Economics Bulletin

Volume 34, Issue 4

Referendum paradox in a federal union with unequal populations: the three state case

Dominique Lepelley
CEMOI, Université de la Réunion

Vincent R Merlin
CREM, CNRS, Université de Caen

Jean-louis Rouet
ISTO UMR 7327 - CNRS/ISTU - BRGM - Université d''Orléans

Laurent Vidu
CREM, Université de Caen

Abstract

In a federal union, a referendum paradox occurs each time a decision taken by representatives elected in separate jurisdictions (districts, states, regions) conflicts with the decision that would have been adopted if the voters had directly given their opinion via a referendum (Nurmi 1999). Assuming that the population is split into three jurisdictions of respective size n1, n2 and n3, we derive exact formulas for the probability of the referendum paradox under the Impartial Culture model. Then we use these results to show that, in our model, allocating seats to the juridictions proportionally to the square root of their size is an apportionment rule that fails to minimize the probability of the referendum paradox in some federations.

This paper is part of the SOLITER project selected in the ``Gouverner, administrer'' program by the French Agence Nationale pour la Recherche (ANR) and has benefited from the ANR-08-GOUV-054 grant.
Citation: Dominique Lepelley and Vincent R Merlin and Jean-louis Rouet and Laurent Vidu, (2014) ''Referendum paradox in a federal union with unequal populations: the three state case'', *Economics Bulletin*, Vol. 34 No. 4 pp. 2201-2207.
Contact: Dominique Lepelley - dominique.lepelley@univ-reunion.fr, Vincent R Merlin - vincent.merlin@unicaen.fr, Jean-louis Rouet - jean-louis.rouet@univ-orleans.fr, Laurent Vidu - laurent.vidu@unicaen.fr.
Submitted: July 24, 2014. **Published:** October 24, 2014.

1 Introduction

In federal unions, decisions are taken via two-tier voting systems: voters choose their representatives in each state (district, region) and these representatives make the final choice, the majority rule being generally used at each level. In such a context, it may happen that the majority of the voters favour an opinion and the majority of the representatives opt for its negation. In social choice theory, this phenomenon is referred to as "referendum paradox" (Nurmi 1999), "compound majority paradox" or "election inversion" (Miller 2012). The referendum paradox is not only a theoretical notion: it has been observed in various real political elections, including the US presidential elections and the French "cantonal" elections (Lahrach and Merlin 2010). Its practical relevance explains why a number of studies have tried to compute its theoretical and empirical likelihood: May (1948), Feix *et al.* (2004), Lahrach and Merlin (2010), Lepelley *et al.* (2011), Miller (2012), among others. All the attempts to compute the theoretical probability of this paradox only deals with the simplest case when a federal union is divided into states with *equal size*. The first objective of the current study is to derive some analytical representations giving the probability of the referendum paradox for federations with three states having (possibly) *unequal sizes*. These representations are based on the widely used Impartial Culture assumption, also called the Independence assumption in game theory (Straffin 1977).

It is worth noticing that considering unequal populations makes the problem notably more complex (and it is the reason why we only consider the three-state case) because we have now to take into consideration the apportionment rule that associates a number of mandates (or a weight) to each state. In practice, the apportionment rules represent a compromise between the federal principle of state equality (one state-one vote) and the one man-one vote principle allocating mandates proportionnally to the population of each state. For example, in the EU, the number of mandated attributed to each state by the treaty of Nice was roughly proportional to the square root of state populations. As we cannot consider all the possible apportionment methods, we will focus our analysis on a particular class of rules, the δ-rules that allocate n_i^δ mandates to state i of population n_i. Though restrictive, this assumption encompasses the pure federalist case ($\delta = 0$ and each state has one mandate), the square root rule case ($\delta = 1/2$), the pure proportionality case ($\delta = 1$) and even the dictatorship of the biggest state ($\delta \to \infty$).

