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Remarks on the proportional distribution in increasing return to scale problems

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Abstract

We study a cooperative problem where agents contribute a certain amount of input in order to obtain a surplus. We assume that the average surplus with respect to the amount contributed is increasing. Within this basic model, a cooperative game is associated and the proportional distribution arises as a natural core allocation. We describe a necessary and sufficient condition for which the core of the game shrinks to the proportional distribution. Furthermore, we characterize axiomatically the proportional distribution by means of three properties: core-selection, core-invariance and resource monotonicity. Finally, we provide a condition that guarantees that the proportional nucleolus coincides with the proportional solution.

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1. Introduction

Many cooperative activities involve the contribution of money, capital or labor by agents. Some models can be found in Mas-Colell (1980), Moulin (1987), Lemaire (1991), Izquierdo and Rafels (2001), or in the complete survey on cost and surplus-sharing problems by Moulin (2002). In this paper we analyze surplus problems with increasing average returns. We analyze the problem from the point of view of a cooperative game where the worth of a coalition is the corresponding surplus obtained. In this context, the proportional distribution with respect to agents' initial contributions arises as a natural core-allocation of the surplus generated.

In Section 2 we introduce the basic model and we provide a necessary and sufficient condition for which the core of the game shrinks to the proportional distribution. In Section 3 we characterize axiomatically the proportional distribution by using three axioms. *Core selection* requires that the solution lies in the core of the cooperative game. *Core invariance* means that two IRS problems with the same core deserve the same solution. *Resource monotonicity* requires that if an agent increases his/her initial contribution the payoff to this player should increase, while the payoff to the rest of agents should not decrease.

We end Section 3 by introducing the proportional nucleolus and comparing it with the proportional distribution . The proportional rule tries to balance and equalize individual average payoffs. The proportional nucleolus is also a core selection rule but tries to balance average revenues of coalitions with respect to their contributions rather than individual average revenues of players. In general, the two approaches differ and we provide a sufficient condition for the coincidence of the proportional solution and the proportional nucleolus.

2. The basic model and the core

Let $N = \{1, 2, \ldots, n\}$ be a set of agents (players) that are engaged in a joint activity. We denote by $\omega_i > 0$ the contribution (capital, labor, effort, inputs) of agent $i \in N$ and by $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ the vector of contributions. For all $S \subseteq N$, we write $\omega(S) = \sum_{i \in S} \omega_i$ and $\omega(\varnothing) = 0$. There is a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ that represents the technology that transforms z units of input into $f(z)$ units of output, with the following assumptions:

$$\begin{aligned} &\text{(a) } f(0) = 0 \text{ and } f(\textstyle\sum_{i=1}^n \omega_i) > 0 \\ &\text{(b) for any } 0 < z_1 \leq z_2 \text{ then } \frac{f(z_1)}{z_1} \leq \frac{f(z_2)}{z_2}. \end{aligned} \tag{1}$$

Condition (b) shows that the function f exhibits increasing returns to scale (IRS) and so that agents have incentive to cooperate. The problem at issue is then to divide the total surplus. We call a problem arising from this situation an IRS problem and we denote it by (N, f, ω).

To analyze an IRS problem we associate to it a cooperative game. A cooperative game is a function v that assigns to each subcoalition of agents $S \subseteq N$ the surplus generated $v(S) \in \mathbb{R}_+$, where $v(\varnothing) = 0$. We name $v(S)$ as the worth of coalition $S \subseteq N$. Given an IRS problem (N, f, ω) the worth of a coalition $S \subseteq N$ is obtained as follows:

$$v_{f,\omega}(S) = f(\sum_{i \in S} \omega_i), \text{ for all } S \subseteq N.$$

Condition (b) above implies that for all pair of coalitions S and T:

$$\text{if } \sum_{i\in S}\omega_i \leq \sum_{i\in T}\omega_i \text{ then } \frac{v_{f,\omega}(S)}{\omega(S)} \leq \frac{v_{f,\omega}(T)}{\omega(T)}. \tag{2}$$

We call a game that arises from this situation to be a *game with increasing returns to scale* or *IRS game* and we denote it by $(N, v_{f,\omega})$. We denote the class of all *IRS games* with player set N by IRG^N. Different types of games formally fit this model: bankruptcy games (Aumann and Maschler, 1982), interpreting claims as contributions; clan games (Potters et al., 1989) or simple games with veto power, assigning equal and strictly positive contributions to the members with veto power and, extending the model, assigning zero contribution to the rest of players; or convex measured games generated from a convex function f (Shapley, 1971). Moreover, IRS games are a subclass of average monotonic games (Izquierdo and Rafels, 2001).

