Economics Bulletin

Volume 34, Issue 3

Moral hazard and optimal insurance contract with a continuum effort

Niousha Shahidi
EDC Paris Business School

Abstract

In the research works, moral hazard is usually represented in two natural states (accident and no accident). In this case, the determination of the optimal contract could be made graphically. The mathematicaing is become more complicated when we consider infinite natural states and efforts under the monotone likelihood ratio property. In fact, the particular form of incentive constraints introduces a non-convex problem. In this paper, under technical conditions we show that the non-convex problem has a solution which is a new result and we determine the optimal contract such that the optimal wealth of the insured is a non-increasing function of the loss.

I am indebted to Professor Rose-Anne Dana, Professor Guillaume Carlier and the anonymous reviewer for helpful comments. Any errors are my own.

Citation: Niousha Shahidi, (2014) "Moral hazard and optimal insurance contract with a continuum effort", *Economics Bulletin*, Vol. 34 No. 3 pp. 1350-1360.

Contact: Niousha Shahidi - niousha.shahidi@edcparis.edu.

Submitted: November 20, 2013. **Published:** July 08, 2014.

1 Introduction

In a Principal-Agent model, we talk of moral hazard when the agent's decision which affects the principal's well-being, is not completely observed by the principal. The principal can therefore act, on his own account, on the agent's action, by means of incentives.

Milgrom and Roberts (1997) write: "the problems of moral hazard appear when an agent (i.e. a vendor, consumer, employee, etc.) is tempted to undertake an ineffective action or supply inaccurate information (leading others to ineffective actions) because his individual interests are not compatible with the collective interests and because neither the information given or the actions can be checked.

The concept of moral hazard was introduced by Arrow (1963) and Drèze (1961).The standard moral hazard model in insurance is as follows. An agent suffers a loss and makes an effort e to avoid it. This effort is not observable by the insurer. Let us recall the insurer's problem. The insurer maximises his profit under two constraints. The first is the constraint of the insured's participation. That means, the level of insured's usefulness must be greater than a level $\bar{u}$, no insurance. The second constraint is the constraint of incentive. This constraint, as its name indicates, encourages the insured to make the optimal effort.

Let us recall the results with two natural states in the world: accident and not accident. Let us suppose that there are two possible efforts: the base effort and the high effort. The probability of each event being realised depends on the effort. The probability of having an accident knowing that the effort is e_1 is bigger than the probability of having an accident knowing that the effort is e_2. The insurer is neutral towards the risk and the insured has an aversion to the risk. If the optimal effort is e_1 the comprehensive insurance is optimal, otherwise it is the partial insurance. In other words, if the base effort is optimal, everything happens as if there were no asymmetry of information between the insurer and the insured and the insured obtains comprehensive insurance. Otherwise the insurer encourages the insured to make the optimal effort by offering him a partial insurance contract.

When there are infinite natural states, Mirrlees (1976) considers a principal-agent model where the agent undertakes an action which cannot be observed by the principal to produce a good outcome. The principal and agent have strong aversion to risk. Furthermore, Mirrlees supposes that the greater the level of production the smaller the probability that the agent has carried out a low level of action. This hypothesis amounts to the monotone likelihood ratio (MLR). Mirrlees shows that, under the MLR property and the convexity of the conditional distribution function of the production in comparison to the agent's action, the optimal remuneration of the agent is a growing function of the level of production. The MLR property was used in the economy of information by Mirrlees (1976), Holmström (1979), Milgrom (1981) and in the economy of the uncertain by Landsberger and Meilijson (1990) and Jewitt (1991) and more recently in adverse-selection by Fluet and Pannequin (1997), Young and Browne (1997) and Shahidi (2009).

In the presence of moral hazard in insurance, if we consider that there are only two natural states (accident and no accident), the determination of the optimal contract could be made sometimes graphically or according to a fomalised problem.

