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Majority Voting Over Lotteries: Conditions for Existence of a Decisive Voter

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Abstract

This note extends known sufficient conditions for existence of a decisive voter in pairwise voting over lotteries. The preferred lottery of such a voter always coincides with the lottery preferred by a majority, meaning voting can be reduced to a decision problem of the decisive voter. The results are useful in solving dynamic models of bargaining and elections, where a binary vote can be expressed as a choice between two lotteries (depending on the discount factor), and voting subgames can be reduced to a decision problem of the decisive voter.

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1. Introduction

In the spatial model of majority voting, Davis, DeGroot, and Hinich (1972) assume an odd number of voters have Euclidean preferences, and they show that if the majority core (the set of alternatives that cannot be improved upon by any majority coalition of voters) is nonempty, then the uniquely defined "core voter" is decisive: one alternative is preferred to another by a majority of voters if and only if it is preferred by the core voter. In the same spirit, Rothstein (1990,1991) defines the notion of order restricted preferences and establishes conditions under which some voter is decisive in the above sense.[1] Banks and Duggan (2006) extend the former result to voting over *lotteries:* assuming voters have expected utility preferences over lotteries with quadratic utilities, one lottery is preferred to another by a majority of voters if and only if it is preferred by the core voter. In the context of a one-dimensional model with quadratic utilities, Lemma A.1 of Cho and Duggan (2003) shows that voter preferences over lotteries are order restricted, which implies the existence of a decisive voter. In this note, I generalize the above conditions for existence of a voter who is decisive over lotteries, relaxing the assumption of quadratic utility and allowing for general voting rules. I do not impose structure on the set of alternatives, but rather on the form of the parameterization of voter utility functions.

As discussed in Banks and Duggan (2006), the decisiveness result—though seemingly a technical lemma in social choice—is useful in the analysis of dynamic models of bargaining and elections, where voting plays an important role yet can significantly complicate the analysis. Specifically, the existence of an individual who is decisive over lotteries allows a voting game to be reduced to a decision problem of the decisive voter, improving analytical tractability in theoretical work and facilitating closed form solutions in modeling applications. The framework of this note is static, whereas in dynamic voting applications, the alternatives under consideration are typically *sequences* (or more generally, lotteries over sequences) of outcomes over time. Nevertheless, the decisiveness result does apply to quite general voting games in which the alternatives considered possess a temporal (as well as stochastic) aspect.

To see how the existence of a voter who is decisive over lotteries is relevant to dynamic voting games, suppose a majority vote is held to decide between two options: one is to choose an alternative x, which is implemented in the current period and remains in place in all future periods; the other is to reject x, in which case the current outcome is y and future outcomes are uncertain. For simplicity, suppose that if y is chosen today, then in each subsequent period, x is implemented with probability p (and remains in place thereafter), and y is chosen with probability $1-p$ (and the same lottery is held in the subsequent period). Assume that preferences over streams of outcomes for a voter i are given by per-period utility u_i and geometric discounting with a common discount factor $\delta \in [0, 1)$. After normalizing by $1 - \delta$, the expected discounted utility to voter i from accepting x is simply $u_i(x)$. The

[1] See also Gans and Smart (1996) for analysis of a single-crossing condition that is equivalent to order restriction.

expected payoff from rejecting x is:

$$(1-\delta)u_i(y)+\delta\left[\begin{array}{l} pu_i(x)+ \\ (1-p)\left[(1-\delta)u_i(y)+\delta\left[\begin{array}{l} pu_i(x)+ \\ (1-p)\left[(1-\delta)u_i(y)+\delta\left[\cdots\right.\right.\end{array}\right.\right.\end{array}\right.$$

Note that the terms multiplying $u_i(x)$ and $u_i(y)$, respectively, sum to

$$\alpha \;=\; \frac{\delta p}{1-\delta+\delta p} \quad \text{and} \quad \beta \;=\; \frac{1-\delta}{1-\delta+\delta p},$$

so the expected discounted utility for the voter is $\alpha u_i(x)+\beta u_i(y)$, which is mathematically equivalent to the expected utility from the lottery with probability α on x and $\beta = 1-\alpha$ on y. Thus, if some voter is decisive over lotteries, then a majority of voters will pass x if and only if it is preferred by the decisive voter, and the voting game can be analyzed simply as a decision problem of the decisive voter.

