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Hedging demand and the certainty equivalent of wealth

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Abstract

This paper casts the opportunity set hedging demand in terms of the certainty equivalent of wealth for an investor who considers both consumption and bequest motives and is constrained to invest his asset proportions of wealth in a convex set. We show that the hedge portfolio exactly balances out the unfavorable impact of the opportunity set on the certainty equivalent of wealth.

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1. Introduction

A profuse empirical literature highlights the predictability of asset returns which is intimately related to the dynamics of the opportunity set.[1] A demand for hedging thus turns out to be an important component of the investor's assets demand. Traditionally, this hedging demand is analyzed and understood in light of the investor’s intermediate consumption (Breeden, 1979; Merton, 1973, 1971) rather than of her terminal wealth. Nevertheless, a large academic literature argues in favor of bequest motives (e.g. Constantinides et al. 2007). This article provides a framework to analyze hedging demand when both consumption and bequest motives are considered.

The opportunity set hedging demand is originally analyzed in terms of the intermediate consumption of investors. Ingersoll (1987), Breeden (1979) and Merton (1973, 1971) show that the hedging demand smoothes out the movements of intermediate consumption. Lioui and Poncet (2001) relate the hedging demand to the fixed income literature in a Markovian framework where the investor has constant relative risk aversion (CRRA) and focus only on bequest motives in a complete market. Detemple and Rindisbacher (2010) extend the work of Lioui and Poncet (2001) to incomplete markets, general utility functions and intermediate consumption using Malliavin calculus. Finally, Munk (2013) relates the hedging demand to the correlation between the portfolio of the investor and state variables in a framework that includes both consumption and terminal wealth.

We rely on the mathematical framework of Cvitanic and Karatzas (1992), that is, a non-necessarily Markovian setting in which investors can restrain their proportions of wealth in a predetermined convex set. As in Lioui and Poncet (2001), we focus on the empirical relevant case where an investor has a constant relative risk aversion towards wealth (Meyer and Meyer, 2005). However, our analysis of the hedging demand departs from that of Lioui and Poncet (2001): we rely on the certainty equivalent of wealth. The certainty equivalent of wealth can be defined in a non-necessarily Markovian framework when both intermediate consumption and terminal wealth are considered.

We show that the hedging demand smoothes out the movements of the certainty equivalent of wealth. Precisely, the hedging demand is the opposite of the (instantaneous) regression of the certainty equivalent of wealth per unit of wealth on traded assets: the hedging demand offsets unfavorable movements of the certainty equivalent per unit of wealth by favorable movements of the wealth of the investor. Moreover, we prove that the hedging demand is proportional to a portfolio that results from the highest correlation, in absolute value, between the wealth of the investor and the certainty equivalent (per unit of wealth).[2] Finally, we show that the weight invested in the hedging portfolio is the complement to unity of investor’s relative risk tolerance. As a consequence, the dividing role of the Bernoulli investor naturally arises in our framework (Breeden, 1979).

We run our illustration in the case of the equity hedging demand (Wachter, 2002; Barberis, 2000). Specifically, we rely on the incomplete market framework of Kim and Omberg (1996). We use our framework to numerically analyze the hump shape of the hedging equity demand as a function of risk aversion (Munk 2008; Wachter, 2002). This hump is loosely attributed to a misspecification of the elasticity of intertemporal substitution in these models (Wachter, 2002).[3] This hump is an interesting feature of equity. Indeed, Munk and

[1] See Merton (1973, 1971) for a definition of the opportunity set. The reader can also refer to Munk and Sørensen (2007) for a modern treatment of asset demand linked to a time-varying opportunity set.
[2] We thank an anonymous referee for this remark.
[3] Precisely, Wachter (2002, Section IV.B) ties the hedging demand to the wealth-consumption ratio and identifies a hump in the wealth-consumption as a function of risk aversion. She explains this hump by the link between risk aversion and elasticity of intertemporal substitution in the time additive framework (Campbell and Viceira, 1999)

Sørensen (2004) show, at least when terminal wealth is considered, that the hedging demand is an increasing function of risk aversion when stochastic interest rates are considered.

