogenous break unit root tests is not restricted to behavior under the null. In the presence of a break, the null distribution shifts leftward. As a result, the null hypothesis is rejected too often even when the alternative is true unless size-corrected critical values are adopted. Thus, while many earlier endogenous break unit root tests can appear more powerful, this outcome is often simply a reflection of the size distortions under the null.

[2] It would be helpful to note that the present paper complements the work of Hassler and Rodrigues (2004) who considered seasonal unit root tests with structural breaks. They note that the LM type seasonal unit root tests are asymptotically robust to seasonal mean shifts of finite magnitude and can overcome the problems of size distortions and power reduction found in other types of seasonal unit root tests when structural breaks are present. Nunes and Rodrigues (2011) also considered LM type tests for seasonal unit roots in the presence of a break in trend under the null and alternative hypotheses.

2. Testing procedures

Consider the following data generating process (DGP) based on the unobserved components model:

$$y_t = \delta' Z_t + X_t, \quad X_t = \beta X_{t-1} + \varepsilon_t, \tag{1}$$

where Z_t contains exogenous variables. The unit root null hypothesis is described by $\beta = 1$. If $Z_t = [1, t]'$, then the DGP is the same as that shown in the no break LM unit root test of Schmidt and Phillips (1992, hereafter SP). We consider two models of structural change. "Model A" is known as the "crash" model, and allows for a one-time change in intercept under the alternative hypothesis. Model A can be described by $Z_t = [1, t, D_t]'$, where $D_t = 1$ for $t \geq T_B+1$ and zero otherwise, T_B is the time period of the structural break, and $\delta' = (\delta_1, \delta_2, \delta_3)$.[3] "Model C" allows for a shift in intercept and change in trend slope under the alternative hypothesis and can be described by $Z_t = [1, t, D_t, DT_t]'$, where $DT_t = t - T_B$ for $t \geq T_B + 1$, and zero otherwise.

According to the LM (score) principle, unit root test statistics are obtained from the following regression:

$$\Delta y_t = \delta' \Delta Z_t + \phi \tilde{S}_{t-1} + u_t, \tag{2}$$

where $\tilde{S}_t = y_t - \tilde{\psi}_x - Z_t\tilde{\delta}$, $t=2,..,T$; $\tilde{\delta}$ are the coefficients in the regression of Δy_t on ΔZ_t; and $\tilde{\psi}_x$ is the restricted MLE of ψ_x $(\equiv \psi + X_0)$ given by $y_1 - Z_1\tilde{\delta}$. Note that the testing regression (2) involves ΔZ_t instead of Z_t. Therefore, ΔZ_t is described by $[1, B_t]'$ in Model A and $[1, B_t, D_t]'$ in Model C, where $B_t = \Delta D_t$ and $D_t = \Delta DT_t$. Thus, B_t and D_t correspond to a change in intercept and trend under the alternative, and to a one period jump and (permanent) change in drift under the null hypothesis, respectively. The unit root null hypothesis is described by $\phi = 0$ and the LM unit root test statistic is given by:

$$\tilde{\tau} = \text{statistic testing the null hypothesis, } \phi = 0. \tag{3}$$

The location of the break (T_B) is determined by searching all possible break points to find the minimum (i.e., the most negative) unit root test statistic as follows:

$$Inf\ \tilde{\tau}(\tilde{\lambda}) = \underset{\lambda}{Inf}\ \tilde{\tau}(\lambda), \quad \text{where } \lambda = T_B/T \text{ and } \lambda \in [0,1]. \tag{4}$$

[3] When $Z_t = [1, t, DT_t^*]'$, the model becomes the "changing growth" Model B, where $DT_t^* = t$ for $t \geq T_B+1$ and zero otherwise. Model B will not be examined here as most economic time series can be adequately described by Model A or C (see, for example, Table VII in Perron, 1989).