2. Majority Voting

Consider an odd number n of voters who have expected utility preferences over lotteries on a set X of outcomes.[2] Assume the preferences of each voter i are given by a von-Neumann-Morgenstern representation of the form

$$U_i(x) \;=\; \alpha_i v(x) - c(x) + \beta_i, \tag{1}$$

where the functions $v\colon X \to \Re$ and $c\colon X \to \Re$ are common to all voters, and $\alpha_i, \beta_i \in \Re$ are parameters that may vary across voters. For simplicity, assume $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, and let $m = \frac{n+1}{2}$ be the voter with the median parameter. Note that for all simple lotteries λ and μ, the function $\delta\colon \Re \to \Re$ defined by

$$\delta(\alpha) \;=\; \left(\sum_{x\in X}[\alpha v(x) - c(x)]\lambda(x)\right) - \left(\sum_{x\in X}[\alpha v(x) - c(x)]\mu(x)\right)$$

is either strictly increasing in α, strictly decreasing in α, or constant in α. This follows directly from the fact that it is affine linear in α.

From this it follows that the median voter, m, is *decisive* over lotteries, in the sense that for all λ and μ, a majority of voters strictly prefer λ to μ if and only if voter m strictly prefers λ to μ. Indeed, let λ and μ be any lotteries, and first assume that a majority of voters strictly prefer λ to μ, so that for a majority of voters i, we have

$$\sum_{x\in X} U_i(x)\lambda(x) \;>\; \sum_{x\in X} U_i(x)\mu(x),$$

[2] The set X may be infinite, but for the sake of simplicity we consider only simple lotteries, which have finite support. All results continue to hold with X replaced by an abstract measure space and summation replaced by integration with respect to arbitrary probability measures.

or equivalently, $\delta(\alpha_i) > 0$. If this does not hold for the median voter m, then there exist voters i and j with $\alpha_i < \alpha_m < \alpha_j$ such that $\delta(\alpha_i) > 0$ and $\delta(\alpha_j) > 0$. But since $\delta(\alpha_i) > 0$ and $\delta(\alpha_m) \leq 0$, it must be that δ is strictly decreasing, but then $\delta(\alpha_j) < 0$, a contradiction. Therefore, $\delta(\alpha_m) > 0$, as desired. Conversely, assume that the median strictly prefers λ to μ, so $\delta(\alpha_m) > 0$. If δ is strictly increasing, then we have $\delta(\alpha_i) > 0$ for all i with $\alpha_m < \alpha_i$; if δ is constant, then we have $\delta(\alpha_i) > 0$ for all i; and if δ is strictly decreasing, then we have $\delta(\alpha_i) > 0$ for all i with $\alpha_i < \alpha_m$. In all cases, a majority of voters strictly prefer λ to μ. Thus, we have established:

Proposition 1 Assume voter preferences have the form in (1). Then the median voter is decisive over lotteries.

A special case is the one-dimensional spatial model with quadratic utilities, where $X = \Re$ and each voter i has a von Neumann-Morgenstern representation $u_i(x) = -(\hat{x}_i - x)^2$. Note that $u_i(x) = -\hat{x}_i^2 + 2\hat{x}_i x - x^2$, so we can set $\alpha_i = 2\hat{x}_i$, $v(x) = x$, $c(x) = x^2$, and $\beta_i = -\hat{x}_i^2$ to obtain a special case of the above framework. For another possibility, let $X = \Re_{++}$ and assume the preferences of voter i are given by the function $\alpha_i \ln(x) - \frac{x^2}{n}$, as might be the case if x is the level of a public good, and the cost of public good is quadratic and divided equally among voters. Of course, we set $v(x) = \ln(x)$ and $c(x) = x^2/n$ to obtain this model as a special case.