Which δ-rule should we choose? A well known answer to this question is based on the notion of equalizing voting power: the best rule is the one which gives to each citizen the same probability of being decisive, i.e. the same probability that his or her vote can change the result of the election (see Felsenthal and Machover, 1998, for a detailed description of these concepts). Penrose (1946, 1952) and Banzhaf (1965) have noticed that, under the Independence assumption, equal treatment in terms of voting power is carried out when each state (district) obtains a number of mandates proportional to the square root of its population. This result, known as the Penrose square root rule, is today a classical reference for many studies on federal unions and two-tier voting systems. Equalizing power is not, however, the only normative criterion that one can imagine to study the merits of various δ-rules. Another criterion, recently suggested by Lahrach and Merlin (2012), is related to referendum paradox: the best δ-rule is the one which minimizes the probability of the referendum paradox. An interesting question is then to know whether the square root rule remains optimal (under the Independence assumption) when this alternative criterion is taken in consideration. Answering this question for the three-state case is the second objective of this note.

2 The general model

Consider a set $I = \{1, \ldots, i, \ldots, N\}$ of N states (or regions, districts, etc.) which have to take decisions altogether in a political union. We assume that n_i voters live in state i, and $\sum_{i=1}^{N} n_i = n$. The vector $\tilde{n} = (n_1, \ldots, n_i, \ldots, n_N)$ describes the repartition of the population among the N states. Without loss of generality, we will assume throughout the paper that $n_1 \geq n_2 \geq \ldots \geq n_N > 0$. Two parties, A and B, compete in each state; the winner in state i is the party who obtains a majority of voters on his side (abstention is not allowed). Each state is represented by a_i mandates in the union, and the winner in state i gets all the mandates. For the sake of simplicity, we set that $a_1 \geq a_2 \geq \ldots \geq a_N \geq 0$, with at least a_1 strictly positive. Thus, the position that is officially adopted by the union is the one which obtains a majority of mandates at the federal level. Notice that we always use throughout the paper the quota of 50% for all the decisions (votes in the states, vote of the delegates and popular vote nationwide).

We assume that the votes from states to states are always drawn independently. Thus, the probabilistic behavior of a given state at the federal level is totally driven by the behavior of its voters. We next assume that each vote is determined by flipping independently a fair coin randomly, i.e. each citizen votes independently from the others and selects among the two issues with equal probability. In game theory, this hypothesis has been called the Independence assumption and has been used by Penrose (1946) and Banzhaf (1965) for measuring the voting power on an elector; it is equivalent to the Impartial Culture (IC) model used in social choice literature for the computation of voting paradox probabilities (Gehrlein 2006). We will restrict our attention to this model in the current paper.

We focus our study on the family of δ-rules. That is, we assume that the vector of mandates, $\tilde{a}$, is entirely characterized by the parameter δ, $\delta \in [0, \infty[$ as $a_i = n_i^{\delta}$ $\forall i = 1, 2, \ldots, N$.

As mentioned above, one of our objectives is to check whether the recommendations we should adopt when we wish to minimize the likelihood of the referendum paradox are compatible with the solution that has been put forward when one wishes to equalize the power of the citizens ($\delta = 0.5$ for Penrose-Banzhaf or IC model).

3 Results for $N = 3$

Without loss of generality, we assume in the following that $\sum_{i=1}^{3} n_i = n = 1$, with $n_1 \geq n_2 \geq n_3 > 0$. The distribution of the mandates is given by $\tilde{a} = (a_1, a_2, a_3)$, with $a_1 \geq a_2 \geq a_3 \geq 0$, and $a_1 > 0$. We consider δ-rules only: $\tilde{a} = (n_1^{\delta}, n_2^{\delta}, n_3^{\delta})$. But, for a weighted majority game with three players, it is well known (see *e.g.* Leech 2002) that any vector $\tilde{a} = (a_1, a_2, a_3)$ can be identified with one of four possible cases that we shall study hereafter.