To correct for autocorrelated errors, we include augmented terms $\Delta\tilde{S}_{t-j}$, j = 1,..., k in (2) as in the standard ADF test. We utilize a general to specific procedure to determine the optimal number of k augmented terms.[4]

3. Asymptotic and invariance properties

To examine the asymptotic distribution of the minimum LM unit root test, we define $V(r)$ as a standard Brownian bridge over the interval [0, 1], and $\underline{V}(r)$ as the demeaned Brownian bridge (see SP, equation (23)). The asymptotic distribution of the minimum LM unit root test in Model A is described as follows.[5]

Theorem 1. *Assume that (i) the data are generated according to (1) with* $Z_t = (1, t, D_t)'$, *(ii) the innovations* ε_t *satisfy the regularity conditions of Phillips and Perron (1988, p. 336), and (iii)* $T_B/T \to \lambda$ *as* $T \to \infty$. *Then, under the null hypothesis that* $\beta = 1$,

$$Inf\ \tilde{\tau}(\tilde{\lambda}) \to \underset{\lambda}{Inf}\,[-\frac{1}{2}\int_0^1 \underline{V}(r)^2]^{-1/2} \quad . \tag{5}$$

Proof in Appendix. [6]

An important implication of Theorem 1 is that the asymptotic distribution in Model A does not depend on the size (δ) or location ($\lambda = T_B/T$) of the break under the null. As a result, in the presence of a break under the null it is not necessary to simulate new critical values in empirical applications since critical values are invariant to the break. This outcome follows from the invariance properties in the exogenous one-break LM unit root test of Amsler and Lee (1995) and is due to the method of de-trending in the LM test. As a result, the one-break minimum LM unit root test is free of the spurious rejections found in the earlier ADF-type endogenous break unit root tests. For example, in these earlier tests the asymptotic null distribution in Model A depends on the location of the break (λ) through the projection residual $W(\lambda, r)$ of a Brownian motion projected onto the subspace generated by $[1, r, d(\lambda,r)]$, where $d(\lambda,r) = 1$ if $r > \lambda$ and 0 otherwise. To make these tests more practical, the authors typically omit B_t under the null and assume $d = 0$ in (6a) and $\alpha_2 = 0$ in (6b) as follows:

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[4] We determine k by following the general to specific procedure suggested in Perron (1989). We begin with a maximum number of lagged first-differenced terms k = 8 and examine the last term to see if it is significantly different from zero at the 10% level (critical value in an asymptotic normal distribution is 1.645). If insignificant, the maximum lagged term is dropped and the model re-estimated with k = 7 terms. The procedure is repeated until either the maximum term is found or k = 0, at which point the procedure stops. This technique has been shown to perform well as compared to other data-dependent procedures to select the number of augmented terms in unit root tests (Ng and Perron, 1995).

[5] Throughout the paper, the symbol "$\to$" denotes weak convergence of the associated probability measure.

[6] See Lee and Strazicich (2003) for a proof of Model C in the context of two breaks in level and trend. The proof is similar for the Model C version of the one break minimum LM unit test and is omitted here to conserve space.

Null $$y_t = \mu_0 + dB_t + y_{t-1} + v_t \quad (6a)$$

Testing Regression $$y_t = \alpha_0 + \alpha_1 t + \alpha_2 B_t + \alpha_3 D_t + \beta y_{t-1} + \sum_{j=1}^{k} c_j \Delta y_{t-j} + e_t\ . \quad (6b)$$

For example, the Zivot and Andrews (1992, ZA hereafter) minimum one-break unit root test statistic depends on the magnitude of the break under the null and exhibits spurious rejections that increase as the break size increases. Thus, while the ZA test is valid if $d = 0$, the test can lead to incorrect inference when $d \neq 0$. Lee and Strazicich (2001) investigate this issue further, and find that regardless of whether B_t is included or excluded in the ZA test regression spurious rejections remain. The problem is that these ADF-type endogenous break unit root tests tend to select the break point incorrectly at T_{B-1}, where bias in estimating β, the coefficient that tests for a unit root, and spurious rejections are the greatest. [7]

While accuracy of estimating the break point with the minimum LM unit root test does not matter under the null, it does matter when the alternative is true. Namely, as Perron (1989) initially showed, failure to allow for an existing structural break leads to a bias in unit root tests that makes it more difficult to reject a false null hypothesis. If the magnitude of the break is large, the minimum LM unit root test estimates the break point fairly well. When the magnitude of the break is small, the break point cannot be accurately estimated, but the test does not suffer a significant loss of power in this case as this is similar to having no break.[8]