In the multidimensional spatial model, where $X \subseteq \Re^d$, and with quadratic utility, voter i's ideal point is a vector $\hat{x} = (\hat{x}_1^i, \ldots, \hat{x}_d^i)$. To extend the analysis to this model, we need to allow for multidimensional parameters. The preferences of voter i are now given by

$$U_i(x) \;=\; \alpha^i \cdot v(x) - c(x) + \beta_i, \qquad (2)$$

where the functions $v\colon X \rightarrow \Re^d$ and $c\colon X \rightarrow \Re$ are common to all voters, and $\alpha^i = (\alpha_1^i, \ldots, \alpha_d^i) \in \Re^d$ and $\beta_i \in \Re$ are parameters that may vary across voters. With multidimensional parameters, we assume the condition *α-symmetry,* which means that voters other than n are paired $\{1,2\}$, $\{3,4\}$, $\ldots$, $\{n-2, n-1\}$ so that the parameter of voter n belongs to the convex hull of the parameters in each pair, i.e., for all $j = 1, \ldots, \frac{n-2}{2}$, we have $\alpha^n \in \mathrm{co}\{\alpha^{2j-1}, \alpha^{2j}\}$. Assuming that n is the unique voter for which this condition holds, we refer to n as the *median voter in all directions.*

Proposition 2 Assume voter preferences have the form in (2) and α-symmetry holds. Then the median voter in all directions is decisive over lotteries.

To prove the proposition, consider any λ and μ. For each voter pair, indexed $j = 1, \ldots, \frac{n-2}{2}$, define the function $\delta^j\colon [0,1] \rightarrow \Re$ by

$$\delta^j(\theta) \;=\; \sum_{x \in X} [(\theta\alpha^{2j-1} + (1-\theta)\alpha^{2j}) \cdot v(x) - c(x)][\lambda(x) - \mu(x)].$$

Assume that a majority of voters strictly prefer λ to μ, and suppose this is not true of the median in all directions. Since a majority of voters strictly prefer λ to μ, there exists j such that both voters $2j-1$ and $2j$ strictly prefer λ to μ, which implies $\delta^j(0) > 0$ and $\delta^j(1) > 0$. By α-symmetry, there exists $\theta \in [0,1]$ such that $\alpha^n = \theta\alpha^{2j-1} + (1-\theta)\alpha^{2j}$. Then $\delta^j(\theta) > 0$, which implies that the median in all directions pefers λ, a contradiction. Conversely, assume the median in all direction strictly prefers λ to μ. If this is not true for $\frac{n-1}{2}$ other voters, then there exists j such that voters $2j-1$ and $2j$ weakly prefer μ to λ, which implies $\delta^j(0) \leq 0$ and $\delta^j(1) \leq 0$, but then α-symmetry implies, as above, that the median in all directions weakly prefers μ, a contradiction.

In the multidimensional spatial model, Plott's (1967) theorem implies that at a majority core alternative (if one exists), one voter must have a zero gradient, and the gradients of the other voters' utility functions must satisfy a strong radial symmetry condition. For the special case of quadratic utility, this reduces to the condition that the core alternative is the ideal point of one voter (the median in all directions, say n) and that the other voters can be paired so that the core alternative belongs to the convex hull of ideal points of each pair of voters. That is, the ideal points of the voters satisfy α-symmetry.

3. General Voting Rules

We first extend the analysis in Proposition 1 to more general voting rules and, in so doing, establishes a result on order restriction of voter preferences. We say voter preferences over lotteries are *order restricted* if for all lotteries λ and μ, and for all voters i, j, and k with $\alpha_i \leq \alpha_j \leq \alpha_k$, if

$$\text{sign}\left(\sum_{x\in X} U_i(x)[\lambda(x)-\mu(x)]\right) \quad = \quad \text{sign}\left(\sum_{x\in X} U_k(x)[\lambda(x)-\mu(x)]\right)$$

then

$$\text{sign}\left(\sum_{x\in X} U_j(x)[\lambda(x)-\mu(x)]\right) \quad = \quad \text{sign}\left(\sum_{x\in X} U_k(x)[\lambda(x)-\mu(x)]\right).$$

That is, if two voters have the same preference over two lotteries, then all voters "between" them agree as well. We have shown in the argument for Proposition 1 that under (1), the difference in expected utility is monotonic in α, which delivers the following result:

Proposition 3 Assume voter preferences have the form in (1). Then voter preferences over lotteries are order restricted.