Eisenhauer (2004) has detremined a general solution for the two-state model of pooling in the presence of moral hazard when the insurer minimized the average risk. In a finite states of losses, for example Hun Seog (2012) has solved a moral hazard problem and found that the indemnity is increasing in loss. The determination of the optimal contract for moral hazard when there are infinite natural states poses a problem. Once the problem is formalised, the problem of optimisation presents some incentive constraints (constraints which prompt the insured to make the optimal effort) which make the problem non-convex. So there are problems of existence of solutions which are not yet resolved in insurance case. In principal-agents problems, Page (1987) has developed an existence theory under general assumptions. In the present article, we deal explicitely with the insurance case and we show that the optimal non-convex problem has a solution under some assumptions which are specific to insurance.

This paper is organised as follows. In section 2, we state our basic assumptions and readjust the MLR property in the context of insurance when there are infinite natural states. In section 3, we present a version of the Mirrlees' result for insurance. We show that if there exists a solution, the optimal wealth of the insured is non-increasing with respect to the loss. Once the insurer's problem introduces a non-convex optimisation problem. This poses a problem of existence of solutions and resolutions. In section 4, we show that the non-convex problem admits a solution and the proposition 4 gives this new result.

2 The model

2.1 Insurance risk

Let $(\Omega, \mathcal{B}, \mathcal{P})$ be a probability space. The market provides insurance contracts for a potential loss. we consider a risk-neutral insurer and a policyholder who has an initial wealth w_0 (limited) and is risk averse, faces a loss X a random variable with support $[0, \bar{x}]$. He chooses an effort level $e \in I\!R^+$ to limit this loss. Let $F(x,e)$ be the conditional distribution function of X knowing that the effort level chosen by the insured is e. We assume that X allows a conditional density $f(x,e)$. We assume that $f(x,e)$ is strictly positive, continuous in x for all e, concave in e for all x and that $f(.,e)$ is bounded.
Let us recall a definition.

Definition 1 *The MLR property is that*
For all (e, e') verifying $e' > e$, the likelihood ratio $\dfrac{f(x,e')}{f(x,e)}$ is non-increasing in x

The MLR property makes it possible to differenciate the efforts: the greater the loss observed the greater the probability that the insured has made a low effort.

Let $f_e(x,e) = \dfrac{\partial f(x,e)}{\partial e}$.

Lemma 1 *The MLR property is equivalent to suppose that $\dfrac{f_e(x,e)}{f(x,e)}$ is non-increasing in x for all e.*

For a proof, see Topkis (1978). The Lemma 1 will be used in the proofs.
An insurance contract is characterised by a pair (I, Π) where Π is the estimated positive premium and $I : [0, \bar{x}] \to I\!R^+$ the indemnity depending on the loss. When the loss is worth x, the random wealth of the insured is

$$S(x) = w_0 - \Pi + I(x) - x$$

It is assumed that the insurer is neutral to risk. When the effort of the insured is e, the insurer's profit function is worth

$$\Pi - \int_0^{\bar{x}} I(x) f(x,e)dx = -\int_0^{\bar{x}} (x + S(x)) f(x,e)dx + w_0$$

The insured is risk averse and his utility is defined as below

$$u(S, \Pi, e) = \int_0^{\bar{x}} U(S(x)) f(x,e)dx - c(e)$$

where $U : I\!R \to I\!R$, strictly increasing, strictly concave and C^1. The function $c : I\!R^+ \to I\!R^+$ assumed C^1, strictly increasing and convex, represents the cost of the effort borne by the insured.

2.2 The insured's constraints

We assume that (S, Π, e) is such that the insured's level of utility is greater than $\bar{u}$ (no insurance case).

$$u(S, \Pi, e) \geq \bar{u}$$

This constraint is called the insured's participation constraint and marked **(CP)**.
Let us assume that e is the optimal effort. Since the insurer cannot observe the effort made by the insured, we assume that the contracts verify the constraint of incentive below, marked **(CI)**

$$u(S,\Pi,e) \geq u(S,\Pi,e') \quad \text{for all } e' \in I\!R^+$$

As its name indicates, this constraint encourages the insured to make the optimal effort because his level of utility is greater than that obtained with a non optimal effort.
The insurer maximises his profit under the constraint of the insured's participation and incentive, i.e. he resolves the problem below.

$$(\mathbf{U0}) \qquad \begin{cases} \sup\limits_{S,\Pi,e} \quad -\displaystyle\int_0^{\bar{x}} (x+S(x))f(x,e)dx \\ u(S,\Pi,e) \geq \bar{u} \qquad (\mathbf{CP}) \\ u(S,\Pi,e) \geq u(S,\Pi,e') \quad \text{for all } e' \in I\!R^+ \ (\mathbf{CI}) \\ \Pi \geq 0 \text{ and } e \geq 0 \end{cases}$$

3 Optimal contract

Let us introduce a new hypothesis
H1 $\quad f_e(x,0)=0$ for all $x \in [0,\bar{x}]$ and $c'(0)>0$.