The remainder of the paper is organized as follows. Section two presents our setting as well as the optimal solutions. Section three focuses on the analysis of our results. Section four concludes and provides possible extensions.

2. Certainty equivalent and portfolio management

We analyze the link between the certainty equivalent of the investor and her hedging demand in the framework of Cvitanic and Karatzas (1992) where the investor is constrained to invest her proportions of wealth in a closed convex set, *K*. For clarity, we reproduce hereafter the setting of Cvitanic and Karatzas (1992).[4]

We consider a complete filtered probability space (Ω, F, Θ, P) endowed with a continuous non decreasing filtration $\Theta \equiv \{F_t : t \in [0,T]\}$. T is a positive constant that represents the end of the economy and $F_T \equiv F$. $z_t, t \in [0;T]$ is an n-dimensional Brownian motion defined on (Ω, F, Θ, P) that represent the risks that our investor faces. $\Theta \equiv \{F_t : t \in [0,T]\}$ can then be understood as the augmented filtration generated by the paths of this Brownian motion. For the remainder of the article, " ' " stands for the transpose symbol and E_t represents the expectation operator conditional on F_t.

We envisage a market that is fictitiously completed with financial assets. The reader can report to Karatzas et al. (1991) for an account on fictitious completion:

$$dP_{0t} = r_t P_{0t} dt \,, \tag{1a}$$

$$dP_t = I_{Pt}\left[\mu_t dt + \sigma_t' dz_t\right]. \tag{1b}$$

P_{0t} designates the risk-free asset with instantaneous risk-free return, r_t, and P_t is the n-dimensional vector of risky assets with instantaneous return, μ_t and volatility matrix, i.e. sensitivity matrix to the chocks of the Brownian motion, σ_t. I_{Pt} stands for the *nxn* diagonal matrix with the vector P_t on its diagonal. In our fictitious complete market, we can define the market price of risk as follows: $\theta_t = \sigma_t^{-1'}\left[\mu_t - r_t 1_n\right]$. 1_n the n-dimensional vector of ones. For later reference, we define the matrix of variance covariance of the market (1a,b): $\Sigma_t = \sigma_t' \sigma_t$.

As mentioned above, the vector of proportions, π, invested in the risky assets is constrained to take its value in a closed convex set: π∈K.[5] For later reference, we consider the support function of the set -K, $\delta(\nu) = \sup_{\pi \in K}\{-\pi'\nu\}$, defined on its effective domain, $\tilde{K} = \{\nu / \delta(\nu) < \infty\}$ (Cvitanic and Karatzas, 1992). Finally, our investor has a constant relative risk aversion, *γ*, and stems satisfaction from consumption, *c*, and bequest motives, *W*:[6]

$$J_t = \sup_{\pi \in K, c} E_t\left[\varepsilon_1 \int_t^T e^{-\alpha(u-t)} u(c_u) du + \varepsilon_2 e^{-\alpha(T-t)} u(W_T)\right], u(x) = x^{1-\gamma} / (1-\gamma) \tag{2a}$$

$$dW_t / W_t = \left[r_t + (\sigma_t \pi_t)' \theta_t\right] dt + (\sigma_t \pi_t)' dz_t. \tag{2b}$$

[4] The reader is invited to report to the article of Cvitanic and Karatzas (1992) for a full presentation of their setting.

[5] The proportion invested in the risk-free asset is the complement to unity to the sum of the proportions invested in the risky assets.

[6] We do not consider background noise in our framework. For an example of investments with background noise, the reader can refer to Osaki (2005).

$\varepsilon_1, \varepsilon_2 \geq 0, \varepsilon_1\varepsilon_2 > 0$ are weighting factors for the utility stemming from consumption and terminal wealth, respectively. The limiting case $\varepsilon_1 = 0$ ($\varepsilon_2 = 0$) represents an investor with utility from terminal wealth (consumption) only. α is the subjective time preference rate of the investor.