3.1 Case 1 : $\tilde{a}^1 = (1, 1, 1)$

All the states have the same power. $\tilde{a}$ is equivalent to $\tilde{a}^1$ if and only if $n_1^{\delta} < n_2^{\delta} + n_3^{\delta}$. This is the most interesting case.

Proposition 1 *Let $P(\tilde{n}, \tilde{a}^1)$ be the likelihood of the referendum paradox for three states of large population under IC (Penrose-Banzhaf assumption) for the distribution $\tilde{n}$ when each state gets one mandate. Then:*

$$P(\tilde{n}, \tilde{a}^1) = \frac{\arccos\left(\sqrt{n_1}\right) + \arccos\left(\sqrt{n_2}\right) + \arccos\left(\sqrt{n_3}\right)}{\pi} - 0.75 \tag{1}$$

Proof: see Appendix.

3.2 Case 2: $\tilde{a}^2 = (2, 1, 1)$

$\tilde{a}$ is equivalent to $\tilde{a}^2$ if and only if $n_1^\delta = n_2^\delta + n_3^\delta$. In case the opinion of states 2 and 3 conflicts with the choice of state 1, a tie breaking rule could be implemented.

This very specific case only occurs when $n_1^\delta = n_2^\delta + n_3^\delta$. Moreover, we have to decide how to interpret a 2:2 deadlock, when state 1 votes for A, while states 2 and 3 endorse B. A latitudinarian interpretation of the definition of the referendum paradox would be to consider this situation is never a paradox, as the popular winner is not defeated with the indirect voting rule. On the opposite, a strict version of the paradox would consider all these situations as paradoxical, if one posits that the popular winner should win with no discussion. In order to derive probabilities, we will adopt a medium term. In case of 2:2 deadlock, we assume that the election is decided by tossing a fair coin, which means that only half of these situations are considered as paradoxical, depending whether or not the popular winner wins the draw.

Proposition 2 *Let $P(\tilde{n}, \tilde{a}^3)$ be the likelihood of the referendum paradox for three states of large population under IC for the distribution $\tilde{n}$ when $\tilde{a} = \tilde{a}^2$. Then:*

$$P(\tilde{n}, \tilde{a}^3) = \frac{2\arccos\left(\sqrt{n_1}\right) + \arccos\left(\sqrt{n_2}\right) + \arccos\left(\sqrt{n_3}\right)}{2\pi} - 0.375 \tag{2}$$

Proof: see Appendix.

3.3 Case 3, $\tilde{a}^3 = (1, 0, 0)$

In that case, state 1 is a dictator. $\tilde{a}$ is equivalent to $\tilde{a}^3$ if and only if $n_1^\delta > n_2^\delta + n_3^\delta$.

Proposition 3 *Let $P(\tilde{n}, \tilde{a}^3)$ be the likelihood of the referendum paradox for three states of large population under IC for the distribution $\tilde{n}$ when state 1 is a dictator. Then:*

$$P(\tilde{n}, \tilde{a}^3) = \frac{\arccos\left(\sqrt{n_1}\right)}{\pi} \tag{3}$$

Proof: see Appendix.

3.4 Case 4. $\tilde{a}^4 = (1, 1, 0)$

Player 3 is a dummy player and in case of opposite opinion for the two decisive states, a tie breaking rule could be implemented. But no δ-rule can encompass this case for the majority rule, as $n_3 > 0$ (see Barthélémy *et al.* 2013).

3.5 Comparisons

By comparing the values given by the formulas derived for the three possible apportionment cases, we are able to find the minimal value of the referendum paradox for each $\tilde{n}$. The corresponding minimal values of the paradox for IC are displayed on Table 1. First, our findings are consistent with the equal population case ($\tilde{n} = (1/3, 1/3, 1/3)$ results (Feix *et al.* 2004). It is also obvious from the proofs that the minimal values can never been be obtained with $\tilde{a}^2$.