Proposition 1 *Let us assume the MLR property and* **H1***, the problem* (**U0**) *is equivalent to the problem* (**U1**) *(where the new constraint* **(CI')** *replaces the constraint* **(CI)***).*

$$(\mathbf{U1}) \qquad \begin{cases} \sup\limits_{S,\Pi,e} -\displaystyle\int_0^{\bar{x}} (x+S(x))f(x,e)dx \\ u(S,\Pi,e) \geq \bar{u} \qquad (\mathbf{CP}) \\ \displaystyle\int_0^{\bar{x}} U(S(x))f_e(x,e)dx - c'(e) = 0 \quad (\mathbf{CI'}) \\ \Pi \geq 0 \ \textit{and } e \geq 0 \end{cases}$$

The proof is given in Appendix.

In what follows, we present (Mirrlees, 1976)'s result applicated to the insurance case. In (Winter, 2001) the result of the next proposition is demonstrated in the presence of finitely many states. The proof of the next proposition is similar to the (Jewitt, 1991) adapted to our insurance model.

Proposition 2 *Let us assume the MLR property and* **H1**. *If* (S^*,Π^*,e^*) *is a solution of* (**U1**)*, then* S^* *is a non-increasing function of* x*. In other words, the optimal wealth of the insured is non-increasing with respect to the loss.*

The proof is given in the Appendix.
Usually the authors donot show that the non-convex problem has a solution (for example see Holmström 1979 or Mirrlees 1976). In the next, we show that under technical conditions there exists a solution.
As we deal with insurance market, we must suppose that there is no overinsurance case ($I(x) \leq x$).

4 The existence of solution

As we have just remarked, we supppose that $I(x) \leq x$. There exists a constant $a \in I\!R$ such that

$$a \leq S(x) \leq w_0 - \Pi \qquad (\mathbf{CA})$$

where $a = w_0 - \Pi - \bar{x}$.

Let us recall the optimal insurance contract if there is no moral hazard.

4.1 No moral hazard case

$$(\mathbf{US}) \qquad \begin{cases} \sup\limits_{S,\Pi,e} \quad -\displaystyle\int_0^{\bar{x}} (x+S(x))f(x,e)dx \\ u(S,\Pi,e) \geq \bar{u} \qquad (\mathbf{CP}) \\ a \leq S(x) \leq w_0 - \Pi \quad (\mathbf{CA}) \\ \Pi \geq 0 \text{ and } e \geq 0 \end{cases}$$

$$\mathbf{H2} \qquad U'(a) = \infty \ \text{ and } \ U'(w_0 - \Pi) = 0.$$

Proposition 3 *Let us assume* **H2**. *If* (S^*, Π^*, e^*) *is a solution of* **(US)**, *then* S^* *is constant.*

The proof is given in the Appendix.
In other words the optimal contract is a deductible one.
In the next, let us consider the problem **(US)** under moral hazard, i.e. we add the constraint $(\mathbf{CI'})$.

4.2 Existence of solution in moral hazard

Let us define the model (**U2**) (the model (**U2**) with the constraint of the bounded wealth)

$$(\mathbf{U2}) \qquad \begin{cases} \sup\limits_{S,\ \Pi,\ e} \quad -\displaystyle\int_0^{\bar{x}} (x+S(x))f(x,e)dx \\ u(S,\Pi,e) \geq \bar{u} \qquad (\mathbf{CP}) \\ \int_0^{\bar{x}} U(S(x))f_e(x,e)dx - c'(e) = 0 \qquad (\mathbf{CI'}) \\ a \leq S(x) \leq w_0 - \Pi \qquad (\mathbf{CA}) \\ \Pi \geq 0 \text{ and } e \geq 0 \end{cases}$$

The next proposition is the principal result of our paper because of the existence of an optimal solution specific to insurance case.

