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Belief functions and preference for flexibility

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Abstract

This note clarifies some deep mathematical connections between the Dempster-Shafer theory belief functions (Dempster, 1967; Shafer, 1976) and preference for flexibility in the tradition of Kreps (1979).

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1 Introduction

Kreps (1979) provides an axiomatisation for "preference for flexibility": necessary and sufficient conditions on the ordering of opportunity sets for consistency with *expected indirect utility (EIU)* maximisation. More recently, Barberà and Grodal (2011) have axiomatised *expected opportunity (EO)* maximisation, in which the decision-maker has a subjective assessment of the probability that a given set of options may vanish between choosing an opportunity set and making her selection from it. Barberà and Grodal restrict attention to antisymmetric binary relations but observe that *any* binary relation on opportunity sets – antisymmetric or otherwise – which is consistent with EO maximisation is also consistent with EIU maximisation. In fact, the converse is also true.

At the heart of the equivalence between EUI and EO is their connection with the theory of belief functions (Dempster, 1967; Shafer, 1976). Nehring (1999) already noted the link between EIU and belief functions. The purpose of this note is to clarify these various mathematical relationships. A convenient synthesis is provided by Theorem 3 below.

2 Belief functions

Let Θ be a non-empty finite set.

Definition 1 *A mapping* $v : 2^{\Theta} \to [0,1]$ *is a* belief function *if there exists a probability (measure)* $m : 2^{\Theta} \to [0,1]$ *such that*

$$v(E) = \sum_{A \subseteq E} m(A)$$

for any $E \subseteq \Theta$*.*

Shafer (1976) refers to m as a the *basic probability assignment (BPA)* for the belief function. Given a belief function v we may recover its BPA via Möbius inversion (Shafer, 1976):

$$m(E) = \sum_{A \subseteq E} (-1)^{|E \setminus A|} v(A)$$

for each $E \subseteq \Theta$.

Definition 2 *The* conjugate *of a belief function* $v : 2^{\Theta} \to [0,1]$ *is the mapping* $v^{*} : 2^{\Theta} \to [0,1]$ *defined as follows:*

$$v^{*}(E) = 1 - v(\Theta \setminus E)$$

for any $E \subseteq \Theta$*. A mapping* $v^{*} : 2^{\Theta} \to [0,1]$ *is a* plausibility function *if it is the conjugate of some belief function.*

Note that $v^* : 2^\Theta \to [0,1]$ is a *plausibility function* iff there exists a probability (measure) $m : 2^\Theta \to [0,1]$ such that

$$v^*(E) \;=\; \sum_{A: A\cap E\neq\varnothing} m(A)$$

for any $E \subseteq \Theta$.

The following question naturally arises: *Given a binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$, what are necessary and sufficient conditions for the existence of a plausibility function v^* that represents $\succsim$ (that is, for which $A \succsim B$ iff $v^*(A) \geq v^*(B)$ for any $A, B \in 2^\Theta$)?*

We shall answer this question shortly, but first let's detour into the literature on ranking opportunity sets.

3 Preference for flexibility

Consider a binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$. Its asymmetric and symmetric parts are denoted by $\succ$ and $\sim$ respectively. As usual, we write $A \succsim B$ for $(A,B) \in \succsim$ and so forth.

Let us interpret subsets of Θ as opportunity sets – budget sets from which a subsequent choice will be made[1] – so $\succsim$ is an ordering of opportunity sets. In this literature it is typical to restrict $\succsim$ to *non-empty* subsets of Θ but we will extend it to all subsets for ease of comparison with belief functions. We also make the conventional assumption that $\Theta \succ \varnothing$.

Kreps (1979) introduced the notion of an *expected indirect utility (EIU)* ordering to reflect uncertainty about future preferences. The following definition adapts Kreps' EIU notion to allow for empty opportunity sets and the $\Theta \succ \varnothing$ convention.