By comparing $P(\tilde{n}, \tilde{a}^1)$ and $P(\tilde{n}, \tilde{a}^3)$, we observe that the former is inferior to the latter (and $a^1 = (1, 1, 1)$ is optimal) if and only if:

$$n_2 \geq \cos^2\left(\frac{\pi}{4} + \arccos\left(\sqrt{n_3}\right)\right) \tag{4}$$

Table 1: The minimal values for the referendum paradox for three states under IC.

$n_2 \downarrow \; n_3 \rightarrow$	0^+	0.05	0.10	0.15	0.20	0.25	0.30	0.333
0^+	$\mathbf{0^+}$	$--$	$--$	$--$	$--$	$--$	$--$	$--$
0.05	**0.0718**	**0.1024**	$--$	$--$	$--$	$--$	$--$	$--$
0.10	**0.1024**	**0.1266**	**0.1476**	$--$	$--$	$--$	$--$	$--$
0.15	**0.1266**	**0.1476**	**0.1666**	$\underline{0.1813}$	$--$	$--$	$--$	$--$
0.20	**0.1476**	**0.1666**	$\underline{\mathbf{0.1845}}$	$\underline{0.1773}$	$\underline{0.1727}$	$--$	$--$	$--$
0.25	**0.1666**	**0.1845**	$\underline{0.1824}$	$\underline{0.1747}$	$\underline{0.1698}$	$\underline{0.1666}$	$--$	$--$
0.30	**0.1845**	$\underline{0.1952}$	$\underline{0.1810}$	$\underline{0.1730}$	$\underline{0.1679}$	$\underline{0.1648}$	$\underline{0.1630}$	$--$
0.333	**0.1959**	$\underline{0.1948}$	$\underline{0.1804}$	$\underline{0.1722}$	$\underline{0.1671}$	$\underline{0.1641}$	$\underline{0.1625}$	$\underline{0.1623}$
0.35	**0.2015**	$\underline{0.1946}$	$\underline{0.1801}$	$\underline{0.1719}$	$\underline{0.1668}$	$\underline{0.1639}$	$\underline{0.1624}$	$--$
0.40	**0.2180**	$\underline{0.1943}$	$\underline{0.1796}$	$\underline{0.1714}$	$\underline{0.1665}$	$--$	$--$	$--$
0.45	**0.2341**	$\underline{0.1941}$	$\underline{0.1994}$	$--$	$--$	$--$	$--$	$--$
0.50^-	$\underline{\mathbf{0.25^-}}$	$--$	$--$	$--$	$--$	$--$	$--$	$--$

In bold: probabilities derived from $P(\tilde{n}, \tilde{a}^3)$.
Underlined: probabilities derived from $P(\tilde{n}, \tilde{a}^1)$.

Then, one may notice that the square root rule, characterized by $\delta = 1/2$, points toward the majority game whenever $\sqrt{n_1} \leq \sqrt{n_2} + \sqrt{n_3}$, and to the dictatorship otherwise. Solving this inequality leads to :

$$n_2 \geq \frac{1}{2} - \frac{n_3}{3} - \frac{1}{2}\sqrt{2\,n_3 - 3\,{n_3}^2} \tag{5}$$

Equation (4) is displayed in bold on Figure 1 for values n_2 and n_3 compatible with our constraints (the interior of the triangle); above it, a^1 is the optimal game, while a^3 enjoys this status below the line. By drawing equation (5), on the same figure, we identify below the dashed line the $\tilde{n}$ that the square root rule associates with the dictatorship of state 1. Clearly, the square root rule fails to be optimal, as the games in between the two curves should be associated with $\tilde{a}^1$. To give an example, consider $\tilde{n} = (0.65,\, 0.30,\, 0.05)$ in Table 1. Using the square root rule leads to the weights $(0.806,\, 0.547,\, 0, 224)$ and the dictatorship of state 1, while $\tilde{a}^1$ is optimal. By integrating the volumes between the two curves, we derive that the square root rule fails to be optimal for 10.38% of the federations. This graphic interpretation suggests that a value slightly smaller than 0.5 would enable us to move the dashed line closer to the optimal curve described by equation (4).