The preceding proposition does not explicitly establish a decisiveness result, but it contains the antecedents of one. Generalizing majority rule, let N denote the set of all voters and $C \subseteq N$ a coalition of voters, and assume the voting rule is given by a collection $\mathcal{W}$ of winning coalitions satisfying: (i) if $C \in \mathcal{W}$ and $C' \supseteq C$, then $C' \in \mathcal{W}$, and (ii) if $C, C' \in \mathcal{W}$, then $C \cap C' \neq \emptyset$. The voting rule is *strong* if for all coalitions C, either $C \in \mathcal{W}$ or $N \setminus C \in \mathcal{W}$. Majority rule, e.g., can be represented by a strong voting rule when n is odd by specifying that $\mathcal{W}$ consist of all majority coalitions. When $\mathcal{W}$ is strong, it is known that there exists a unique parameter value α such that

$$\{i \mid \alpha_i \geq \alpha\} \in \mathcal{W} \quad \text{and} \quad \{i \mid \alpha_i \leq \alpha\} \in \mathcal{W},$$

and that $\alpha = \alpha_i$ for some voter i. Assuming this is true for just one voter, we refer to i as the *core voter*. Extending earlier terminology, we say voter i is *decisive* over lotteries if for all λ and μ,

$$\sum_{x\in X} U_i(x)\lambda(x) \quad > \quad \sum_{x\in X} U_i(x)\mu(x)$$

holds if and only if there is some $C \in \mathcal{W}$ such that

$$\text{for all } j \in C,\ \sum_{x\in X} U_j(x)\lambda(x) \quad > \quad \sum_{x\in X} U_j(x)\mu(x).$$

Standard arguments (which are omitted) yield the following:

Corollary Assume voter preferences have the form in (1). If the voting rule is strong, then the core voter is decisive over lotteries.

Next, we extend Proposition 2 to allow for a general voting rule $\mathcal{W}$, as above. We say *generalized α-symmetry* holds if α^n belongs to the convex hull of parameters of members of all winning coalitions, i.e., for all $C \in \mathcal{W}$, we have $\alpha^n \in \text{co}\{\alpha^j \mid j \in C\}$. Assuming n is the unique voter for which this condition holds, we refer to voter n is the *core voter*. This is consistent with the usage of this term for the special case of the one-dimensional parameterization.

Proposition 4 Assume voter preferences have the form in (2) and generalized α-symmetry holds. If the voting rule is strong, then the core voter is decisive over lotteries.

Consider any λ and μ. Suppose the core voter n strictly prefers λ to μ, but there is no coalition $C \in \mathcal{W}$ all of whose members share this preference. Then because $\mathcal{W}$ is strong, the coalition

$$C \quad = \quad \left\{ j \mid \sum_{x\in X} U_j(x)\lambda(x) \leq \sum_{x\in X} U_j(x)\mu(x) \right\}$$

is decisive. For all $j \in C$, we have

$$\sum_{x\in X} \alpha^j \cdot v(x)[\lambda(x) - \mu(x)] \quad \leq \quad \sum_{x\in X} c(x)[\mu(x) - \lambda(x)],$$

and as in the proof of Proposition 3, the right-hand side of the above inequality is constant in β^j. Because C is winning, generalized α-symmetry implies $\alpha^i \in \text{co}\{\alpha^j \mid j \in C\}$, and we conclude that

$$\sum_{x\in X} \alpha^n \cdot v(x)[\lambda(x) - \mu(x)] \quad \leq \quad \sum_{x\in X} c(x)[\mu(x) - \lambda(x)],$$

contradicting the strict preference of the core voter. Conversely, if all members of a winning coalition C strictly prefer λ to μ, then we have

$$\sum_{x\in X} \alpha^j \cdot v(x)[\lambda(x) - \mu(x)] \quad > \quad \sum_{x\in X} c(x)[\mu(x) - \lambda(x)]$$

for all $j \in C$, and generalized α-symmetry implies that the core voter shares this strict preference.