Definition 3 *A binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$ is an EIU order if there exists a finite set S, a probability (measure) $p : S \to [0,1]$ and a utility function $u_s : \Theta \to \mathbb{R}_+$ for each $s \in S$ satisfying $u_s(\theta) > 0$ for at least one $\theta \in \Theta$, such that $\succsim$ is represented by the function $v : 2^\Theta \to \mathbb{R}_+$ defined as follows:*

$$v(A) \;=\; \begin{cases} \sum_{s\in S} p(\{s\}) \left[\max_{\theta\in A} u_s(\theta)\right] & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$$

Kreps (1979) provides an answer to the following question: *What are necessary and sufficient conditions for a binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$ to be an EIU order?*

It turns out that this question is mathematically equivalent to the previous one. Before proving this fact, let us mention yet another equivalent question.

[1]Following Kreps (1979), it may be convenient to think of the elements of Θ as meal options (chicken, fish, steak, ...) and subsets of Θ as menus. The binary relation $\succsim$ ranks menus from which a subsequent meal choice will be made. Imagine, for example, ranking restaurants for a dinner booking some time in the future.

In a recent paper, Barberà and Grodal (2011) introduce the concept of an *expected opportunity (EO)* order to reflect uncertainty about whether all options from a given opportunity set will still be available when the time comes to choose a member of this set.[2] Our definition differs slightly from theirs, since we require only $\Theta \succ \varnothing$, not the stronger condition that $A \succ \varnothing$ for all $A \neq \varnothing$, which they impose.

Definition 4 *A binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$ is an EO order if there exists a utility function $u : \Theta \to \mathbb{R}_{++}$ and a probability (measure) $\mu : 2^\Theta \to [0,1]$ such that $\succsim$ is represented by the function $v : 2^\Theta \to \mathbb{R}_+$ defined as follows:*

$$v(A) = \begin{cases} \sum_{E \subseteq \Theta} \mu(E) \left[\max_{\theta \in A \cap E} u(\theta)\right] & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$$

(with the convention that the maximum of any function over the empty set is zero).

One may think of $\mu(E)$ as the probability that only elements from E will still be available at the time of choosing an element from the opportunity set. The restriction $u(\theta) > 0$ for all $\theta \in \Theta$ ensures that $v(\Theta) > v(\varnothing)$.

It is natural to ask the following question: *What are necessary and sufficient conditions for a binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$ to be an EO order?*

Barberà and Grodal (2011) provide a partial answer: one that applies only to the class of antisymmetric binary relations.[3] However, a complete answer is readily obtained.

4 A synthesis

To address the three questions posed above, consider the following axioms (assumed to hold for any $A, B, C \in 2^\Theta$):

(A0) $\succsim$ is a weak order (i.e., complete and transitive).

(A1) *(monotonicity)* If $A \subseteq B$, then $B \succsim A$.

(A2) *(strict expansion monotonicity)* If $B \subset A$, $A \cap C = \emptyset$ and $A \succ B$, then $A \cup C \succ B \cup C$ (where $\subset$ denotes a proper subset).

(A2′) *(strict contraction monotonicity)* If $C \subseteq B \subset A$ and $A \succ B$, then $A \setminus C \succ B \setminus C$.

(A3) *(non-triviality)* $\Theta \succ \emptyset$.

[2] Perhaps steak will not be available on the night you dine at your chosen restaurant.

[3] A binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$ is *antisymmetric* if $A \succsim B$ and $B \succsim A$ imply $A = B$ (for any $A, B \subseteq \Theta$).

Wong et al. (1991) prove the following result.[4] Since it plays a central role in our analysis, we include a proof. Our argument is a simplified version of that in Wong et al. (1991).

Theorem 1 (Wong et al., 1991) *A binary relation $\succsim \subseteq 2^{\Theta} \times 2^{\Theta}$ may be represented by a belief function if and only if it satisfies (A0), (A1), (A2) and (A3).*

Proof: The "only if" part is straightforward. For the "if" part, we argue in three steps.