4. Simulation results

This section provides critical values and simulation results for the one-break minimum LM unit root test. To perform our simulations, we generate pseudo-iid $N(0,1)$ random numbers using the Gauss (version 11.0) RNDNS procedure, where the DGP has the form described in equation (1). Initial values of y_0 and ε_0 are assumed to be zero and σ_ε^2 is assumed to equal 1. All simulations are performed using 5,000 replications with $T = 100$ and a break at $T_B = 50$. Critical values for Models A and C are provided in Table 1. Since critical values for Model C depend (somewhat) on the location of the break, we provide critical values for a variety of break locations. Critical values at additional break points can be interpolated.

[7] In an alternative approach, Harvey, Leybourne, and Newbold (2001) suggest modifying the one-break ADF-type endogenous unit root test by moving the break point forward one period to $\hat{T}_B^* = 1 + \hat{T}_B$, where $\hat{T}_B^*$ is the revised break point and $\hat{T}_B$ is the estimated break.

[8] Asymptotic properties for the one-break minimum LM unit root test are similar for Model C, except that the test statistic is no longer invariant to the location of a break under the null. However, simulation results show that even though the minimum LM test for Model C is not invariant to the location of a break under the null, it is nearly so and remains free of spurious rejections. See Lee and Strazicich (2003) for discussion of the asymptotic properties of Model C in the context of two breaks in level and trend.

Table 1. Critical Values of the One-Break Minimum LM Unit Root Test

Model A

1%	5%	10%
-4.239	-3.566	-3.211

Model C

λ	1%	5%	10%
.1	-5.11	-4.50	-4.21
.2	-5.07	-4.47	-4.20
.3	-5.15	-4.45	-4.18
.4	–5.05	-4.50	-4.18
.5	-5.11	-4.51	-4.17

Note: All critical values were derived in samples of size T = 100. Critical values in Model C (intercept and trend break) depend (somewhat) on the location of the break ($\lambda = T_B/T$) and are symmetric around λ and (1-λ). Model C critical values at additional break locations can be interpolated.

Properties of size and power, and accuracy of estimating the break, are examined in Table 2 for Model A.[9] Simulations are first performed for the case where the unit root null hypothesis is true ($\beta = 1$), and then where the alternative is true ($\beta = 0.8$). The size (frequency of rejections under the null) and power (frequency of rejections under the alternative) properties of the test are evaluated at the 5% significance level in each case.[10]

Table 2. Rejection Rates and Frequency of Estimated Break Points

Test	δ_3	5% Rej.	Emp. Crit.	Frequency of Estimated Break Points in the Range						
				T_B-5 ~ T_B -2	T_B-1	$\mathbf{T_B}$	T_B+1	T_B+2 ~ T_B+5	*T_B± 10*	*T_B± 30*
(a) Size Under the Null ($\beta = 1$)										
LM	0	.057	-3.62	.048	.015	.013	.010	.054	*.259*	*.721*
	4	.046	-3.53	.020	.006	.325	.005	.019	*.446*	*.809*
	6	.050	-3.56	.023	.013	.401	.009	.022	*.519*	*.832*
	8	.049	-3.56	.035	.019	.448	.018	.035	*.598*	*.861*
	10	.039	-3.48	.051	.031	.480	.029	.046	*.682*	*.877*
(b) Power Under the Alternative ($\beta = 0.8$)										
LM	0	.710	-5.20	.057	.012	.014	.015	.062	*.305*	*.745*
	4	.581	-4.91	.040	.026	.553	.027	.048	*.746*	*.907*
	6	.537	-4.70	.041	.028	.737	.028	.047	*.908*	*.962*
	8	.492	-4.64	.041	.018	.834	.017	.036	*.967*	*.982*
	10	.454	-4.63	.026	.014	.898	.013	.024	*.985*	*.991*

Note: All simulations were performed for Model A in samples of size T = 100.