4 Conclusion

Based on the Penrose-Banzhaf (or IC) hypothesis, which assumes that each voter chooses between A and B by tossing a fair coin, the exact formulas we have obtained for $N = 3$ demonstrate that $\delta = 0.5$ **is not** optimal for minimizing the probability of the referendum paradox. Thus, the main conclusion of this note is that the square root rule, which stands for a long time as the only normative recommendation for voting in federations, can be seriously contested.

For $N > 3$, the number of states becomes too important to give a complete enumeration for all the cases. Moreover, obtaining formulas for more than three states, though technically possible for $N = 4$ and $N = 5$, as in (Feix *et al.* 2004), would be cumbersome. So, the search for an optimal allocation rule among the family of δ-rules when the number of states is higher than three has to rely on computer simulations.

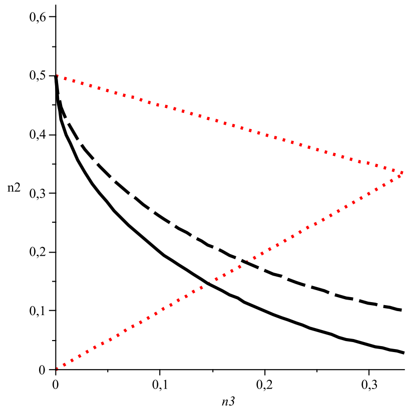

Figure 1: The boundaries between the majority game and dictatorship for the minimization of the referendum paradox. Domains for the square root rule and the optimal δ rule under IC.

References

Banzhaf J.R. III. (1965) Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review*, 13, 317–343.

Barthélémy F., Lepelley D., Martin M. (2013) On the likelihood of dummy players in weighted majority games. *Social Choice and Welfare*, 41, 263-279

Feix M., Lepelley D., Merlin V., Rouet J.L. (2004) The probability of US presidential type election paradoxes. *Economic Theory, 23*, 227–258.

Felsenthal D.S., Machover M. (1998) *The Measurement of voting power*. Edward Elgar Publishing.

Gerhlein W.V. (2006) *Condorcet's Paradox*. Springer.

Lahrach R., Merlin V. (2010) Assessing the probability of the referendum paradox: the French local election case. Working paper.

Lahrach R., Merlin V. (2012) Which voting rule minimizes the probability of the referendum paradox? Lessons from French data. In Felsenthal D.S. and Machover M. Eds, *Electoral systems. Paradoxes, Assumptions and Procedures*, 129–150. Springer

Leech D. (2002) Power indices as an aid to institutional design : the general apportionment problem. Warwick Economic Research Papers, nr 648

Lepelley D., Merlin V., Rouet J.-L. (2011) Three ways to compute accurately the probability of the referendum paradox. *Mathematical Social Sciences*, 62, 28–33.

May K., (1948) Probabilities of certain election results. *The American Mathematical Monthly* 55, 203–209.

Miller N.R. (2012) Election inversions by the U.S. Electoral College. In Felsenthal, D.S. and Machover, M. Eds, *Electoral systems. Paradoxes, Assumptions and Procedures*, 93–128. Springer

Nurmi H. (1999) *Voting paradoxes, and how to deal with them ?* Springer.

Penrose L.S. (1946) The elementary statistics of majority voting, *Journal of the Royal Statistical Society*, 109(I), 53–57.

Penrose L.S. (1952) *On the objective study of crowd behavior*. London: H.K. Lewis and Co.

Straffin P.D. (1977) Homogeneity, independence and power indices. *Public Choice*, 30, 107–118.