Proposition 4 *Let us assume the MLR property,* **H1** *and* **H2**. *The problem* (**U2**) *has an optimal solution* (S^*, Π^*, e^*). *For any optimal solution, there exists an optimal solution with the same premium and the same level of effort such that the optimal wealth of the insured is a non-increasing function of the loss and gives the same profit to the insurer and the same utility to the insured.*

The proof is given in the Appendix.

5 Conclusion

In the presence of moral hazard, when there are infinite natural states, we gave the proposition of the existence of solutions to our non-convex problem. Our demonstration is based on the concept of decreasing rearrangement of random variables and the Hardy Littlewood inequality.We established that for all optimal contract, there exists an optimal solution such that the optimal wealth of the insured is a decreasing function of the loss and gave the same utility level to the insured and the same profit to the insurer. An other concept of the asymmetry of information which we can consider is adverse selection. In this model, the insurer proposes a set of contracts and he ignores the type of the agents. The high risks obtain full coverage while the low risks' contract is an optimal solution of a non-convex problem such that the associated wealth is non-increasing with respect to the loss.

Aknowledgement: I am indebted to Professor Rose-Anne Dana, Professor Guillaume Carlier and the anonymous reviewer for helpful comments. Any errors are my own.

References

Arrow, K.J. (1963) "Uncertainty and the Welfare Economics of Medical Care "*American Economic Review* **53**, 941-973.

Drèze, J. (1961) *Les fondements logiques de l'utilité cardinale et de la probabilité subjective*, La Décision: 73-87 Translated as Chapter 3 of Drèze, *Essays on Economic Decision Under Uncertainty.* Cambridge University Press.

Eeckhoudt, L. and C. Gollier (1995) "Demand for Risky Assets and the Monotone Probability Ratio Order" *Journal of Risk and Uncertainty* **11**, 113-122.

Eisenhauer, J. G. (2004) "Risk pooling in the presence of moral hazard",*Bulletin of Economic Research* **56(1)** , 107-111.

Fluet, C. and P. Pannequin (1997) "Insurance Contracts Under Adverse Selection with Random Loss Severity" document de travail, Chaire d'économie et d'économétrie de l'assurance (EHESS-ENSAE-Delta) **30**.

Gollier, C. (2001) *The economics of risk and time*, MIT Press.

Hardy, G.H., J.E. Littlewood and G. Polya (1934 reprinted in 1997) *Inequalities*, Cambridge University Press.

Holmström, B. (1979) "Moral Hazard and observability" *The Bell Journal of Economics* **19**, 74-91.

Hun Seog, S. (2012) "Moral Hazard and Health Insurance When Treatment Is Preventive" *The Journal of Risk and Insurance*, **79(4)**, 1017-1038.

Jewitt, I. (1991) "Applications of Likelihood Ratio Orderings in Economics" *Stochastic Orders* **19**, 74-91.

Jewitt, I. (1989) "Justifying the First-Order Approach to Principal-Agent Problems" *Econometrica* **56**, 1177-1190.

Landsberger, M. and I. Meilijson (1990) "Demand for Risky Financial Assets: Portfolio Analysis", *Journal of Economic Theory* **50**, 204-213.

Lehmann, E. L. (1991) *Testing Statistical Hypotheses*, Wiley, New York.

Milgrom, P. R. (1981) "Good news and bad news: Representation theorems and applications" *The Bell Journal of Economics* **12**, 380-391.

Milgrom, P. and J. Roberts (1997). Economie, organisation et management. Paris - Bruxelles: De Boeck Université.

Mirrlees, J. (1999) "The Theory of Moral Hazard and Unobservable Behaviour: Part I.", *Review of Economic Studies* **66**, 3-21.

Mirrlees, J. (1979) "The Implication of Moral Hazard for Optimal Insurance", in Seminar given at Conference held in honor of Karl Borch, Bergen, Norway, mimeo.

Mirrlees, J. (1976) "The optimal Structure of Incentives and Authority within an Organization" *The Bell Journal of Economics* **7(1)**, 105-131.

Page, F. (1987) "The existence of optimal contracts in the principal-agent model" *Journal of Mathematical Economics* **16**, 157-167.