It is easy to see that any game with increasing returns to scale is superadditive. Let $S, T \subseteq N$, $S \cap T = \varnothing$, then

$$\begin{aligned} v_{f,\omega}(S) + v_{f,\omega}(T) &= \frac{v_{f,\omega}(S)}{\omega(S)} \cdot \omega(S) + \frac{v_{f,\omega}(T)}{\omega(T)} \cdot \omega(T) \\ &\leq \frac{v_{f,\omega}(S\cup T)}{\omega(S\cup T)} \cdot \omega(S) + \frac{v_{f,\omega}(S\cup T)}{\omega(S\cup T)} \cdot \omega(T) = v_{f,\omega}(S\cup T). \end{aligned}$$

An efficient distribution of the worth of the grand coalition $v(N)$ is a vector $x = (x_i)_{i\in N}$ where x_i is the payoff of agent $i \in N$ such that $\sum_{i\in N} x_i = v(N)$. In the sequel and for all $S \subseteq N$, we write $x(S) = \sum_{i\in S} x_i$. The set of imputations of a cooperative game v with player set N is defined as

$$I(v) = \{x \in \mathbb{R}^N \mid x_i \geq v(\{i\}), \text{ for all } i \in N, \text{ and } x(N) = v(N)\},$$

while its corresponding core is defined as

$$C(v) = \{x \in \mathbb{R}^N \mid x(S) \geq v(S), \text{ for all } S \subseteq N, \text{ and } x(N) = v(N)\}.$$

The core of an IRS game is always non-empty. The proportional rule $P(N, f, \omega) = (p_i)_{i\in N}$ arises as a natural candidate to be a solution of the problem, where

$$p_i = \frac{f(\omega(N))}{\omega(N)} \cdot \omega_i = \frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega_i, \text{ for all } i = 1, 2, \ldots, n.$$

The proportional solution is obviously efficient and it can be easily checked that satisfies all core inequalities. This is, for all $S \subseteq N$, we have

$$p(S) = \frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega(S) \geq \frac{v_{f,\omega}(S)}{\omega(S)} \cdot \omega(S) = v_{f,\omega}(S),$$

where the inequality follows from (2). In general the core of an IRS game is quite large. The following lemma shows that the marginal contribution of an agent is always attainable within the core of the game.

Lemma 1 *Let $(N, v_{f,\omega})$ be an IRS game. Then, for every $i \in N$ there exists $x \in C(v_{f,\omega})$ such that $x_i = v_{f,\omega}(N) - v_{f,\omega}(N \setminus \{i\})$.*

Proof. Take $x \in \mathbb{R}^N$ defined as $x_i = v_{f,\omega}(N) - v_{f,\omega}(N \setminus \{i\})$ and $x_k = \omega_k \cdot \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})}$, for all $k \in N \setminus \{i\}$. It is easy to see that $x(N) = v_{f,\omega}(N)$. Moreover, by (2), we have that for any $S \subseteq N$, $i \notin S$, $x(S) = \omega(S) \cdot \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} \geq \omega(S) \cdot \frac{v_{f,\omega}(S)}{\omega(S)} = v_{f,\omega}(S)$. On the other hand, if $i \in S$ we have

$$\begin{aligned}
x(S) &= x_i + x(S \setminus \{i\}) = v_{f,\omega}(N) - v_{f,\omega}(N \setminus \{i\}) + \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} \cdot \omega(S \setminus \{i\}) \\
&= v_{f,\omega}(N) - \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} \cdot \omega(N \setminus \{i\}) + \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} \cdot \omega(S \setminus \{i\}) \\
&= v_{f,\omega}(N) - \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} \cdot \omega(N \setminus S) \\
&= \frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega(N) - \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} \cdot \omega(N \setminus S) \\
&\geq \frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega(N) - \frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega(N \setminus S) = \frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega(S) \geq v_{f,\omega}(S).
\end{aligned}$$

□

Using the above Lemma we show that the core of an IRS game shrinks to a single point (and so to the proportional solution) if the average return of the grand coalition and of coalitions of $n-1$ agents are all equal.