We define the certainty equivalent of wealth per unit of wealth: $ce_t = u^{-1}(J_t)/W_t$, and, for later reference, simply refer to it as the certainty equivalent of wealth. An important special case of our setting is the incomplete market framework, where $K = \{\pi \in R^n / \pi_k = 0, k = m+1,..,n\}$: the investor invests freely in the first m assets but cannot or does not want to invest in the remaining assets. We are now ready to state the main result of this manuscript:

Theorem 1. *Our investor's portfolio can be decomposed into a mean-variance portfolio and a certainty equivalent hedge portfolio with weights 1/γ and 1-1/γ, respectively:*

$$\begin{bmatrix} \pi_{ft} \\ \pi_t \end{bmatrix} = \frac{1}{\gamma}\begin{bmatrix} 1 - 1_n^{'} \pi_{MVt} \\ \pi_{MVt} \end{bmatrix} + \left[1 - \frac{1}{\gamma}\right]\begin{bmatrix} 1 - 1_n^{'} \pi_{cet} \\ \pi_{cet} \end{bmatrix}. \tag{3a}$$

The mean-variance portfolio is such that:

$$\pi_{MVt} = \Sigma_t^{-1}\left[\mu_t - r_t 1_n + \nu_t\right]. \tag{3b}$$

The certainty equivalent hedge portfolio is such that:

$$\pi_{Cet} = -\Sigma_t^{-1}\sigma_t^{'}\sigma_{ce_t}, \tag{3c}$$

where, $\nu_t \in \tilde{K}$ *is a process such that* $\delta(\nu_t) + \pi_t^{'}\nu_t = 0$ *and* σ_{ce_t} *is the volatility vector of the certainty equivalent per unit of wealth.*

Proof. See appendix.

In the important case of the incomplete market, we show in the appendix that *v* is nil for the part that corresponds to the traded assets while the vector of expected returns, the matrix of variance covariance and the matrix of volatilities reduce to those of the traded assets. This important special case justifies the name given to the mean-variance and certainty equivalent hedge portfolios as well as the subsequent comments of theorem 1.[7]

First, π_{MVt} involves the excess return of the risky assets divided by their matrix of variance covariance. As a consequence, π_{MVt}/γ corresponds to the mean-variance wealth proportion invested in the risky assets (Munk and Sørensen, 2007; Breeden, 1979; Merton, 1973, 1971). Therefore, $[1 - 1/\gamma]\pi_{cet}$ matches the proportion invested in the risky assets to hedge against the changes in the opportunity set. Note that this proportion vanishes out in the case of a Bernoulli type investor ($\gamma = 1$). Furthermore, a more (less) risk averse investor, $\gamma < 1$ ($\gamma > 1$), than the Bernoulli investor will short (buy) the certainty equivalent hedge portfolio to invest more (less) in the speculative mean-variance portfolio. As a consequence, an investor with a risk aversion higher than unity, $\gamma > 1$, is actually a hedging investor.

Second, the interpretation of the hedge portfolio, Eq. (3c), is straightforward and highlights the role of the certainty equivalent of wealth. Indeed, Eq. (3c) demonstrates that the hedge portfolio is the opposite of a regression coefficient: the hedge portfolio mimics the opposite of the movements of the certainty equivalent per unit of wealth. As a consequence, an unfavorable change in the certainty equivalent per unit of wealth is compensated for by an increase in the value of the wealth of the investor. In addition, we prove in the appendix that the risky proportions π_{cet} are proportional to the portfolio that maximizes, in absolute value,

[7] Because of the obvious link displayed by theorem 1 between the risk-free asset and risky assets, we focus our analysis on the risky proportions.

the correlation between the value of its associated strategy and the wealth certainty equivalent.

Third, our framework provides a novel interpretation of the usual increasing pattern, in absolute value, of the hedge portfolio as a function of horizon. Indeed, the increasing behavior of the utility function combined with the fact that any given strategy can be achieved by a strategy with a longer horizon guaranties that the certainty equivalent is an increasing function of the investment horizon. In our arbitrage-free framework, an investor is not better off with a longer investment horizon. As a consequence, the increasing pattern of the certainty equivalent must be compensated by an increase in its volatility. Theorem 1 proves that, as far as the hedging term is concerned, the investment horizon only impacts the demand in assets through the volatility of the certainty equivalent.