Rogerson, W.P. (1985) "The First-Order Approach to Principal-Agent Problems" *Econometrica* **53**, 1357-1367.

Shahidi, N. (2009) "Competitive equilibrium in insurance markets under adverse selection and non-expected utlity" *The Journal of Risk* **12(1)**, 79-99.

Topkis, D. (1978) "Minimizing a submodular function on a lattice" *Operations Research* **26**, 305-21.

Young, V.R. and M.J. Browne (1997) "Explaining insurance policy provisions via adverse selection" *Geneva Papers on Risk and Insurance Theory* **22**, 121-134.

Winter, R. A. (2001 reprinted in 2012) *Optimal Insurance under Moral Hazard* in Handbook of Insurance edited by Georges Dionne.

Appendix

PROOF OF PROPOSITION 1 The effort e satisfies the constraint of incitation **(CI)** if and only if there exists a solution to the below problem

$$\sup_{e \geq 0} u(S, \Pi, e) \tag{1}$$

As $f(x,e)$ is concave in e and c is convex, $u(S, \Pi, .)$ is concave. Let e^* be a solution of (1). If $e^* > 0$, it is solution of

$$\int_0^{\bar{x}} U(S(x)) f_e(x,e) dx - c'(e) = 0 \quad (\mathbf{CI'})$$

The effort zero could not be a solution of **(U1)**. If $e^* = 0$ is an optimal solution of (**U1**) then the constraint (**CI′**) becomes, under **H1**, $c'(0) = 0$ which is impossible. Under **H1**, the problem (**U1**) is equivalent to the problem (**U0**).■

PROOF OF PROPOSITION 2 Let us suppose that (S^*, Π^*, e^*) is a solution of **(U1)**. As U is concave, if S^* is a solution of (**U1**), there exists $\lambda \geq 0$ and $\mu \in I\!R$ such that S^* is solution of

$$\begin{aligned}\sup_{S, \Pi \geq 0, e \geq 0} \quad -\int_0^{\bar{x}} (x + S(x)) f(x,e) dx \quad &+ \quad \lambda \left(\int_0^{\bar{x}} U(S(x)) f(x,e) dx - c(e) - \bar{u} \right) \\ &+ \quad \mu \left(\int_0^{\bar{x}} U(S(x)) f_e(x,e) dx - c'(e) \right)\end{aligned}$$

For all $x \in [0, \bar{x}]$, S^* is then solution of

$$\sup_S -(x + S) f(x,e) + \lambda U(S) f(x,e) + \mu U(S) f_e(x,e)$$

then

$$\frac{1}{U'(S^*(x))} = \lambda + \mu \frac{f_e(x,e^*)}{f(x,e^*)}$$

S^* is a function of $\dfrac{f_e(x,e^*)}{f(x,e^*)}$, constant if $\mu = 0$, non increasing if $\mu < 0$, non decreasing if $\mu > 0$. If S^* is a constant ($S^* = \bar{S}$), the constraint **(CI')** becomes

$$U(\bar{S}) \int_0^{\bar{x}} f_e(x,e^*)dx = c'(e^*)$$

As $\displaystyle\int_0^{\bar{x}} f(x,e)dx = 1$ for all e, $\displaystyle\int_0^{\bar{x}} f_e(x,e)dx = 0$ for all e. Moreover, $c'(e) > 0$ for all e involves that the constraint (CI') is not satisfied. If S^* is a non increasing function of $\dfrac{f_e(x,e^*)}{f(x,e^*)}$,

$$cov(U(S^*(X)), \frac{f_e(X,e^*)}{f(X,e^*)}) \leq 0$$

The previous inequality is equivalent to

$$\int_0^{\bar{x}} U(S^*(x)) \frac{f_e(x,e^*)}{f(x,e^*)} f(x,e^*)dx \leq \left(\int_0^{\bar{x}} U(S^*(x)) f(x,e^*)dx \right) \left(\int_0^{\bar{x}} \frac{f_e(x,e^*)}{f(x,e^*)} f(x,e^*)dx \right) \tag{2}$$