Step I. By (A0) and (A1) there exists a representation $f : 2^{\Theta} \to \mathbb{R}$ that is monotone with respect to set inclusion. It is clearly WLOG to assume $f(\emptyset) = 0$ and that, for any A with $A \succ B$ for all $B \subset A$,

$$f(A) \geq -\sum_{B \subset A} (-1)^{|A \backslash B|} f(B) \tag{1}$$

The idea is to obtain a representation with enough "space" between the values assigned to distinct indifference classes that the gap can be filled with non-negative basic probability assignments.[5] In particular, (1) implies that

$$\sum_{B \subseteq A} (-1)^{|A \backslash B|} f(B) \geq 0$$

when $f(A) > f(B)$ for all $B \subset A$.

Step II. We now use (A1) and (A2) to show that $m(A) = 0$ for any A with $f(A) = f(D)$ for some $D \subset A$, where m is the Möbius inverse of f. Thus:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \backslash B|} f(B) \tag{2}$$

Choose some $\theta \in A \diagdown D$ and let $D' = A \backslash \{\theta\}$. Then we may re-write (2) as follows:

$$m(A) = \sum_{B \subseteq D'} (-1)^{|A \backslash B|} \left[f(B) - f(B \cup \{\theta\}) \right]$$

We claim that $f(B \cup \{\theta\}) - f(B) = 0$ for all $B \subseteq D'$. We have $f(B \cup \{\theta\}) \geq f(B)$ by (A1). If $f(B \cup \{\theta\}) > f(B)$, then we deduce $f(D' \cup \{\theta\}) > f(D')$ (directly if $B = D'$ or by (A2) if $B \subset D'$). But $A = D' \cup \{\theta\}$, so $f(A) = f(D)$ and (A1) imply $f(D' \cup \{\theta\}) = f(D')$, which gives the required contradiction. Hence, $m(A) = 0$.

Combining Steps I and II, we have a representation f for $\succsim$ whose Möbius inverse m satisfies $m(A) \geq 0$ for all A.

[4] They call (A2) *partial monotonicity*.

[5] Suppose that $0 = f_0 < f_1 < \cdots < f_n$ denote the distinct values of f. Taking each $i = 1, 2, ..., n$ in turn, add a suitable non-negative constant to all f_j with $j \geq i$ to ensure that (1) holds for all A with $f(A) = f_i$ and $f(B) < f_i$ for all $B \subset A$.

Step III. By (A3), $f(\Theta) > 0$, so let

$$m' = \frac{1}{f(\Theta)} m.$$

Then m' is a BPA whose associated belief function represents $\succsim$.

This completes the proof. $\square$

An answer to our first question is now easily obtained:

Corollary 1 *A binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$ may be represented by a plausibility function if and only if it satisfies (A0), (A1), (A2′) and (A3).*

Proof: It is straightforward to verify the "only if" part. For the converse, suppose that $\succsim$ satisfies (A0), (A1), (A2′) and (A3). Define another binary relation $\succsim^* \subseteq 2^\Theta \times 2^\Theta$ as follows: for any $A, B \in 2^\Theta$, $A \succsim^* B$ iff $B^c \succsim A^c$. It is easy to verify that $\succsim^*$ satisfies (A0), (A1), (A2) and (A3). It follows (Theorem 1) that there is a belief function that represents $\succsim^*$. It is obvious that its conjugate represents $\succsim$. $\square$

The following additional corollary – in conjunction with our main result (Theorem 3 below) – provides a new (and arguably simpler) proof of Theorem 4 in Nehring (1999). A *concave capacity* is a mapping $v : 2^\Theta \to [0,1]$ satisfying $v(\varnothing) = 0$, $v(\Theta) = 1$ and the following two conditions for any $A, B \in 2^\Theta$:

$$A \subseteq B \quad \Rightarrow \quad v(A) \leq v(B)$$

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$$

Corollary 2 *A binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$ may be represented by a concave capacity if and only if it satisfies axioms (A0), (A1), (A2′) and (A3).*