3. Illustration

We consider the financial market of Wachter (2002) and Kim and Omberg (1996), where the investor can choose between investing in an instantaneously risk-free asset with constant interest rate and a risky asset that stands for an equity index. The equity index has a market price of risk, which dynamically reverts to its long term mean and is negatively but imperfectly correlated with the innovations of the equity index.[8] The time-*t* price of the equity index and its market price of risk are denoted by S_t and λ_t, respectively:

$$\frac{dS_t}{S_t} = [r + \sigma_S \lambda_t]dt + \sigma_S dz_{St}, \tag{4a}$$

$$d\lambda_t = \kappa[\lambda_{lm} - \lambda_t]dt + \sigma_\lambda dz_{\lambda t}, \tag{4b}$$

where dz_{St} and $dz_{\lambda t}$ are the instantaneous increments of correlated one dimensional Brownian motions and stand for the innovations of our economy. Their correlation is denoted by, ρ, $\rho = E_t[dz_{\lambda t} dz_{St}]$. σ_S is the constant volatility of the equity price and r denotes the constant instantaneously risk-free interest rate. The market price of risk follows a mean-reverting process of constant speed of adjustment κ , long term mean λ_{lm} and volatility σ_λ .

The imperfect correlation between the equity price and its market price of risk results in market incompleteness. Liu (2007) shows that our framework leads to closed-form solutions provided that preferences are restricted to bequest motives:[9]

$$J_t = \sup_{\pi_s, t \le s \le T} E_t[u(W_T)], \tag{5a}$$

$$dW_t / W_t = [r + \sigma_S \pi_t \lambda_t]dt + \sigma_S \pi_t dz_{St}, \tag{5b}$$

where π_t stands for the proportion of wealth invested in the equity index. Theorem 1 demonstrates that the analysis of the mean-variance component is straightforward in our setting. As a consequence, we restrict our study to the hedging demand in equity. Because of the time-homogenous Markovian nature of our framework, we restrict our analysis at time 0: $\pi_{Ce} \equiv \pi_{Ce0}$. Direct computation proves that: $\pi_{Ce} = (\rho\sigma_\lambda / \sigma_S)[A_\lambda(T)\lambda_t + A_0(T)]$ where, $A_\lambda(T), A_0(T)$ are two deterministic functions of the horizon available from the authors upon request. The base-case parameters, in monthly unit, of the financial market, Eqs. (4a,b), are obtained from Wachter (2002) and Barberis (2000) and are given in Table 1.

[8] The equity market price of risk can be thought about as a dividend-price ratio (Wachter, 2002). Barberis (2000) finds a correlation coefficient of -0.93 between the dividend-price ratio and the equity price.

[9] α only leads to a positive multiplicative constant in Eq (5a) and does not need to be defined for program (5a,b).

Table 1. Base-case parameters in monthly unit and in %

r	**σ**	**λ_{lm}**	**κ**	**σ_λ**	**ρ**
0.14	4.36	7.88	2.26	1.89	-90.00

Table (1) reports the base case parameters in % and monthly unit used for the numerical illustrations.

Since our aim is to study the opportunity hedging demand as a function of risk aversion, we consider a hedging investor, i.e $\gamma \geq 1$. Indeed, in line with the empirical findings in Meyer and Meyer (2005) and most allocation studies,[10] investors are more risk averse than the Bernoulli investor. Moreover, as shown by Theorem 1, a more aggressive investor than the Bernoulli investor will short the hedge portfolio to buy more of the speculative portfolio: intuitively, this feature seems unlikely. We present our analysis for an investment horizon T=20 years.

Figure 1. Opportunity set hedging risky proportion as a function of risk aversion *γ*

Figure 1a) plots the hedging proportion $[1-1/\gamma]\pi_{Ce}$ and Figure 1b) the certainty equivalent hedge proportion π_{Ce} . Both proportions are displayed as a function of relative risk aversion, γ, varying from 1 to 10 for four values of the equity market price of risk: λ= - λ_{lm} (plain line), λ=0 (dash-dot line), λ= λ_{lm} (dotted line) and λ=2 λ_{lm} (dash line). λ_{lm} stands for the long term mean of the equity market price of risk. Investment horizon is set to 20 years. Parameters are given in Table 1.