Proposition 1 *Let $(N, v_{f,\omega})$ be an IRS game. Then*

$$C(v_{f,\omega}) = \{P(N, f, \omega)\} \textit{ if and only if } \frac{v_{f,\omega}(N)}{\omega(N)} = \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})}, \textit{ for all } i \in N.$$

Proof. ($\Rightarrow$) By Lemma 1, for any agent $i \in N$ the marginal contribution is attainable within the core. Since the core consists only in the proportional distribution, this marginal contribution is attained at this point and so, for any $i \in N$

$$\frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega_i = v_{f,\omega}(N) - v_{f,\omega}(N \setminus \{i\}) = \frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega_i + \left[\frac{v_{f,\omega}(N)}{\omega(N)} - \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})}\right] \cdot \omega(N \setminus \{i\}),$$

which implies $\frac{v_{f,\omega}(N)}{\omega(N)} = \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})}$.

($\Leftarrow$) It is well-known that for a cooperative game with a non-empty core, if the vector assigning each agent his marginal contribution is efficient, then the core of the game contains a single point. Hence, take $x = (v_{f,\omega}(N) - v_{f,\omega}(N \setminus \{i\}))_{i \in N}$ and notice that $x = \left(\frac{v_{f,\omega}(N)}{\omega(N)} \cdot \omega_i\right)_{i \in N} = P(N, f, \omega)$. Since the proportional distribution is an efficient vector we are done. □

Notice that the condition that guarantees the proportional distribution is the unique core element says that no agent is *essential* to obtain the higher average return $\frac{v_{f,\omega}(N)}{\omega(N)}$.

3. An axiomatization of the proportional solution

Proposition 1 shows that the core of an IRS game might narrow quickly if the average return of coalitions of n and $n-1$ agents are getting closer. This situation might happen when agents increase their contribution and average returns are stagnated beyond some contribution threshold. In this case the proportional solution arises as the unique rule that makes compatible to select a core allocation and weakly increases the payoff to players when contributions increase.

Next we introduce three axioms that characterize the proportional rule. The first one states that the rule picks out an allocation from the core of the associated game (*Core selection*). The second one states that if the core of the game remains unchanged when we modify the function f (the technology) while contributions stay the same then solution will not change (*Core invariance*). Finally, *Resource monotonicity* requires that an increasing of the contribution by some player should benefit this player while not harming the rest of players.

The *rationale* for adoption of these axioms might read as follows. Regarding Core selection notice that the core of this kind of problems is always non-empty and then it represents the unique set of imputations of the game that are not dominated in the sense of von Neumann and Morgenstern[1]. Moreover, core allocations of an IRS game are the unique payoff distributions that are robust in front of objections and counter-objections in the sense of Davis-Maschler (Davis and Maschler, 1963) and in the sense of Mas-Colell (Mas-Colell, 1989). In other words, any allocation not in the core of the game can be objected without any possible counter-objection. This is a direct consequence of the fact that any IRS game is an average monotonic game (Izquierdo and Rafels, 2001). All these arguments lead us to focus on core allocations and discard imputations not in the core of the game. Concerning Core invariance, the contributions of players determine an initial *status quo* among agents. On the other hand, as we have argued before, players choose among core elements which can be viewed as the set of alternatives in this decision problem. If the same set of agents faces two IRS problems with the same initial contributions and the same set of alternatives there is no reason to discriminate between the final solution in both problems, and this is what Core invariance says. Finally, we respect to Resource monotonicity, the axiom enhances the robustness of an allocation rule in front of variations of the input contributed: if all players benefit from an increasing of the contribution of some player, any agent has no justification to block higher investment levels and all players have incentives to contribute more.

A allocation rule α on the domain of IRS problems is a rule that assigns to every IRS problem (N, f, ω) a unique payoff vector $\alpha(N, f, \omega) \in \mathbb{R}^N$.

Axiom 1 Core selection *A solution* α *is a* core selection *if, for all IRS problem* (N, f, ω) *it holds*

$$\alpha(N, f, \omega) \in C(v_{f,\omega}).$$

Axiom 2 Core invariance *A solution* α *is* core invariant *if for all pair of IRS problems* (N, f, ω) *and* (N, f', ω) *with the same initial contribution vector* ω*, it holds that*

$$\text{if } C(v_{f,\omega}) = C(v_{f',\omega}), \text{ then } \alpha(N, f, \omega) = \alpha(N, f', \omega).$$

[1]An imputation x of a game (N, v) dominates another imputation y if there exists a coalition $S \subseteq N$, $S \neq \varnothing, N$, such that $x_i > y_i$, for all $i \in S$ and $x(S) \leq v(S)$.