We have just remarked that $\int_0^{\bar{x}} f_e(x,e)dx = 0$ for all e. The inequality (2) is then equivalent to

$$\int_0^{\bar{x}} U(S^*(x)) f_e(x,e^*)dx \leq 0$$

which contradicts the constraint (CI') because $c'(e^*) > 0$. Then S^* is a non decreasing function of $\dfrac{f_e(x,e^*)}{f(x,e^*)}$ (and $\mu > 0$). According to the MLR property, the function $\frac{f_e(x,e)}{f(x,e)}$ is non increasng in x for all e. S^* is then a non increasing function of x. ■

PROOF OF PROPOSITION 3 Let us assume that (S^*, Π^*, e^*) be a solution of (**US**). As U is concave, there exists $\lambda \geq 0$ such that S^* is a solution of

$$\sup_{a \leq S \leq w_0 - \Pi, \ \Pi \geq 0, \ e \geq 0} W(S, \Pi, e, \lambda)$$

where

$$W(S, \Pi, e, \lambda) = -\int_0^{\bar{x}} (x + S(x)) f(x,e)dx + \lambda (\int_0^{\bar{x}} U(S(x)) f(x,e)dx - c(e) - \bar{u})$$

The optimal solution S^* of **(US)** is obtained by maximising pointwise

$$\sup_{a \leq S \leq w_0 - \Pi} V(S) \tag{3}$$

where $V(S) = -(x + S) + \lambda U(S)$. Let us remark that

$$V'(S) = -1 + \lambda U'(S)$$

According to **H2**, we could remark that if $\lambda > 0$

$$V'(a) > 0, \ V'(w_0 - \Pi) < 0$$

Furthermore, as V' is trictly decreasing in S then for all x, S^* is a solution of

$$\frac{1}{U'(S^*(x))} = \lambda$$

S^* is then a constant. If $\lambda = 0$ then $S^*(x) = a$. In any case, S^* is a constant. ∎

PROOF OF PROPOSITION 4 Before giving the proof , let us recall the definition of a non decreasing rearrangement, the inequality of (Hardy *et al*, 1934) and Helly's theorem.

Definition 2 *Let $t : [0, \bar{x}] \to [0, \bar{y}]$, be a measurable function with respect to μ (a nonatomic Borel probability measure on $[0, \bar{x}]$), there exists a unique non-decreasing function $\tilde{t}$, which is called the non-decreasing rearrangement of t with respect to μ such that*

$$\mu(\{\tilde{t} \leq \alpha\}) = \mu(\{t \leq \alpha\}), \text{ for all } \alpha \in [0, \bar{y}] \tag{4}$$

Let us recall that (4) is equivalent to the equimeasurability of t and $\tilde{t}$, which is defined as:

$$\int_0^{\bar{x}} f(t(x))d\mu(x) = \int_0^{\bar{x}} f(\tilde{t}(x))d\mu(x) \text{for every measurable functionf}$$

Lemma 2 *Let μ be a measure such that $d\mu = f(x)dx$ where f is positive, $f \in L^1$ and $\int d\mu = 1$. Let $t_i : [0, \bar{x}] \longrightarrow [0, \bar{y}]$ for $i = 1, 2$ be measurable with respect to μ. Let $\underline{t_1}$ be the non-increasing rearrangement of t_1 with respect to μ and let $\bar{t_i}$ be the non-decreasing rearrangement of t_i for $i = 1, 2$ with respect to μ. We then have:*

$$\int_0^{\bar{x}} \bar{t_1}(x)\bar{t_2}(x)d\mu(x) \geq \int_0^{\bar{x}} t_1(x)t_2(x)d\mu(x) \geq \int_0^{\bar{x}} \underline{t_1}(x)\bar{t_2}(x)d\mu(x)$$

Let us recall Helly's theorem:

Theorem 1 *Let $F = \{f(x)\}$ be a sequence of non-decreasing functions, defined on $[0, \bar{x}]$. If every function is bounded by the same constant:*

$$|f(x)| \leq K, f \in F, 0 \leq x \leq \bar{x},$$

then there exists a subsequence of functions $\{f_n(x)\}$ in F, which converges pointwise to a non-decreasing function $\phi(x)$.