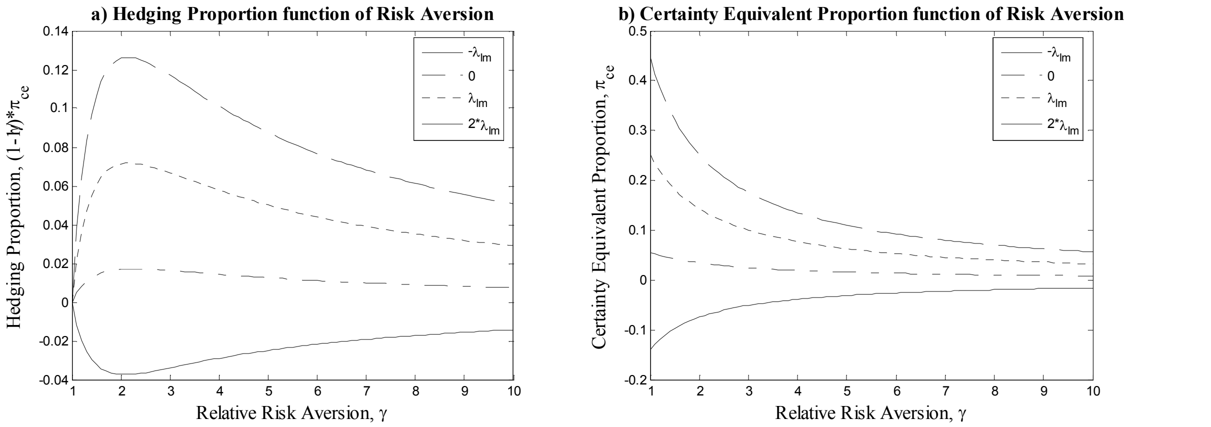

Figure 1a) plots the hedging demand proportion, $[1-1/\gamma]\pi_{ce}$ as a function of the relative risk aversion. We clearly notice a hump for every value of the equity market price of risk under consideration, i.e., for an equity market price of risk varying from the opposite of its long term value to the double of its long term mean value. The hedging demand depends on γ through the weight, $1\text{-}1/\gamma$, whose behavior as a function of γ is straightforward, as well as on the certainty equivalent hedge proportion, π_{ce} displayed in Figure 1b. Figure 1b shows a symmetric pattern of π_{ce} with respect to λ. It is increasing and negative for negative values of λ while decreasing and positive for positive values of λ - since our investor is unconstrained, the change of sign of λ can and does lead to symmetric positions. As a consequence, whatever the values of λ, π_{ce} is, in absolute value, a decreasing function of γ. Indeed, an unconstrained infinite risk-averse investor will not invest except for time preference motives (Munk and Sørensen, 2004). As a consequence, for negative values of λ, the hump underlined in Figure 1a is explained by the unconstrained nature of the investor, i.e., a negative hedging position while for positive values of λ, the hump is caused by opposite impacts of risk aversion on the hedging demand. On the one hand, a higher risk aversion reduces the volatility of the certainty equivalent of wealth. On the other hand, an investor with a higher risk aversion increases her hedging demand and thus increasingly invests in the opportunity set hedging portfolio through the weight 1-1/γ.

[10] For an account of this allocation studies, the reader can refer to the references in Munk and Sørensen (2007).

4. Concluding remarks

This article focalizes on the interpretation of the opportunity set hedging demand in a framework where the investor is constrained to invest in a closed convex set determined at the beginning of the investment. Our setting encompasses the important incomplete market case and our investor considers both consumption and bequest motives. We focus on the case of constant relative risk aversion. We show that the hedge portfolio protects the investor against unfavorable impacts of the opportunity set on the certainty equivalent of wealth and that the weight invested in this portfolio is an increasing function of risk aversion. We illustrate our decomposition in the case of the equity hedging demand.