Axiom 3 Resource monotonicity *A solution* α *is* resource monotonic *if for every pair of contribution vectors* $\omega \in \mathbb{R}^N_{++}$ *and* $\omega' \in \mathbb{R}^N_{++}$ *such that* $\omega'_i > \omega_i$*, for some player* $i \in N$*, and* $\omega'_j = \omega_j$*, otherwise, it holds*

$$\alpha_j(N, f, \omega') \geq \alpha_j(N, f, \omega), \text{ for all } j \in N,\ j \neq i \text{ and } \alpha_i(N, f, \omega') > \alpha_i(N, f, \omega).$$

Next theorem states that the only solution that satisfies the three aforementioned axioms is the proportional distribution.

Theorem 1 *The proportional solution* $P(N, f, \omega)$ *is the unique allocation rule on the domain of IRS problems that satisfies Core selection, Resource monotonicity and Core invariance.*

Proof. Clearly the proportional solution satisfies the three axioms. Let α be a solution satisfying the three axioms and let (N, f, ω) be an arbitrary IRS problem. Then, let us define the following function f': $f'(z) = f(z)$, if $z \leq \omega(N)$ and $f'(z) = \dfrac{f(\omega(N))}{\omega(N)} \cdot z$, for all $z > \omega(N)$. Clearly f' satisfies (1). Moreover, let $i \in N$ be an arbitrary agent and define now $\omega' \in \mathbb{R}^N_{++}$ as $\omega'_i = \omega(N) > \omega_i$ and $\omega'_j = \omega_j$, for all $j \neq i$. By Axiom 3, $\alpha_i(N, f', \omega) < \alpha_i(N, f', \omega')$ and $\alpha_j(N, f', \omega) \leq \alpha_j(N, f', \omega')$, for all $j \neq i$. Then, we have that for all $j \in N$, $j \neq i$

$$\begin{aligned} \alpha_j(N, f', \omega) &\leq \alpha_j(N, f', \omega') \leq v_{f',\omega'}(N) - v_{f',\omega'}(N \setminus \{j\}) \\ &= \omega'_j \cdot \frac{v_{f',\omega'}(N)}{\omega'(N)} = \omega_j \cdot \frac{v_{f,\omega}(N)}{\omega(N)} = P_j(N, f, \omega), \end{aligned} \tag{3}$$

where the first inequality follows from Axiom 3 and the second one from Axiom 1. Taking now $i' \in N$, $i' \neq i$ and repeating the same argument, we can also deduce that

$$\alpha_i(N, f', \omega) \leq P_i(N, f, \omega). \tag{4}$$

Since α is a core selection it is efficient. Hence, by (3) and (4), $\alpha(N, f', \omega) = P(N, f, \omega)$. Finally, by Axiom 2, and since $C(v_{f,\omega}) = C(v_{f',\omega})$, we conclude $\alpha(N, f, \omega) = \alpha(N, f', \omega) = P(N, f, \omega)$ and we are done. ☐

Next we show that no axiom in our characterization is implied by the others. To this end we introduce several solutions satisfying all axioms but one

Example 1 *The egalitarian solution* $E(N, f, \omega) = \left(\dfrac{f(\omega(N))}{n}\right)_{i \in N}$ *satisfies Resource monotonicity and Core invariance, but it is not a core selection as the next numerical example shows.*

Consider a three-agent problem where $\omega = (\omega_1, \omega_2, \omega_3) = (10, 20, 20)$ *and* $f(z) = 2z$ *if* $z \geq 30$ *and* $f(z) = z$ *if* $0 < z < 30$*. By Theorem 1 the core of the associated game* $(N, v_{f,\omega})$ *shrinks to the proportional distribution* $C(N, v_{f,\omega}) = \{(20, 40, 40)\}$*, but* $E(N, f, \omega) = (\frac{100}{3}, \frac{100}{3}, \frac{100}{3})$*.*

Example 2 *The equal marginal solution (EM) is defined as* $EM(N,f,\omega) = \frac{\sum_{k=1}^{n} x^k}{n}$, *where for each* $k = 1,\ldots,n$ *we have* $x^k = \left(x_i^k\right)_{i\in N} \in \mathbb{R}^N$ *with* $x_k^k = v_{f,\omega}(N) - v_{f,\omega}(N\setminus\{k\})$ *and* $x_i^k = \frac{v_{f,\omega}(N\setminus\{k\})}{\omega(N\setminus\{k\})}\cdot\omega_i$, *for all* $i \neq k \in N$.