Proof: The "if" part follows since a plausibility function is a concave capacity, as is easily checked. For the converse, only (A2′) is non-trivial to verify. Concavity implies

$$v(A) + v(B \diagdown C) \leq v(A \diagdown C) + v(B)$$

when $C \subseteq B \subset A$. Thence $v(A \diagdown C) > v(B \diagdown C)$ when $v(A) > v(B)$. $\square$

We may now prove our main result:

Theorem 3 *Given a binary relation $\succsim \subseteq 2^\Theta \times 2^\Theta$ the following are equivalent:*

1. *$\succsim$ satisfies (A0), (A1), (A2′) and (A3).*

2. *$\succsim$ can be represented by a plausibility function.*

3. *$\succsim$ is an EIU order.*

4. *$\succsim$ is an EO order.*

Proof: Corollary 1 establishes the equivalence of (1) and (2). It is straightforward to verify that (3) implies (1) and that (4) implies (1). We complete the proof by showing that (2) implies (3) and also (4).

Let $\succsim$ be represented by the conjugate of the belief function v and let m denote the Möbius inverse of v. Then

$$A \succsim B \quad \Leftrightarrow \quad \sum_{E:E\cap A\neq\varnothing} m(E) \geq \sum_{F:F\cap B\neq\varnothing} m(F)$$

Defining

$$u_E(\theta) = \begin{cases} 1 & \text{if } \theta \in E \\ 0 & \text{otherwise} \end{cases}$$

for each $E \subseteq \Theta$, we observe that

$$\sum_{E\cap A\neq\varnothing} m(E) = \sum_{E\subseteq\Theta} m(E)\left[\max_{\theta\in A} u_E(\theta)\right] = \sum_{E\subseteq\Theta} m(E)\left[\max_{\theta\in A\cap E} u_\Theta(\theta)\right]$$

The second equality gives an EO representation, while the first, together with $S = 2^\Theta$, gives an EIU representation. □

The equivalence of (1) and (4) strengthens Theorem 1 in Barberà and Grodal (2011).

Corollary 2 (Barberà and Grodal, 2011, Theorem 1) *Let $\succsim \subseteq 2^\Theta \times 2^\Theta$ be an antisymmetric binary relation that satisfies (A0). Then the following are equivalent:*

(i) $\succsim$ satisfies (A1).

(ii) $\succsim$ is an EO order.

Proof: If $\succsim$ is an antisymmetric weak order that satisfies (A1) then it clearly satisfies (A2′) and (A3) as well. By Theorem 3 we deduce that (i) implies (ii). The converse is trivial. □

5 Discussion

The equivalence between (1) and (3) in Theorem 3 is the famous result of Kreps (1979), modulo the additional requirement of non-triviality. To see this, note that the contrapositive of (A2′) says: for any $A, B, C \in 2^\Theta$ with $B \subset A$ and $A \cap C = \emptyset$, $B \succsim A$ implies $B \cup C \succsim A \cup C$. Given (A0) and (A1), we may drop the restriction $A \cap C = \emptyset$.[6] From here, given (A1), it is innocuous to further modify (A2′) as follows: for any $A, B, C \in 2^\Theta$ with $B \subseteq A$, $B \sim A$ implies $B \cup C \sim A \cup C$, which is Kreps' condition (1.5).

Nehring (1999, Proposition 2) demonstrates the equivalence of (2) and (3). Barberà and Grodal (2011) observe that (4) implies (3). The converse is easily deduced from Nehring (1999, Proposition 1(ii)). Combining the work of Nehring, Kreps and Barberà and Grodal, one can therefore reconstruct all of Theorem 3. However, to the best of our knowledge, this reconstruction has not been explicitly done in the literature. Theorem 3 provides a useful synthesis, and a streamlined proof.

Theorem 3 also highlights the important related work of Wong et al. (1991), which seems not to be well-known within the decision theory community, as well as the usefulness of Möbius inversion for studying EIU or EO maximisation. The latter reinforces a point already made by Nehring (1999), but our arguments provide new insights into these connections.

References

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[6] Nehring (1999) calls the resulting property of $\succsim$ *ordinal submodularity.*