5. Appendix

Cvitanic and Karatzas (1992) proves that optimal control of program (2a,b) can be couched as a function of a state price density as in the complete market case – see Munk and Sørensen (2007) for an example of the derivation of optimal quantities for complete markets:

$$c_{\nu u} = \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\alpha}{\gamma}(u-t)} \frac{W_t}{E_t\left[\int_t^T \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\alpha}{\gamma}(u-t)} H_{\nu u}^{1-\frac{1}{\gamma}} du + \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\alpha}{\gamma}(T-t)} H_{\nu T}^{1-\frac{1}{\gamma}}\right]} H_{\nu u}^{-\frac{1}{\gamma}}, \tag{A1}$$

$$W_{\nu T} = \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\alpha}{\gamma}(T-t)} \frac{W_t}{E_t\left[\int_t^T \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\alpha}{\gamma}(u-t)} H_{\nu u}^{1-\frac{1}{\gamma}} du + \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\alpha}{\gamma}(T-t)} H_{\nu T}^{1-\frac{1}{\gamma}}\right]} H_{\nu T}^{-\frac{1}{\gamma}}. \tag{A2}$$

The state price density is $H_{\nu u} = \exp\left(-\int_t^u r_x + \delta(\nu_x)du - \frac{1}{2}\int_t^s \left\|\theta_{\nu x}\right\|^2 du - \int_t^u \theta_{\nu x}{}' dz_x\right)$, the process ν_u is given as in theorem 1 and $\theta_{\nu u}$ is a modified market price of risk: $\theta_{\nu t} = \theta_t + \sigma_t^{-1'} \nu_t$.

We replace optimal controls given by Eq. (A1, A2) in the the budget constraint, $H_{\nu s} W_s = E_s\left[\int_s^T H_{\nu u} c_{\nu u} du + H_{\nu T} W_{\nu T}\right]$, to get optimal wealth as $W_s = \frac{W_t}{Q_{\nu t}} e^{-\frac{\alpha(s-t)}{\gamma}} H_{\nu s}^{-\frac{1}{\gamma}} Q_{\nu s}$, with $Q_{\nu t} = E_t\left[\int_t^T \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\alpha}{\gamma}(u-t)} H_{\nu u}^{1-\frac{1}{\gamma}} du + \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\alpha}{\gamma}(T-t)} H_{\nu T}^{1-\frac{1}{\gamma}}\right]$. We apply Ito lemma to optimal wealth and identify its volatility part with the volatility part of the value of the portfolio given by Eq. (2b):

$$\sigma_t \pi_t = \frac{1}{\gamma}\left[\theta_t + \sigma_t^{-1'} \nu_t\right] + \sigma_{Q_{\nu t}}, \tag{A3}$$

where $\sigma_{Q_{\nu t}}$ is the volatility (vector) of $Q_{\nu t}$.

We replace optimal controls in the value function defined by Eq. (2a) and arrange terms to get: $J_t = u(W_t)Q_{\nu t}^{\gamma}$. The certainty equivalent per unit of wealth is then simply a function of $Q_{\nu t}$: $ce_t = Q_{\nu t}^{\gamma/[1-\gamma]}$. We apply Ito lemma to the preceding equation to get: $\sigma_{Q_{\nu t}} = -\left[1 - \frac{1}{\gamma}\right]\sigma_{ce_t}$, where σ_{ce_t} is the volatility of the certainty equivalent. We arrange terms in (A3) to obtain:

$$\pi_t = \frac{1}{\gamma}\Sigma_t^{-1}\left[\mu_t - r_t 1_n + \nu_t\right] - \left[1-\frac{1}{\gamma}\right]\Sigma_t^{-1}\sigma_t^{'}\sigma_{ce_t}, \quad (A4)$$

Theorem 1 directly follows from Eq. (A4) by considering $\pi_{ft} = 1 - 1_n{}'\pi_t$.