Let us consider the problem (**U3**). This new problem is similar to (**U2**) where the constraint **(CI')** is replaced by the constraint **(CI")**.

$$(\mathbf{U3}) \qquad \begin{cases} \sup\limits_{S,\ \Pi,\ e} \; -\int_0^{\bar{x}} (x + S(x))f(x, e)dx \\ u(S, \Pi, e) \geq \bar{u} \qquad (\mathbf{CP}) \\ \int_0^{\bar{x}} U(S(x))f_e(x, e)dx \geq c'(e) \qquad (\mathbf{CI}'') \\ a \leq S(x) \leq w_0 - \Pi \qquad (\mathbf{CA}) \\ \Pi \geq 0 \text{ and } e \geq 0 \end{cases}$$

First, let us verify that (S^*, Π^*, e^*) is a solution of **(U2)** if and only if (S^*, Π^*, e^*) is a solution of **(U3)**. Let V_i be the function value of the problem **(Ui)** for $i = 3, 4$. the difference between the problems **(U2)** and **(U3)** is based on the second constraint which appears with an inequality in **(U3)** instead of an equality in **(U2)**. Then, $V_4 \geq V_3$.
Let us verify that if (S^*, Π^*, e^*) is a solution of **(U3)**, it is then a solution of **(U2)**. For this, we have to

demonstrate that the constraint **(CI")** is bounded. Let us suppose that it is not bounded, (S^*, Π^*, e^*) is then a solution of

$$\sup_{a \leq S \leq w_0 - \Pi,\ \Pi \geq 0,\ e \geq 0} W(S, \Pi, e, \lambda)$$

defined in the proof of Proposition 3 (because the associated multiplier to (CI'') is equal to zero). The optimal wealth S^* is then a constant, in this case $(\mathbf{CI''})$ is equivalent to $c'(e^*) \leq 0$, this inequality could not be satisfied (see the proof of Proposition 2). We then have $V_4 \leq V_3$ which implies that $V_4 = V_3$.

Let us verify that if (S^*, Π^*, e^*) satisfies the constraints **(CP)**, **(CI")** and **(CA)**, then $(\tilde{S^*}, \Pi^*, e^*)$, where $\tilde{S^*}$ is the non-increasing rearrangement (see Définition 2) of S^* with respect to $f(x, e^*)dx$, satisfies also the constraints.

- Since S^* and $\tilde{S^*}$ have the same probability distribution,

$$u(S^*, \Pi^*, e^*) = u(\tilde{S^*}, \Pi^*, e^*)$$

  Then, $(\tilde{S^*}, \Pi^*, e^*)$ satisfies **(CP)**.

- To verify the constraint $(\mathbf{CI''})$, let us use Lemma 2 of Hardy et al.. According to the MLR property, the function $\frac{f_e(x,e)}{f(x,e)}$ is non-increasing in x for all e. We obtain then

$$\int_0^{\bar{x}} U(\tilde{S^*}(x)) \frac{f_e(x, e^*)}{f(x, e^*)} f(x, e^*)dx \geq \int_0^{\bar{x}} U(S^*(x)) \frac{f_e(x, e^*)}{f(x, e^*)} f(x, e^*)dx \geq c'(e^*).$$

  $(\tilde{S^*}, \Pi^*, e^*)$ satisfies then the constraint **(CI")**.

- Moreover $\tilde{S^*}$ satisfies the constraint **(CA)**.

We have just proved that $(\tilde{S^*}, \Pi^*, e^*)$ satisfies all the constraints of the problem (**U3**). Let us prove that it gives the same profit to the insurer as (S^*, Π^*, e^*). Since S^* and $\tilde{S^*}$ have the same probability distribution

$$\int_0^{\bar{x}} (x + \tilde{S^*}(x)) f(x, e^*)dx = \int_0^{\bar{x}} (x + S^*(x)) f(x, e^*)dx$$

We have just proved that $(\tilde{S^*}, \Pi^*, e^*)$ gives the same profit to the insurer as (S^*, Π^*, e^*).

Whithout loss of generality, we can suppose that S^* is non-increasing. We have to prove that the problem **(U3)** has a solution assuming this hypothesis.