This rule satisfies Core selection since every vector x^k, $k \in N$, *is a core element (see Lemma 1). It also satisfies Core invariance, since if* $C(v_{f,\omega}) = C(v_{f',\omega})$, *we have that* $v_{f,\omega}(N) = v_{f',\omega}(N)$ *and* $v_{f,\omega}(N\setminus\{k\}) = v_{f',\omega}(N\setminus\{k\})$, *for all* $k \in N$, *where this last equality holds since the worth of any coalition of* $n-1$ *agents is attainable within the core (see Lemma 1). However, it does not satisfy Resource monotonicity as next example shows.*

Consider a three-agent problem where $\omega = (\omega_1,\omega_2,\omega_3) = (10,20,20)$ *and* $f(z) = 2z$ *if* $z \geq 40$ *and* $f(z) = z$ *if* $0 < z < 40$. *The equal marginal solution for this problem is* $EM(N,f,\omega) = (\frac{40}{3},\frac{130}{3},\frac{130}{3})$. *However if agent 1 increases his contribution up to* $\omega_1' = 20$, *while* $\omega_2' = 20$ *and* $\omega_3' = 20$, *we have* $EM(N,f,\omega') = (40,40,40)$ *but the payoff to agents 2 and 3 has decreased.*

Example 3 *Consider the rule* α *defined as* $\alpha(N,f,\omega) = EM(N,f,\omega)$, *if* f *satisfies that*

$$f(x+z) - f(x) \leq f(y+z) - f(y), \text{ for all } 0 \leq x < y \text{ and all } z \geq 0 \tag{5}$$

while $\alpha(N,f,\omega) = P(N,f,\omega)$, *otherwise. This rule satisfies Core selection since both the proportional and the equal marginal rules are core selection rules. As we have commented before, the proportional rule is resource monotonic and so is the equal marginal solution if* f *satisfies condition (5). To check it take* ω *and* ω' *such that (w.l.o.g.)* $\omega_1' > \omega_1$ *and* $\omega_i' = \omega_i$, *for all* $i \in N$, $i \neq 1$. *Notice that*

$$\begin{aligned} EM_1(N,f,\omega') &= \frac{1}{n}\left[v_{f,\omega'}(N) - v_{f,\omega'}(N\setminus\{1\}) + \textstyle\sum_{k\neq 1}\frac{v_{f,\omega'}(N\setminus\{k\})}{\omega'(N\setminus\{k\})}\cdot\omega_1'\right] \\ &> \frac{1}{n}\left[v_{f,\omega}(N) - v_{f,\omega}(N\setminus\{1\}) + \textstyle\sum_{k\neq 1}\frac{v_{f,\omega}(N\setminus\{k\})}{\omega(N\setminus\{k\})}\cdot\omega_1\right] \\ &= EM_1(N,f,\omega), \end{aligned}$$

where the strict inequality follows since $v_{f,\omega'}(N) > v_{f,\omega}(N)$, $v_{f,\omega'}(N\setminus\{1\}) = v_{f,\omega}(N\setminus\{1\})$ *and* $\frac{v_{f,\omega'}(N\setminus\{k\})}{\omega'(N\setminus\{k\})} \geq \frac{v_{f,\omega}(N\setminus\{k\})}{\omega(N\setminus\{k\})}$, *for all* $k \in N$, $k \neq 1$. *Now take* $i \in N$, $i \neq 1$. *Then,*