Regarding the case of the incomplete market, Karatzas et al. (1991) show that σ can be partitioned as follows: *σ=[S R]*, where S is the volatility matrix of the traded assets and R is the volatility matrix of the fictitious assets that complete the market, such that S'R and R'S are nil matrixes. As a consequence, using block matrixes multiplication, the matrix of variance Σ is block diagonal: $\Sigma = \begin{bmatrix} \Sigma_S & 0_{m\,x\,n-m} \\ 0_{n-m\,x\,m} & \Sigma_R \end{bmatrix}$, where $\Sigma_S = S'S$ and $\Sigma_R = R'R$, as well as its inverse: $\Sigma^{-1} = \begin{bmatrix} \Sigma_S^{-1} & 0_{m\,x\,n-m} \\ 0_{n-m\,x\,m} & \Sigma_R^{-1} \end{bmatrix}$. In addition, Karatzas et al. (1991) have shown that the expected return of assets could also be partitioned between traded and fictitious assets: $\mu_t{}' = \begin{bmatrix} \mu_{St}{}' & \mu_{Rt}{}' \end{bmatrix}$. Finally, we denote by $\nu_t{}' = \begin{bmatrix} \nu_{St}{}' & \nu_{Rt}{}' \end{bmatrix}$, the parts of ν corresponding to the traded and the fictitious assets, respectively. Cuoco (1997) has shown that $\nu_{St} = 0_m$. We apply theses partitioning results to Eq. (A4) denoting by $\pi_t{}' = \begin{bmatrix} \pi_{St}{}' & \pi_{Rt}{}' \end{bmatrix}$ the parts of the demand invested in the risky and the fictitious assets, respectively:

$$\pi_{St} = \frac{1}{\gamma}\pi_{SMVt} + \left[1-\frac{1}{\gamma}\right]\pi_{Scet}, \quad (A5)$$

with $\pi_{SMVt} = \Sigma_{St}^{-1}\left[\mu_{St} - r_t 1_m\right]$and $\pi_{Scet} = -\Sigma_{St}^{-1}S_t^{'}\sigma_{ce_t}$. Similarly, computing the part of Eq. (A4) linked to the fictitious assets combined with the fact that the demand in fictitious assets is nil at equilibrium, $\pi_{Rt} = 0_{n-m}$, proves that ν_{Rt} is solution of the following implicit equation:

$$0_{n-m} = \frac{1}{\gamma}\Sigma_{Rt}^{-1}\left[\mu_t - r_t 1_n + \nu_{Rt}\right] - \left[1-\frac{1}{\gamma}\right]\Sigma_{Rt}^{-1}R_t^{'}\sigma_{ce_t}. \quad (A6)$$

To show that the hedging portfolio demand in risky asset is proportional to the portfolio that maximizes the correlation between the certainty equivalent and the wealth of the investor, we adapt a proof from Munk (2013, p. 94).[11] Let us call ζ, the correlation we want to maximize. This correlation will be maximized, in absolute value, when its square is maximized. Direct computation shows that:

$$\zeta_t^2 = \left(\pi_t^{'}\sigma_t^{'}\sigma_{cet}\right)^2 / \left(\pi_t^{'}\Sigma_t\pi_t\,\sigma_{cet}^{'}\sigma_{cet}\right). \quad (A7a)$$

The first order condition of (A7a) with respect to π to leads to:

$$\sigma_t^{'}\sigma_{cet}\pi_t^{'}\Sigma_t\pi_t = \pi_t^{'}\sigma_t^{'}\sigma_{cet}\Sigma_t\pi_t\ . \quad (A7b)$$

We multiply each side of (A7b) by Σ_t^{-1} to obtain:

$$\Sigma_t^{-1}\sigma_t^{'}\sigma_{cet}\pi_t^{'}\Sigma_t\pi_t = \pi_t^{'}\sigma_t^{'}\sigma_{cet}\pi_t\ . \quad (A7c)$$

We multiply (A7c) by $1_n^{'}$ to compute the sum of terms of (A7c) and use the fact that by the definition of portfolio, $1_n^{'}\pi_t = 1$:

$$1_n^{'}\Sigma_t^{-1}\sigma_t^{'}\sigma_{cet}\pi_t^{'}\Sigma_t\pi_t = \pi_t^{'}\sigma_t^{'}\sigma_{cet}. \quad (A7d)$$

We divide (A7c) by the sum of its components and get:

[11] This proof demonstrates that the correlation between the wealth of the investor and the state variable is maximized for an investment in the hedging portfolio.

$$\pi_t = \frac{\Sigma_t^{-1}\sigma_t^{'}\sigma_{cet}}{1_n^{'}\Sigma_t^{-1}\sigma_t^{'}\sigma_{cet}}. \quad \text{(A8)}$$

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