[9] To conserve space, size and power properties are reported only for Model A. For detailed simulations of Model C in a two-break framework please see Lee and Strazicich (2003).

[10] Copies of the computer code to run the minimum LM unit root test for Model A and Model C are available on the web site http://www.cba.ua.edu/~jlee/gauss/.

The size property simulation results are reported in Table 2 (a). For example, with no break under the null (δ_3 = 0), column 3 indicates a 5.7% rejection rate, which is close to the nominal size of 5%. In the presence of a unit root with break (i.e., $\delta_3 \neq 0$), the LM unit root test statistic is relatively stable with approximately correct size across all break magnitudes. Desirable size properties can also be observed when examining the 5% empirical critical values in column 4, where the critical values are mostly invariant to the magnitude of a break under the null. Overall, we see that the one-break minimum LM unit root test has approximately the correct size and is free of spurious rejections in the presence of a unit root with break. This is true even for a relatively large break size of $\delta_3 = 10$.

The power property simulation results are reported in Table 2 (b). For the case of no break, δ_3 = 0, we see that the power to reject the null when the alternative is true is relatively high at 71%. As the magnitude of the break increases, the power of the test decreases (from 58% when δ_3 = 4 to 45% when δ_3 = 10), but remains relatively strong.

We next examine the accuracy of estimating the break point. Frequency of estimating the break point at different locations is shown in columns 5-11 in Table 2 (a) and (b). Setting the break at T_B = 50 in the DGP, the frequency of estimating the break at $T_B - 5$ to $T_B - 2$, $T_B - 1$, T_B, $T_B + 1$, $T_B + 2$ to $T_B + 5$, $T_B \pm 10$, and $T_B \pm 30$ is reported for different magnitudes of the break term, δ_3. As the magnitude of a break under the null increases, the frequency of estimating the break point correctly at T_B increases (becoming 48% at δ_3 = 10). Under the alternative, the story is similar, only more pronounced. As the size of the break increases, the minimum LM unit root test estimates the break point accurately with increasing frequency (90% at δ_3 = 10). These results make a sharp contrast when compared with the earlier endogenous one-break unit root tests that tend to estimate the break point incorrectly where bias and spurious rejections are the greatest; see Lee and Strazicich (2001).

5. Conclusion

This paper formally considers a minimum LM unit root test that endogenously determines one structural break in level and trend. Properties of the test were described and critical values presented. The one-break minimum LM unit root test tends to estimate the break point correctly and is free of spurious rejections. By combining the one-break LM unit root test presented here with the two-break LM unit root test in Lee and Strazicich (2003), researchers can more accurately consider the correct number of breaks in unit root tests.

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Appendix

Proof of Theorem 1

We employ the functional limit theory used in Phillips and Perron (1988) and utilize the results of Zivot and Andrews (1992, ZA hereafter) on continuity of the composite functional. First, we consider the following regression:

$$\Delta y_t = \delta(\lambda)'\Delta Z_t(\lambda) + \phi(\lambda)\tilde{S}_{t-1}(\lambda) + e_t , \qquad t = 2,..,T , \tag{A.1}$$

where $\tilde{S}_t(\lambda) = \sum_{j=2}^{t}\varepsilon_j - (\tilde{\delta}(\lambda)' - \delta(\lambda)')(Z_t(\lambda) - Z_1(\lambda))$, and the vector $Z_t(\lambda)$ includes deterministic terms such that $Z_t(\lambda) = [1, t, D_t]'$. Let $S_t = \sum_{j=2}^{t}\varepsilon_j$ and $[rT]$ be the integer part of rT, for $r \in [0,1]$. Following a procedure similar to ZA, we let $P_{\Delta Z}(\lambda) = \Delta z_T(\lambda)[\Delta z_T(\lambda)'\Delta z_T(\lambda)]^{-1}\Delta z_T(\lambda)$, and $M_{\Delta Z}(\lambda) = I - P_{\Delta Z}(\lambda)$, where $\Delta z_T(\lambda) = (\Delta z_{1,T}(\lambda),..,\Delta z_{T,T}(\lambda))'$. Pre-multiplying (A.1) by $M_{\Delta Z}(\lambda)$, we obtain:

$$M_{\Delta Z}(\lambda)\Delta Y = \phi(\lambda) M_{\Delta Z}(\lambda)\tilde{S}_1(\lambda) + M_{\Delta Z}(\lambda)\, e , \tag{A.2}$$

where $\Delta Y = (\Delta y_2,.., \Delta y_T)'$, $\tilde{S}_1(\lambda) = (\tilde{S}_1(\lambda),..,\tilde{S}_{T-1}(\lambda))'$ and $e = (e_{2,..,}e_T)'$. Then, the $Inf\ \tilde{\tau}(\tilde{\lambda})$ statistic can be written as:

$$Inf\ \tilde{\tau}(\tilde{\lambda}) = \inf_{\lambda}\ [T^{-2}\tilde{S}_1(\lambda)' M_{\Delta Z}(\lambda)\, \tilde{S}_1(\lambda)]^{-1/2}[T^{-1}\tilde{S}_1(\lambda)' M_{\Delta Z}(\lambda)\, e] / s_T(\lambda) , \tag{A.3}$$

where $s_T(\lambda)$ is the corresponding standard error of the regression. We then obtain:

$$T^{-2}\tilde{S}_1(\lambda)' M_{\Delta Z}(\lambda)\, \tilde{S}_1(\lambda) = \sigma^2 \int_0^1 [S_T(r) - P_{\Delta Z}(\lambda)\, S_T(r)]^2\, dr , \tag{A.4}$$

$$T^{-1}\tilde{S}_1(\lambda)' M_{\Delta Z}(\lambda)\, e = \sigma^2 \int_0^1 S_T(r)dS_T(r) - \sigma^2 \int_0^1 P_{\Delta Z}(\lambda)\, S_T(r)dS_T(r) . \tag{A.5}$$

The effect of applying $M_{\Delta Z}(\lambda)$ or $P_{\Delta Z}(\lambda)$ to the above expressions is twofold; one is to demean the process, and the other is to de-trend the structural dummy effect. We can establish the result that the effect of de-trending the structural break in the minimum LM unit root test vanishes asymptotically. To see this, we note that:

$$\int_0^1 \Delta z_T(\lambda,s)\Delta z_T(\lambda,s)'ds = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{A.6}$$

This is so because the term B_t in ΔZ_t is asymptotically negligible, whereas $\Delta Z_t = [1, B_t]'$. Then, it is clear that:

$$\int_0^1 [S_T(r) - P_{\Delta Z}(\lambda)\, S_T(r)]^2\, dr = \sigma^2 \int_0^1 \underline{V}_T(r)^2\, dr , \tag{A.7}$$

where $\underline{V}_T(r)$ is the demeaned Brownian bridge, $\underline{V}_T(r) = V_T(r) - \int_0^1 V_T(r)dr$. Therefore, using the results in Schmidt and Phillips (1992, p. 286) and (A.7), we can establish the limiting distribution of the minimum LM unit root test. In particular, the test does not depend on λ. The remaining procedure of the proof is to show continuity of a composite function. We simply follow ZA and express the $Inf\ \tilde{\tau}(\tilde{\lambda})$ t-statistic as:

$$Inf\ \tilde{\tau}(\tilde{\lambda}) = g[S_T(r), \underline{V}_T(r), \int_0^1 S_T(r)dS_T(r), \int_0^1 P_{\Delta Z}(\lambda)\, S_T(r)dS_T(r), s^2] + o_p(1)\ , \qquad \text{(A.8)}$$

where $g = h^*[h[H_1(\bullet), H_2(\bullet), s_T(\lambda)]]$, with $h^*(m) = Inf\ m(\bullet)$ for any real function $m(\bullet)$, and $h[m_1, m_2, m_3] = m_1^{-1/2}m_2/m_3$. The functionals H_1 and H_2 are defined by (A.4) and (A.5). Continuity of h^* and h is proved in ZA. The case with $Z_t(\lambda) = [1, t, D_t, tD_t]'$ can be similarly considered while the expression for $\underline{V}_T(r)$ is changed accordingly as shown in Lee and Strazicich (2003).