4. Discussion

Because Proposition 3 and its corollary (and the extension in Proposition 4) provide sufficient conditions, it is natural to consider the possibility of weaker preference restrictions under which the decisiveness result might hold. Banks and Duggan (2006) give a one-dimensional example in which voter utilities are a concave, decreasing function of distance from their ideal points, but no voter is decisive over lotteries, so sufficient conditions for a decisive voter must involve restrictions on the curvature of utilities. Note that the key step in the sufficiency argument of Propositions 1 and 3 is that the difference in the expected utility from one lottery compared to another for a voter with parameter α, i.e., $\delta(\alpha)$, is a monotonic function of α. (At a finer level of detail, what is crucial is that as we vary α, the sign of $\delta(\alpha)$ can change at most once.) To understand the limits of the argument, suppose that the utility from x for a voter with parameter α is a general function $w(\alpha v(x))$ of the product $\alpha v(x)$. Given lotteries λ and μ, the difference in expected utility is then

$$\sum_{x\in X} w(\alpha v(x))\lambda(x) - \sum_{x\in X} w(\alpha v(x))\mu(x),$$

and the rate of change with respect to α is then

$$\sum_{x\in X} v(x)w'(\alpha v(x))\lambda(x) - \sum_{x\in X} v(x)w'(\alpha v(x))\mu(x). \tag{3}$$

Because the derivative w' is written as a general function, we cannot disentangle the parameter α from the choice x to sign the change in expected utility.

Now suppose that the derivative has the form $w'(\alpha v(x)) = s(\alpha)t(x)$ for some functions s and t with the sign of $s(\alpha)$ constant; this is the case in the earlier analysis, where w is the identity function. Then the rate of change in (3) becomes

$$s(\alpha)\left(\sum_{x\in X} t(x)\lambda(x) - \sum_{x\in X} t(x)\mu(x)\right),$$

and the difference in expected utility changes sign at most once. The above decomposition of the derivative is of course not a general property; rather, it determines a functional equation that is satisfied by a special class of functions. Consider the possibility of specifying three functions, f, g, and h, such that the equation $f(xy) = g(x)h(y)$ holds identically for $x, y > 0$. This is a functional equation of Pexider, and it is satisfied (see Theorem 4 (p.144) of Aczél (1966)) for all and only functions of the following form: either f and g (or f and h) are identically zero, or

$$f(z) = abz^c, \;\; g(z) = az^c, \;\; h(z) = bz^c,$$

where a, b, and c are parameters. In our application, f corresponds to the derivative w'. Thus, $w'(z) = abz^c$, and therefore we have two cases: for $c \neq -1$, we have (up to an additive constant) $w(z) = \frac{ab}{1+c} z^{1+c}$, and for $c = -1$, we have $w(z) = ab\ln(z)$.

This suggests the following functional forms for the voters' von Neumann-Morgenstern representations: consider

$$U_i(x) \;\; = \;\; a(\alpha_i v(x))^b - c(x) + \beta_i, \tag{4}$$

or, with the restrictions that $\alpha_i > 0$ and $v(X) \subseteq \Re_{++}$,

$$U_i(x) \;=\; a\ln(\alpha_i v(x)) - c(x) + \beta_i, \tag{5}$$

where the functions $v\colon X \to \Re$ and $c\colon X \to \Re$ and the parameters $a, b \in \Re$ with $b \neq 0$ are common to all voters, and $\alpha_i, \beta_i \in \Re$ may vary across voters. Note, however, that (4) can be obtained from (1) by a suitable relabeling of parameters. Specifically, given a utility function of the form in (4), define $\hat{\alpha}_i = \alpha^b$ for each voter and $\hat{v}(x) = av(x)^b$ to map this into our earlier functional form. As well, given a utility function of the form in (5), we can define $\hat{\alpha}_i = \ln(\alpha_i)$, $\hat{v}(x) = a$ for all x, and $\hat{c}(x) = a\ln(v(x)) - c(x)$ to again obtain a special case of (1). Thus, the apparent generality of (4) and (5) is spurious. The decisiveness results may hold for functional forms beyond those considered here, but it appears that scope for further general results is quite narrow.

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