$$\begin{aligned} EM_i(N,f,\omega') &= \frac{1}{n}\left[v_{f,\omega'}(N) - v_{f,\omega'}(N\setminus\{i\}) + \textstyle\sum_{k\neq i}\frac{v_{f,\omega'}(N\setminus\{k\})}{\omega'(N\setminus\{k\})}\cdot\omega_i'\right] \\ &= \frac{1}{n}\Bigg[f(\omega(N) + (\omega_1' - \omega_1)) - f(\omega(N\setminus\{i\}) + (\omega_1' - \omega_1)) \\ &\qquad + \textstyle\sum_{k\neq i}\frac{v_{f,\omega'}(N\setminus\{k\})}{\omega'(N\setminus\{k\})}\cdot\omega_i'\Bigg] \\ &\geq \frac{1}{n}\left[f(\omega(N)) - f(\omega(N\setminus\{i\})) + \textstyle\sum_{k\neq i}\frac{v_{f,\omega}(N\setminus\{k\})}{\omega(N\setminus\{k\})}\cdot\omega_i\right] \\ &= EM_i(N,f,\omega), \end{aligned}$$

where the inequality holds by taking in (5) $x = \omega(N\setminus\{i\})$, $y = \omega(N)$ *and* $z = \omega_1' - \omega_1$. *However, the solution it is not Core invariant as it illustrates the three-agent problem* *where* $\omega = (\omega_1, \omega_2, \omega_3) = (50, 50, 100)$ *and* $f(z) = \frac{1}{2}z$*, if* $0 \leq z < 100$*, and* $f(z) = 1.5z - 100$*, if* $100 \leq z$*. Since* f *is convex it satisfies (5) and so* $\alpha(N, f, \omega) = (47\frac{2}{9}, 47\frac{2}{9}, 105\frac{5}{9})$*. Now consider the function* f' *defined as* $f'(z) = f(z)$ *if* $0 \leq z < 200$ *and* $f'(z) = z$*, if* $200 \leq z$*. The function* f' *does not satisfy (5) and so* $\alpha(N, f, \omega) = (50, 50, 100)$*. However notice that* $v_{f,\omega} = v_{f',\omega}$ *and so* $C(N, v_{f,\omega}) = C(N, v_{f',\omega})$*.*

The proportional solution is a rule that gives a constant reward to each unit contributed. It reflects a normative approach. However, the process of selecting an allocation generates conflicts between coalitions: if a coalition receives a better reward is because other coalitions reduce their payoff. The nucleolus (Schmeidler, 1969), a solution that comes from the field of cooperative game theory, tries to equalize the looses of coalitions whose payoff enter in conflict (looses with respect to their respective worth). This approach is more in line with a positive point of view where it might be expected that the bargaining between agents ends by equalizing these looses. To better adapt this well-known solution to our model we consider not looses but average looses with respect the units contributed by a coalition. The reason to take this approach is that agents in this model often argue and bargain based on average revenues rather than on absolute revenues. In this sense the proportional excess reflects the gap between the average worth and the average payoff assigned to a coalition. This idea of weighting the excesses of coalitions is not new and has already analyzed by several authors: Wallmeier (1984), Lemaire (1984) and Derks and Haller (1999). We call the solution that tries to equalize these average looses as the proportional nucleolus. Let's formally define this solution.

Given an allocation vector $x \in I(N, v_{f,\omega})$ and a coalition $\varnothing \neq S \subseteq N$, $S \neq N$, the proportional excess of S at x relative to $v_{f,\omega}$ is

$$\frac{v_{f,\omega}(S) - x(S)}{\omega(S)} = \frac{v_{f,\omega}(S)}{\omega(S)} - \frac{x(S)}{\omega(S)}.$$

As we have commented before, notice that the second expression of the proportional excess reflects the difference between the average worth of a coalition and its corresponding average payoff at x.

Given an allocation $x \in \mathbb{R}^N$, the proportional excess vector of x, $\Theta^p(x)$ is the vector whose components are the proportional excesses of the $2^n - 2$ subcoalitions of N(except N and $\varnothing$) ordered in a non-increasing way. This is

$$\Theta^p(x) = \left(\frac{v_{f,\omega}(S_k) - x(S_k)}{\omega(S_k)}\right)_{k=1,\ldots,2^n-2}$$

such that $S_k \subseteq N$, $S_k \neq \varnothing, N$, for all $k = 1, \ldots, 2^n - 2$, and $\frac{v_{f,\omega}(S_k) - x(S_k)}{\omega(S_k)} \geq \frac{v_{f,\omega}(S_{k+1}) - x(S_{k+1})}{\omega(S_{k+1})}$, for all $k = 1, \ldots, 2^n - 3$.