According to the foregoing, there exists a maximizing sequence (S_n^*, Π_n^*, e_n^*) where $S_n^*(x) = w_0 - \Pi_n^* + I_n^*(x) - x$ such that S_n^* is a non-increasing function of x. We have that (S_n^*, Π_n^*, e_n^*) satisfies the constraints of the problem **U3** and

$$\lim_{n \to \infty} - \int_0^{\bar{x}} (x + S_n^*(x)) f(x, e_n^*)dx = \sup_{S, \Pi, e} - \int_0^{\bar{x}} (x + S(x)) f(x, e)dx$$

where (S, Π, e) satisfies the constraints **(CP)**, **(CI")** and **(CA)**.

Let us prove that there exists a subsequence, noted in the same way, (S_n^*, Π_n^*, e_n^*) that converges to $(\bar{S}, \bar{\Pi}, \bar{e})$ such that the function $\bar{S} : [0, \bar{x}] \to [a, w_0 - \bar{\Pi}]$ is a non-increasing function of x and $(\bar{S}, \bar{\Pi}, \bar{e})$ satisfies the constraints **(CP)**, **(CI")** and **(CA)**.

- Let us remark that $0 \leq \Pi_n^* \leq w_0 - a$, hence the sequence Π_n^* is bounded and there exists then a subsequence Π_n^* that converges to $\bar{\Pi}$.

- The sequence of functions S_n^* is non-increasing in x and it is bounded because $a \leq S_n^*(x) \leq w_0$. According to Helly's theorem, there exists a subsequence of functions S_n^* which converges to a function $\bar{S} : [0, \bar{x}] \to [a, w_0 - \bar{\Pi}]$ non-increasing in x.

- Let us prove that e_n^* is bounded. Let us suppose that there exists a subsequence $e_n^* \to_{n\to\infty} \infty$. As $S_n^*(x) \leq w_0$ and S_n^* satisfies the constraint (**CP**), we obtain

$$U(w_0) - c(e_n^*) \geq u(S_n^*, \Pi_n^*, e_n^*) \geq \bar{u}$$

When $n \to \infty$,

$$U(w_0) - c(\infty) \geq \bar{u}$$

which is impossible because $c(\infty) = \infty$. There exists then a subsequence e_n^* which converges to $\bar{e}$. Let us remark that $\lim\limits_{n\to\infty} e_n^* \neq 0$. If $\lim\limits_{n\to\infty} e_n^* = 0$, the constraint (**CI**$''$) becomes under **H1**, $c'(0) \leq 0$ which is impossible.

We have just proved that thers exists a subsequence (S_n^*, Π_n^*, e_n^*) which converges respectively to $(\bar{S}, \bar{\Pi}, \bar{e})$ such that the function $\bar{S} : [0, \bar{x}] \to [a, w_0 - \bar{\Pi}]$ is non-increasing in x. It remains to prove that $(\bar{S}, \bar{\Pi}, \bar{e})$ satisfies the constraints **(CP)**, **(CI")** and **(CA)**.

As S_n^* and $f(., e_n^*)$ are bounded for all n, using the Theorem of Lebesgue's convergence,

$$\lim_{n\to\infty} u(S_n^*, \Pi_n^*, e_n^*) = u(\bar{S}, \bar{\Pi}, \bar{e}) \text{ for } i = 1, 2.$$

Using the previous limit, as (S_n^*, Π_n^*, e_n^*) satisfies the constraint (**CP**), $(\bar{S}, \bar{\Pi}, \bar{e})$ also satisfies this constraint. Doing a similar reasoning $(\bar{S}, \bar{\Pi}, \bar{e})$ satisfies the constraint (**CI**$''$). Otherwise, $(\bar{S}, \bar{\Pi}, \bar{e})$ satisfies the constraint (**CA**). Furthermore, using again the Theorem of Lebesgue's convergence

$$\begin{aligned} \lim_{n\to\infty} -\int_0^{\bar{x}} (x + S_n^*(x)) f(x, e_n^*) dx &= -\int_0^{\bar{x}} (x + \bar{S}(x)) f(x, \bar{e}) dx \\ &= \sup_{S,\Pi,e} -\int_0^{\bar{x}} (x + S(x)) f(x, e) dx \end{aligned}$$

where (S, Π, e) satisfies the constraints (**CP**), (**CI**$''$) and (**CA**). We obtain that $(\bar{S}, \bar{\Pi}, \bar{e})$ is a solution of (**U3**). ■