The proportional nucleolus[2] is defined as the unique allocation $\eta^p(N, v_{f,\omega}) = x^* \in I(N, v_{f,\omega})$ such that

$$\Theta^p(x^*) \leq_{Lex} \Theta^p(x),$$

[2]Given $x, y \in \mathbb{R}^n$, we say $x <_{Lex} y$ if there is some $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j$ for $1 \leq j < i$. Also we say $x \leq_{Lex} y$ if $x <_{Lex} y$ or $x = y$.

for all $x \in I(N, v_{f,\omega})$. Hence, the proportional nucleolus tries to minimize the maximum proportional excesses. The proportional rule and the proportional nucleolus do not coincide in general. However, the normative and the positive approach converge when the average return of all coalitions of $n-1$ agents do coincide.

Proposition 2 *Let $(N, v_{f,\omega})$ be an IRS game. If $\frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} = \frac{v_{f,\omega}(N \setminus \{j\})}{\omega(N \setminus \{j\})}$, for all $i, j \in N$, then*

$$P(N, f, \omega) = \eta^p(N, v_{f,\omega}).$$

Proof. Let $P(N, f, \omega) = (p_i)_{i \in N}$. For all $i, j \in N$, we have

$$\begin{aligned}\frac{v_{f,\omega}(N \setminus \{i\}) - p(N \setminus \{i\})}{\omega(N \setminus \{i\})} &= \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} - \frac{p(N \setminus \{i\})}{\omega(N \setminus \{i\})} \\ &= \frac{v_{f,\omega}(N \setminus \{j\})}{\omega(N \setminus \{j\})} - \frac{p(N \setminus \{j\})}{\omega(N \setminus \{j\})} = \frac{v_{f,\omega}(N \setminus \{j\}) - p(N \setminus \{j\})}{\omega(N \setminus \{j\})}.\end{aligned}$$

Moreover, for all $i \in N$ and for all $\varnothing \neq S \subseteq N$, $S \neq N$ it holds that $S \subseteq N \setminus \{j\}$, for some $j \in N$ and

$$\begin{aligned}\frac{v_{f,\omega}(N \setminus \{i\}) - p(N \setminus \{i\})}{\omega(N \setminus \{i\})} &= \frac{v_{f,\omega}(N \setminus \{j\}) - p(N \setminus \{j\})}{\omega(N \setminus \{j\})} \\ &= \frac{v_{f,\omega}(N \setminus \{j\})}{\omega(N \setminus \{j\})} - \frac{p(N \setminus \{j\})}{\omega(N \setminus \{j\})} \\ &\geq \frac{v_{f,\omega}(S)}{\omega(S)} - \frac{p(S)}{\omega(S)} = \frac{v_{f,\omega}(S) - p(S)}{\omega(S)}.\end{aligned}$$

The above inequality shows that the maximum proportional excesses at the proportional allocation are attained at all coalitions of $n-1$ agents.

Consider now an imputation $x \in I(N, v_{f,\omega})$ different from the proportional allocation $x \neq P(N, f, \omega)$. Since x is efficient, there exists an agent $i \in N$ such that $x_i > p_i$ and so $x(N \setminus \{i\}) < p(N \setminus \{i\})$. At this allocation x the proportional excess of coalition $N \setminus \{i\}$ is

$$\begin{aligned}\frac{v_{f,\omega}(N \setminus \{i\}) - x(N \setminus \{i\})}{\omega(N \setminus \{i\})} &= \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} - \frac{x(N \setminus \{i\})}{\omega(N \setminus \{i\})} \\ &> \frac{v_{f,\omega}(N \setminus \{i\})}{\omega(N \setminus \{i\})} - \frac{p(N \setminus \{i\})}{\omega(N \setminus \{i\})} = \frac{v_{f,\omega}(N \setminus \{i\}) - p(N \setminus \{i\})}{\omega(N \setminus \{i\})}.\end{aligned}$$

This implies that the proportional excess of $N \setminus \{i\}$ at x is strictly larger that the maximum excess at $P(N, f, \omega)$. Hence, we conclude that $\Theta^p(P(N, f, \omega)) <_{Lex} \Theta^p(x)$, for all $x \in I(N, v_{f,\omega})$, $x \neq P(N, f, \omega)$. Therefore, it follows that $\eta^p(N, v_{f,\omega}) = P(N, f, \omega)$. □

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